# Lecture Notes: Principles of Mathematical Analysis

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ABSTRACT. This is a lecture notes on REAL ANALYSIS, we mainly refers the book of Rudin, Walter *PRINCIPLES OF MATHEMATICAL ANALYSIS*.

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# Real Number System

The real number system is an ordered field with the least-upper-bound property. First let us consider the concept of order, which will make the least upper bound concept meaningful.

## 1. Ordered Sets

**Definition 1.1.** Let $S$ be a set. An *order* on $S$ is a relation, denoted by $<$, with the following two properties:

(1) If $x \in S$ and $y \in S$, then one and only one of the statements

$$x < y, \quad x = y, \quad y < x$$

is true.

(2) If $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

We usually call $S$ is an *ordered set* in this case.

The advantage of ordered set is that in which we can compare two elements with the help of the order. Especially we have the definition of *bounded* .

**Definition 1.2.** Suppose $S$ is an ordered set, and $E \subset S$. If there exists a $\beta \in S$ such that $x \leq \beta$ for every $x \in E$, we say that $E$ is *bounded above* , and call $\beta$ an upper bound of $E$.

The most core concept is the *lest upper bound* of set E.

**Definition 1.3.** Suppose $S$ is an ordered set, $E \subset S$, and $E$ is bounded above. Suppose there exists an $\alpha \in S$ with the following properties:

(1) $\alpha$ is an upper bound of $E$.

(2) If $\gamma < \alpha$ then $\gamma$ is not an upper bound of $E$.

then $\alpha$ is called the *least upper bound* of $E$ or the *supremum* of $E$, and we write

$$\alpha = \sup E.$$

**Definition 1.4.** An ordered set $S$ is said to have *least-upper-bound-property* if the following is true:

If $E \subset S$, $E$ is not empty, and $E$ is bounded above, then $\sup E$ exists in $S$.

## 2. Fields

An other aspect of real number system is that it is a field.

**Definition 2.1.** A *field* is a set $F$ with two operations, called *addition* and *multiplication* , which satisfy the following so-called "field axioms" (A), (M) and (D):

**(A) Axioms for addition**

(A1) If $x \in F$ and $y \in F$, then their sum $x + y$ is in $F$.

(A2) Addition is commutative: $x + y = y + x$ for all $x, y \in F$.

(A3) Addition is associative: $(x + y) + z = x + (y + z)$ for all $x, y, z \in F$.

(A4) $F$ contains an element $\mathbf{0}$ such that $\mathbf{0} + x = x$ for every $x \in F$.

(A5) To every $x \in F$ corresponds an element $-x \in F$ such that

$$x + (-x) = 0.$$

**(M) Axioms for multiplication**

(M1) If $x \in F$ and $y \in F$, then their product $xy$ is in F.
(M2) Multiplication is commutative: $xy = yx$ for all $x, y \in F$.
(M3) Multiplication is associative: $(xy)z = x(yz)$ for all $x, y, z \in F$.
(M4) $F$ contains an element $1 \neq 0$ such that $1x = x$ for every $x \in F$.
(M5) If $x \in F$ and $x \neq 0$ then their exists an element $1/x \in F$ such that

$$x \cdot (1/x) = 1.$$

**(D) The distributive law**

$$x(y + z) = xy + xz$$

hold for all $x, y, z \in F$.

**Remark 2.2.** The most basic property of the addition and multiplication of a field is the cancelation law:

(1) if $x + y = x + z$ then $y = z$. Which implies

$$x + y = x \Rightarrow y = 0, \quad x + y = 0 \Rightarrow y = -x, \quad \text{and } -(-x) = x.$$

(2) If $x \neq 0$ and $xy = xz$ then $y = z$. Which implies analogous properties with addition.

With the distribution law we have the following statements, for any $x, y, z \in F$:

(1) $0x = 0$.
(2) If $x \neq 0$ and $y \neq 0$ then $xy \neq 0$.
(3) $(-x)y = -(xy) = x(-y)$.
(4) $(-x)(-y) = xy$.

## 3. Ordered Fields

**Definition 3.1.** An *ordered field* is a field $F$ which is also an ordered set, such that

(1) $x + y < x + z$ if $x, y, z \in F$ and $y < z$;
(2) $xy > 0$ if $x \in F$, $x > 0$ and $y > 0$.

If $x > 0$, we call $x$ *positive* ; if $x < 0$, $x$ is *negative* .

**Proposition 3.2.** *The following statements are true in every ordered field.*

*(1) If $x > 0$ then $-x < 0$ and vice versa.*
*(2) If $x > 0$ and $y < z$ then $xy < xz$.*
*(3) If $x < 0$ and $y < z$ then $xz < xy$.*
*(4) If $x \neq 0$ then $x^2 > 0$. In particular, $1 > 0$.*
*(5) If $0 < x < y$ then $0 < 1/y < 1/x$.*

## 4. The Real Field

The core theorem of this chapter is the existence of real field.

**Theorem 4.1.** *There exists an ordered field $R$ which has the least-upper-bound property. Moreover, $R$ contains $Q$ as an subfield.*

**Remark 4.2.** Generally, there are three methods to construct the real number systems. In the book of Landau[Lan51] and Thurston[Thu56], they entirely devoted to number systems. Chapter 1 of Knopp's book[Kno28] contains a more leisurely description of how $R$ can be obtained form $\boldsymbol{Q}$. Another construction, in which each real number is defined to be an equivalence class of Cauchy sequences of rational numbers, is carried out in Sec. 5 of the book by Hewitt and stromberg[**HS65**].

The cuts in $\boldsymbol{Q}$ which Rudin's book used were invented by Dedekin. The construction of $\boldsymbol{R}$ form $\boldsymbol{Q}$ by means of Cauchy sequences is due to Cantor. Both Cantor and Dedekind published their constructions in 1872.

PROOF. We will use the *cuts* method of Dedekin to constructing $\boldsymbol{R}$ form $\boldsymbol{Q}$. The members of $\boldsymbol{R}$ will be certain subsets of $\boldsymbol{Q}$, called cuts.
**setp**1. [Construction of cuts]

**Definition 4.3.** A cut is any set $\alpha \subset Q$ with the following three properties

(1) $\alpha$ is not empty and $\alpha \neq Q$.
(2) If $p \in \alpha$, $q \in Q$ and $q < p$, then $q \in \alpha$.
(3) If $p \in \alpha$, then $p < r$ for some $r \in \alpha$.

Note that (3) simply says that $\alpha$ has no largest member, and (2) implies two facts which will be used freely:

(1) If $p \in \alpha$ and $q \notin \alpha$ then $p < q$.
(2) If $r \notin \alpha$ and $r < s$ then $s \notin \alpha$.

**setp**2. [Define an order "$<$" of Cuts] Define "$\alpha < \beta$" to mean: $\alpha$ is an proper subset of $\beta$, ie., $\alpha \subset \beta$ and $\alpha \neq \beta$.

Check that this truely defines an order on the cuts.

**setp**3. [$\boldsymbol{R}$ has the least-upper-bound property] Let $A$ be a nonempty subset of $\boldsymbol{R}$, and assume that $\beta \in R$ is an upper bound of $A$. Define $\gamma$ to be the union of all $\alpha \in A$. In other words, $p \in \gamma$ if and only if $p \in \alpha$ for some $\alpha \in A$. We shall prove that $\gamma \in R$ and that $\gamma = \sup A$.

Since $A$ is not empty, there exists an $\alpha_0 \in A$. This $\alpha_0$ is not empty. Since $\alpha_0 \in \gamma$, $\gamma$ is not empty. Next, $\gamma \subset \beta$, Since $\alpha \subset \beta$ for every $\alpha \in A$, therefore $\gamma \neq Q$. Thus $\gamma$ satisfies property (1) in the definition of cuts. To prove (2) and (3), pick $p \in \gamma$. Then $p \in \alpha_1$ for some $\alpha_1 \in A$. If $q < p$, then $q \in \alpha_1$, hence $q \in \gamma$; this proves (2). If $r \in \alpha_1$ is so chosen that $r > p$, we see that $r \in \gamma$, since $r \in \alpha_1 \subset \gamma$, and therefore $\gamma$ satisfies (3).

Thus $\gamma \in R$.

It is clear that $\alpha \leq \gamma$ for every $\alpha \in A$.

Suppose $\delta < \gamma$. Then there is an $s \in \gamma$ and that $s \notin \delta$. Since $s \in \gamma$, $s \in \alpha$ for some $\alpha \in A$. Hence $\delta < \alpha$, and $\delta$ is not an upper bound of $A$.

This gives the desired result: $\gamma = \sup A$.

**setp**4. [Definition of addition] If $\alpha \in R$ and $\beta \in R$ we define $\alpha + \beta$ to be the set of all sums $r + s$, where $r \in \alpha$ and $s \in \beta$.

We define $0^*$ to be the set of all negative rational numbers. The $0^*$ will play the role of $0$.

For a fixed $\alpha \in R$. Let $\beta$ be the set of all $p$ with the following property:

$$\text{There exists } r > 0 \text{ such that } -p - r \notin \alpha.$$

The $\beta$ will play the role of $-\alpha$.

Check that the axiom of addition satisfied.

**setp**5. [Cancelation law] The cancelation law of addition satisfied, especially we have

$$\text{If } \alpha, \beta, \gamma \in R \text{ and } \beta < \gamma, \text{ then } \alpha + \beta < \alpha + \gamma.$$

The first requirement of ordered field hold.

It also follows that $\alpha > 0^*$ if and only if $-\alpha < 0^*$.

**setp**6. [Definition of multiplication] We first confine ourself to $\boldsymbol{R}^+$, ie., the set of all $\alpha \in R$ with $\alpha > 0^*$.

If $\alpha, \beta \in R^+$, we define $\alpha\beta$ to be the set of all $p$ such that $p \leq rs$ for some choice of $r \in \alpha$, $s \in \beta, r, s > 0$.

We define $1^*$ to be the set of all $q < 1$.

Then the axioms of multiplication and distribution law of a field($R^+$) hold. Note that the second requirement of ordered field holds:

$$\text{If } \alpha > 0^* \text{ and } \beta > 0^* \text{ then } \alpha\beta > 0^*.$$

We complete the definition of multiplication by setting $\alpha 0^* = 0^* \alpha = 0^*$, and by setting

$$\alpha\beta = \begin{cases} (-\alpha)(-\beta) & \text{if } alpha < 0^*, \beta < 0^*, \\ -[(-\alpha)\beta] & \text{if } alpha < 0^*, \beta > 0^*, \\ -[\alpha \cdot (-\beta)] & \text{if } alpha > 0^*, \beta > 0^*. \end{cases}$$

Check that the axioms of multiplication and distribution law of $R$ satisfied.

Till now, we have completed the proof that $R$ is an ordered field with the least-upper-bound property.

**setp**7. [Contains $Q$ as a subfield] We associate with each $r \in Q$ the set $r^*$ which consists of all $p \in Q$ such that $p < r$. It is clear that $r^* \in R$, and satisfy the following relations:

(1) $r^* + s^* = (r+s)^*$,
(2) $r^* s^* = (rs)^*$,
(3) $r^* < s^*$ if and only if $r < s$.

The above properties says that the map which corresponding each $r \in Q$ with the cuts $r^* \in R$ preserves sums, products and order. In other words, The ordered field $Q$ is *isomorphic* to the ordered field $Q^*$ whose elements are the rational cuts.

*It is this identification of $Q$ with $Q^*$ which allows us to regard $Q$ as a subfield of $R$.*

This complete the whole theorem. □

**Remark 4.4.** Form $Q$ to $R$, there is an identification, and note that the same phenomenon occurs when the real numbers are regarded as a subfield of the complex field, and it also occurs at a much more elementary level, when the integers are identified with a certain subset of $Q$.

In fact, *any two ordered fields with the least-upper-bound property are isomorphic*. The first part of Theorem 4.1 therefore characterizes the real field $R$ completely.

As a good illustration of the least-upper-bound property we give the following important properties of real number system.

**Proposition 4.5** (*Archimedean property* ). *If $x, y \in R$, and $x > 0$, then there is a positive integer $n$ such that*

$$nx > y.$$

PROOF. Consider the set $\{nx\}_{n=1}^{\infty}$ and use the least-upper-bound property to derive a contradiction. □

**Proposition 4.6** (*Density of $Q$ in $R$* ). *If $x, y \in R$ and $x < y$, then there exists a $p \in Q$ such that $x < p < y$.*

PROOF. Apply the Archimedean property, we have

$$n(y - x) > 1$$

for some positive integer $n$.

Suppose that there is a rational number $m/n$ satisfy $x < m/n < y$, then

$$nx < m < ny, \quad \text{and } 1 + nx < ny.$$

Thus if we can choose an integer $m$, such that

$$nx < m \leq 1 + nx < ny,$$

then we have down. But this is equal to the existence of an integer $m$ which satisfy

$$m - 1 \leq nx < m.$$

Which can be derived from the Archimedean property. □

## 5. The Complex Field

**Definition 5.1.** A *complex number* is an ordered pair $(a, b)$ of real numbers with the following *addition* and *multiplication*

$$(a, b) + (c, d) \stackrel{\text{def}}{=} (a + c, b + d), \quad (a, b)(c, d) \stackrel{\text{def}}{=} (ac - bd, ad + bc).$$

We always write $(0, 0) = 0$, $(1, 0) = 1$, $(0, 1) = i$ and $(a, 0) = a$, Where the $i$ has the property $i^2 = -1$ and $(a, b) = a + bi \stackrel{\text{def}}{=} (a, 0) + (b, 0)(0, 1)$.

CHAPTER 3

# Basic Topology

There are some basic properties of real number system, first it is an uncountable set, while $Q$ is a countable set. Also it is a topology space, what's more it is a metric space. All these properties will be concerned in this chapter.

## 1. Metric Spaces

**Definition 1.1.** A set $X$, whose elements we shall call *points* , is said to be a *metric space* if with any two points $p$ and $q$ of $X$ there is associated a real number $\mathrm{d}(p, q)$, called the *distance* from $p$ to $q$, such that

(1) $\mathrm{d}(p, q) > 0$ if $p \neq q$; $\mathrm{d}(p, p) = 0$;
(2) $\mathrm{d}(p, q) = \mathrm{d}(q, p)$;
(3) $\mathrm{d}(p, q) \leq \mathrm{d}(p, r) + \mathrm{d}(r, q)$, for any $r \in X$.

Any function with these properties is called a *distance function* , or a *metric* .

**Example.** The most basic and important example is the Euclidean spaces $R^k$, especially $R^1$(the real line) and $R^2$(the complex plane); the distance in $R^k$ is defined by

$$\mathrm{d}(x, y) = |x - y| = \sqrt{\sum_{i=1}^{k}(x_i - y_i)^2}, \quad x, y \in R^k.$$

Other examples are the spaces of continuous function $\mathscr{C}(K)$ and square-integrable function spaces $\mathscr{L}^2(\mu)$.

**Remark 1.2.** It is important to observe that every subset $Y$ of a metric space $X$ is a metric space in its own right, with the same distance function.

**Definition 1.3.** we always call $(a, b)$ *segment* and $[a, b]$ an *interval* . If $x \in R^k$ and $r > 0$, then the *open (or closed) ball* $B$ with center at $x$ and radius $r$ is defined to be the set of all $y \in R^k$ such that $|y - x| < r$ (or $|y - x| \leq r$).

We call a set $E \subset R^k$ *convex* if for any $x, y \in R^k$, and any $0 \leq \lambda \leq 1$, we have

$$\lambda x + (1 - \lambda)y \in E.$$

Clearly, all balls are convex.

**Definition 1.4.** Let $X$ be a metric space. All points and sets mentioned below are understood to be elements and subsets of $X$.

(1) A *neighborhood* of $p$ is a ball $B_r(p)$ centered at $p$ and with radius $r$.
(2) A point $p$ is a *limit point* of the set $E$ if every neighborhood of $p$ contains a point of $E$, which is distinct from $p$. All limit points of $E$ will be denoted $E'$. The *closure* of $E$ is the set $\bar{E} = E \bigcap E'$.
(3) $E$ is *closed* if every limit point is still in $E$.
(4) $E$ is *open* if every point of $E$ has a neighborhood $N$, which is contained in $E$.
(5) Suppose $Y \subset X$ and $E \subset Y$. We say that $E$ is *open related to* $Y$ , if to each point $p \in E$, there is associated a ball $B_r(p)$ such that whenever $q \in B_r(p) \bigcap Y$, we have $q \in Y$.

(6) $\boldsymbol{E}$ is *perfect* if $\boldsymbol{E}$ is closed and if every point of $\boldsymbol{E}$ is a limit point of $\boldsymbol{E}$.
(7) $\boldsymbol{E}$ is *dense* in $\boldsymbol{X}$ if every point of $\boldsymbol{X}$ is a limit point of $\boldsymbol{E}$, or a point of $\boldsymbol{E}$ (or both).

The following propositions are almost clearly to see.

**Proposition 1.5.** *Suppose $\boldsymbol{E}$ is a subset of a metric space $\boldsymbol{X}$.*

(1) *every neighborhood of a limit point of $\boldsymbol{E}$ contains infinitely many points of $\boldsymbol{E}$.*
(2) *A set $\boldsymbol{E}$ is open if and only if its complement $\boldsymbol{E}^c$ is closed.*
(3) *open and close sets both closed under finite intersection and union, whereas open sets are closed under any union, and closed sets are closed under any intersection.*
(4) *$\bar{\boldsymbol{E}}$ is a closed set, and it is the smallest closed set that contains $\boldsymbol{E}$.*
(5) *Suppose $\boldsymbol{Y} \subset \boldsymbol{X}$. A subset $\boldsymbol{E}$ of $\boldsymbol{Y}$ is open relative to $\boldsymbol{Y}$ if and only if $\boldsymbol{E} = \boldsymbol{Y} \bigcap \boldsymbol{G}$ for some open subset $\boldsymbol{G}$ of $\boldsymbol{X}$.*

The interesting connection with real numbers will be the following

**Theorem 1.6.** *Let $\boldsymbol{E}$ be a nonempty set of real numbers which is bounded above. Let $\boldsymbol{y} = \sup \boldsymbol{E}$. Then $\boldsymbol{y} \in \bar{\boldsymbol{E}}$. Hence $\boldsymbol{y} \in \boldsymbol{E}$ if $\boldsymbol{E}$ is closed.*

## 2. Compactness

**Definition 2.1.** By an *open cover* of a set $\boldsymbol{E}$ in a metric space $\boldsymbol{X}$ we mean a collection $\{\boldsymbol{G}_\alpha\}$ of open subsets of $\boldsymbol{X}$ such that $\boldsymbol{E} \subset \bigcup_\alpha \boldsymbol{G}_\alpha$.

The basic properties of compactness are stated in the following theorem.

**Theorem 2.2.** *Suppose $\boldsymbol{X}$ is a metric space, $\boldsymbol{Y} \subset \boldsymbol{X}$.*

(1) *a subset $\boldsymbol{K} \subset \boldsymbol{Y}$ is compact relative to $\boldsymbol{Y}$ if and only if it is compact in $\boldsymbol{X}$.*
(2) *Compact subsets of metric spaces are closed. Since metric spaces are Hausdorff space.*
(3) *Closed subsets of compact sets are compact.*

The most important property of compact sets is that it has the so-called *finite intersection property* , which stated as a theorem in the following:

**Theorem 2.3.** *If $\{\boldsymbol{K}_\alpha\}$ is a collection of compact subsets of a metric space $\boldsymbol{X}$ with the finite intersection property, ie., the intersection of every finite sub-collection of $\{\boldsymbol{K}_\alpha\}$ is non-empty, then $\bigcap_\alpha \boldsymbol{K}_\alpha$ is non-empty too.*

*The most famous case is the intersection of a sequence of non-empty decreasing compact sets will be non-empty.*

**Example.** The *$\boldsymbol{k}$-cell or $\boldsymbol{k}$-cube* is compact. It can be showed with a continuous half-divide of the $\boldsymbol{k}$-cell to derive a contradiction.

In $\boldsymbol{R}^{\boldsymbol{k}}$, as a special case of metric space. There are some equivalent statement of compactness.

**Theorem 2.4.** *If a set $\boldsymbol{E}$ in $\boldsymbol{R}^{\boldsymbol{k}}$ has one of the following three properties, then it has the other two:*

(1) *$\boldsymbol{E}$ is closed and bounded.*
(2) *$\boldsymbol{E}$ is compact.*
(3) *Every infinite subset of $\boldsymbol{E}$ has a limit point in E.*

**Corollary 2.5** (Weierstrass). *Every bounded infinite subset of $\boldsymbol{R}^{\boldsymbol{k}}$ has limit point in $\boldsymbol{R}^{\boldsymbol{k}}$.*

### 3. Perfect Sets

One fact about perfect set is that any non-empty perfect set in $R^k$ will be uncountable.

In fact, if $E$ is a countable perfect subset of $R^k$. Without lost of generality, let $E = \{x_i\}_{i=1}^{\infty}$. We can construct a sequence of neighborhoods $\{V_n\}$ as follows.

Let $V_1$ be any neighborhood of $x_1$. Suppose that $V_n$ has been constructed, so that $V_n \bigcap E$ is not empty. It must contains infinite point of $E$, since every point of $E$ is a limit point of $E$. Thus there is a neighborhood $V_{n+1}$ such that $\bar{V}_{n+1} \subset V_n$, $x_n \notin \bar{V}_{n+1}$ and $V_{n+1} \bigcap E \neq \emptyset$.

Put $K_n = \bar{V}_n \bigcap E$. Since $\bar{V}_n$ is closed and bounded, $\bar{V}_n$ is compact. Clearly, $\bigcap_{i=1}^{\infty} K_i$ is empty (note that $K_n \subset E$). But $\{K_n\}$ has the finite intersection property, this is a contradiction.

**Example** (Cantor set)**.** Let $E_0$ be the interval $[0, 1]$. Remove the segment $(\frac{1}{3}, \frac{2}{3})$, and let $E_1$ denote the union of intervals

$$[0, \frac{1}{3}], \quad [\frac{2}{3}].$$

Remove the middle of thirds of these intervals, and $E_2$ be the union of intervals

$$[0, \frac{1}{9}], \quad [\frac{2}{9}, \frac{3}{9}], \quad [\frac{6}{9}, \frac{7}{9}], \quad [\frac{8}{9}, 1].$$

Continuing in this way, we obtain a sequence of compact sets $E_n$, clearly it has the following properties:

(1) $E_1 \supset E_2 \supset \cdots$;
(2) $E_n$ is the union of $2^n$ intervals, each of length $3^{-n}$;
(3) the removed intervals are

$$\left(\frac{3k+1}{3^m}, \frac{3k+2}{3^m}\right), \quad k, m \in Z^+.$$

The set

$$P = \bigcap_{n=1}^{\infty} E_n,$$

is called the *Cantor sets* .

It can be shown that $P$ is a non-empty perfect closed compact set. One of the most interesting properties of the Cantor set is that it provides us with an example of an uncountable set of measure zero.

### 4. Connected Sets

**Definition 4.1.** Two subsets $A$ and $B$ of a metric space $X$ are side to be *separated* if both $A \bigcap \bar{B}$ and $\bar{A} \bigcap B$ are empty. A set $E \subset X$ is said to be *connected* if $E$ is not a union of two nonempty separated sets.

The connected subset of the real line have a particularly simple structure.

**Theorem 4.2.** *A subset $E$ of the real line $R^1$ is connected if and only if it has the following property: If $x \in E$, $y \in E$, and $x < z < y$, then $z \in E$.*

CHAPTER 4

# Numerical Sequences and Series

In this chapter, we always assume our sequence is complex, except clearly stated.

## 1. Convergent Sequences

**Definition 1.1.** A sequence $\{p_n\}$ in a metric space $X$ is said to *converge* if there is a point $p \in X$ such that for every $\varepsilon > 0$, there is an integer $N$, such that $n \geq N$ implies that $\mathrm{d}(p_n, p) < \varepsilon$. In this case, we also say that $\{p_n\}$ is converges to $p$, and $p$ is the limit of $\{p_n\}$, write

$$\lim_{n\to\infty} p_n = p.$$

If $\{p_n\}$ is not converge, it is said to *diverge* .

The well known properties about convergent sequences is stated in the following proposition.

**Proposition 1.2.** *Let $\{p_n\}$ be a sequence in a metric space $X$.*

(1) *$\{p_n\}$ converges to $p \in X$ if and only if every neighborhood of $p$ contains $p_n$ for all but finitely many $n$.*
(2) *The limit of a convergent sequence is unique.*
(3) *a convergent sequence is bounded.*
(4) *If $E \subset X$ and if $p$ is a limit point of $E$, then there is a sequence $\{p_n\}$ in $E$ such that $p = \lim_{n\to\infty} p_n$.*
(5) *The limit operation is commutable with the* four arithmetic operation *, ie., addition, minus, product and division.*
(6) *If $x_n = (x_n^1, x_n^2, \cdots, x_n^k) \in R^k$, then $\{x_n\} = (x^1, x^2, \cdots, x^k)$ converges to $x \in R^k$ if and only if $\lim_{n\to\infty} x_n^i = x^i$ for every $i = 1, 2, \cdots, k$.*

## 2. Subsequences

**Definition 2.1.** Given a sequence $\{p_n\}$, consider a sequence $n_k$ of positive integers, such that $n_1 < n_2 < \cdots$. Then the sequence $\{p_{n_i}\}$ is called a *subsequence* of $\{p_n\}$.

It is clear that $\{p_n\}$ is converges to $p$ if and only if every subsequence of $\{p_n\}$ converges to $p$.

A other fact about subsequence is the following theorem.

**Theorem 2.2.** *The subsequential limits of a sequence $\{p_n\}$ in a metric space $X$ form a closed subset of $X$.*

PROOF. Let $E^*$ be the set of all subsequential limits of $\{p_n\}$ and let $q$ be a limit point of $E^*$. We need to show that $q \in E^*$.

Choose $n_1$ so that $p_{n_1} \neq q$. (if no such $n_1$ exists, then $E^*$ has only one point, and there is nothing to prove.) Put $\delta = \mathrm{d}(q, p_{n_1})$. Suppose $n_1, n_2, \cdots, n_{i-1}$ are chosen. Since $q$ is a limit point of $E^*$, there is an $x \in E^*$ with $\mathrm{d}(x, q) < 2^{-i}\delta$. Since $x \in E^*$, there is an $n_1 > n_{i-1}$ such that $\mathrm{d}(x, p_{n_i}) < 2^{-i}\delta$. Thus

$$\mathrm{d}(q, p_{n_i}) < 2^{1-i}\delta,$$

for $i = 1, 2, \cdots$. This says that $\{p_{n_i}\}$ converges to $q$. Hence $q \in E^*$. □

## 3. Cauchy Sequences

**Definition 3.1.** A sequence $\{p_n\}$ in a metric space $X$ is said to be a *cauchy sequence* if for every $\varepsilon > 0$ there is an integer $N$ such that $\mathrm{d}(p_n, p_m) < \varepsilon$ for any $m, n \geq N$.

**Definition 3.2.** A metric space in which every Cauchy sequence converges is said to be *complete* .

**Proposition 3.3.** *Let $X$ be a metric space.*

1. *Every convergent sequence is a Cauchy sequence.*
2. *If $X$ is compact, then every Cauchy sequence converge to some point of $X$.*
3. *$R^k$ is a complete metric space.*

## 4. Bounded Sequence and Monotonic Sequences

We already know that every convergent sequence is bounded, but the converse is not true in general. However, if we require that the sequence is monotonic, decrease or increase, the converse will be true.

**Theorem 4.1.** *Suppose $\{s_n\}$ is monotonic. Then $\{s_n\}$ converges if and only if it is bounded.*

## 5. Upper and Lower Limits

**Definition 5.1.** Let $\{s_n\}$ be a sequence of real number. Let $E$ be the set of numbers $x$ (include $+\infty$ and $-\infty$) such that $s_{n_k} \to x$ for some subsequence $\{s_{n_k}\}$. Define

$$s^* = \sup E, \text{ and } s_* = \inf E.$$

They are called the *upper and lower limits* of $\{s_n\}$, respectively. We also use the notation

$$\limsup_{n\to\infty} s_n = s^*, \quad \liminf_{n\to\infty} s_n = s_*.$$

**Theorem 5.2.** *Let $\{s_n\}$ be a sequence of real numbers. Let $E$ and $s^*$ have the same definition as above. Then $s^*$ has the following two properties:*

1. *$s^* \in E$.*
2. *If $x > s^*$, then there is an integer $N > 0$ such that $n \geq N$ implies $s_n < x$.*

*Move over, these two properties also define the upper limits of $\{s_n\}$. Analogous result is true for $s_*$.*

It's quite clearly to see that the $\limsup$ and $\liminf$ operation preserve the inequality.

## 6. Series

We need not to constrain ourselves to the real-valued series, if it do, we will explicitly state.

**Definition 6.1.** Given a sequence $\{a_n\}$, we use the notation

$$\sum_{n=P}^{q} a_n \quad (p \leq q)$$

to denote the sum $a_p + ap + 1 + \cdots + a_q$. With $\{a_n\}$ we associate a sequence $\{s_n\}$, where

$$s_n = \sum_{k=1}^{n} a_k.$$

For $\{s_n\}$ we also use the symbolic expression

$$a_1 + a_2 + a_3 + \cdots$$

or, more concisely

$$\sum_{n=1}^{\infty} a_n.$$

Called *infinite series* . The numbers $s_n$ are called *partial sums* of the series. If $\{s_n\}$ converges to $s$, we say that the series *converges* , and write

$$\sum_{n=1}^{\infty} a_n = s.$$

The number is called the sum of the series.

If $\{s_n\}$ diverges, the series $\{a_n\}$ is said to *diverge* .

**Remark 6.2.** It is quite clear that every theorem about sequences can be stated in terms of series, since $a_n = s_n - s_{n-1}$, $n \geq 2$, $a_1 = s_1$, and vice versa.

Especially the Cauchy criterion for convergent sequence can be restated in the following form:

**Theorem 6.3.** *$\sum a_n$ converges if and only if for every $\varepsilon > 0$ there is an integer $N$ such that*

$$\left| \sum_{k=n}^{m} \right| \leq \varepsilon$$

*for every $m \geq n \geq N$.*

Particularly, the the term of convergent series must tend to $0$ as $n$ tends to $\infty$.

**Corollary 6.4.** *If $\{a_n\}$ converges, then $\lim_{n\to\infty} a_n = 0$.*

As a consequence of the bounded monotonic theorem (cf. theorem 4.1), we have that

**Theorem 6.5.** *A series of nonnegative terms* [1]*converges if and only if its partial sums form a bounded sequence.*

**6.1. Test Method of Series.** To test a series is convergent or divergent, we have the so called "*comparison test* ".

**Theorem 6.6.** *Suppose $\{a_n\}$, $\{c_n\}$, $\{d_n\}$ are series.*

*(1) If $|a_n| \leq c_n$ for $n \geq N_0$, where $N_0$ is some fixed integer, and if $\sum c_n$ converges, then $\sum a_n$ converges.*
*(2) If $a_n \geq d_n$ for $n \geq N_0$, and if $\sum d_n$ diverges, then $\sum a_n$ diverges.*

The comparison test is a very useful one; The most used non-negative series is the *geometry series* .

**Theorem 6.7.** *The series $\{a_n\}$, $a_n = x^n$ converges for $x \in [0, 1)$, and diverges other wise.*

**Problem.** Suppose $a_1 \geq a_2 \geq a_3 \cdots \geq 0$. Then the series $\sum_{n=1}^{\infty} a_n$ converges if and only if the series

$$\sum_{k=0}^{\infty} 2^k a_{2^k}$$

converges.

Use the above result to conclude that $\sum \frac{1}{n^p}$ converges if $p > 1$ and diverges if $P \leq 1$.

Take a thought at the converges or diverges of the following series

$$\sum_{n=1}^{\infty} \frac{1}{n(\log n)^p}, \quad \sum_{n=3}^{\infty} \frac{1}{n \log n \log\log n}, \quad \sum_{n=3}^{\infty} \frac{1}{n \log n (\log\log n)^2}, \cdots .$$

[1]nonnegative terms means that every term $a_n > 0$.

There are two derived ways in application of the comparison test, ie., the *root test* and the *ratio test* .

**Theorem 6.8** (Root Test)**.** *Given* $\sum a_n$*, put* $\alpha = \limsup_{n\to\infty} \sqrt[n]{|a_n|}$*, then*

(1) *if* $\alpha < 1$*,* $\sum a_n$ *converges;*
(2) *if* $\alpha > 1$*,* $\sum a_n$ *diverges;*
(3) *if* $\alpha = 1$*, the test gives no information.*

**Theorem 6.9** (Ratio Test)**.** *The series* $\sum a_n$*,*

(1) *converges if* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$*,*
(2) *diverges if* $\limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| \geq 1$ *for all* $n \geq n_0$*, where* $n_0$ *is some fixed integer.*

**Remark 6.10.**

- The ratio test is frequently easier to apply than the root test, however, the root test has wider scope, since we have

  **Theorem 6.11.** *For any sequence* $\{c_n\}$ *of positive number,*

  $$\liminf_{n\to\infty} \frac{c_{n+1}}{c_n} \leq \liminf_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \sqrt[n]{c_n} \leq \limsup_{n\to\infty} \frac{c_{n+1}}{c_n}.$$

- Neither of the two test is subtle with regard to divergence, and they are both for absolute convergence.

### 6.2. Power Series.

**Definition 6.12.** Given a sequence $\{c_n\}$ of complex numbers, the series

$$\sum_{n=0}^{\infty} c_n z^n$$

is called a *power series* . The numbers $c_n$ are called the *coefficients* of the series.

The behavior of the series $\sum c_n z^n$ of course depending on the coefficients $c_n$. In fact, we can show that it will converges if $z$ in the inner of a circle, and diverges in the exterior. The radius of the circle called the *radius of convergence* of the series.

**Theorem 6.13.** *Given the power series* $\sum c_n z^n$*, put*

$$\alpha = \limsup_{n\to\infty} \sqrt[n]{|c_n|}, \quad R = \frac{1}{\alpha}.$$

*Then* $\sum c_n z^n$ *converges if* $|z| < R$*, and diverges if* $|z| > R$*.*

Next we will investigate the series of the form $\sum a_n b_n$, the following lemma play an important role

**Lemma 6.14** (*Partial Summation Formula* )**.** *Given two sequences* $\{a_n\}$*,* $\{b_n\}$*, put*

$$A_n = \sum_{k=0}^{n} a_k$$

*if* $n \geq 0$*; Put* $A_{-1} = 0$*. Then, if* $0 \leq p \leq q$*, we have*

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q-1} A_n(b_n - b_{n+1}) + A_q b_q - A_{p-1} b_p.$$

PROOF. The results can be derived directly from

$$\sum_{n=p}^{q} a_n b_n = \sum_{n=p}^{q} (A_n - A_{n-1}) b_n = \sum_{n=p}^{q} A_n b_n - sum_{n=p-1}^{q-1} A_n b_{n+1}.$$

□

When the $\{b_n\}$ is monotonic, we have the following theorems

**Theorem 6.15.** *Suppose*

(1) *the partial sums* $A_n$ *of* $\sum A_n$ *form a bounded sequence;*
(2) $b_0 \geq b_1 \geq b_2 \geq \cdots$*;*
(3) $\lim_{n\to\infty} b_n = 0$*.*

*Then* $\sum a_n b_n$ *converges.*

**Corollary 6.16** (*Alternating Series* ; Leibnitz)**.** *Suppose*

(1) $|c_1| \geq |c_2| \geq |c_3| \geq \cdots$*;*
(2) $c_{2m-1} \geq 0, c_{2m} \leq 0 \quad (m = 1, 2, 3, \cdots)$*,;*
(3) $\lim_{n\to\infty} c_n = 0$*.*

*Then* $\sum c_n$ *converges.*

**Corollary 6.17.** *Suppose the radius of convergence of* $\sum c_n z^n$ *is* $1$*, and suppose* $c_0 \geq c_1 \geq c_2 \geq \cdots$*,* $\lim_{n\to\infty} c_n = 0$*. Then* $\sum c_n z^n$ *converges at every point on the circle* $|z| = 1$*, except possibly at* $z = 1$*.*

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