Relay Circuit Analysis

. . . by "odd-even" algebra

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A UNIQUE APPROACH to circuit design is presented in this article. Based on a new but easily understood system, the principles outlined have considerable practical value in simplifying circuits as well as in determining their condition for any combination of relay positions

TO DETERMINE whether a complex circuit—electrical or hydraulic—is open or closed has previously required a type of mathematics with which few engineers are familiar. To avoid introducing this higher mathematics, a simple type of mathematics, called *odd-even algebra*, can be used. This algebra differs from conventional algebra in two ways: (1) only whole numbers are used and (2) every even number is considered to be the same as zero while every odd number is considered to be the same as the number 1. This second rule makes several interesting changes in the appearance of the laws, e.g., $1 + 1 = 0$, but they are easily understood.

BASIC VOCABULARY: In translating from circuits to formulas it is necessary to relate the two possible positions of a switch to the two possible representative numbers and also to give the elementary formulas for series connection and parallel connection. These facts are matters of definition, and are as follows:

1. A closed switch is represented by an odd number
2. An open switch is represented by an even number
3. If X represents one switch and Y represents another, then the system consisting of the two switches in series is represented by XY
4. If X represents one switch and Y represents another, then the system consisting of the two switches in parallel is represented by $X + Y + XY$

The foregoing basic facts are illustrated in *Fig.* 1. For example, if two closed switches are in parallel, the first switch may be represented by 3 and the second by 5. The number for the parallel connection will be $3 + 5 + 15 = 23$. As 23 is an odd number, the parallel circuit is itself closed. In practice, it is convenient to let 1 represent all odd numbers and 0 represent all even numbers, but this is a convenience rather than a necessity.

LAWS OF ODD-EVEN ALGEBRA: The multiplication table for odd-even algebra consists of only four products which agree with the customary algebraic laws:

$0 \times 1 = 0$; an even number times an odd number is an even number

$1 \times 0 = 0$; an odd number times an even number is an even number

$0 \times 0 = 0$; an even number times an even number is an even number

$1 \times 1 = 1$; an odd number times an odd number is an odd number

The addition table also has four elements. The first three are identical with ordinary addition. The fourth law of addition appears different because 2

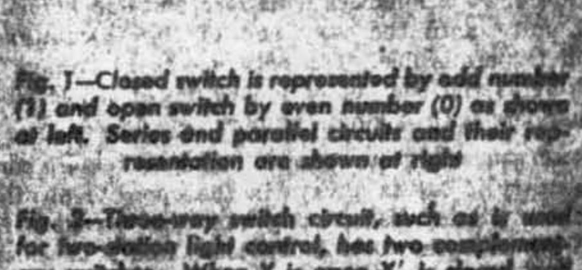

Fig. 1—Closed switch is represented by odd number (1) and open switch by even number (0) as shown at left. Series and parallel circuits and their representation are shown at right

Fig. 2—Three-way switch circuit, such as is used for two-station light control, has two complementary switches. When X is open X' is closed, and vice versa. Complete circuit is represented by X + Y + 1 from which it is seen that a change in either X or Y changes the entire circuit

Fig. 3—Simple bridge circuit with relays A, B, C, and D. Formula for circuit is given in Equation [illegible]

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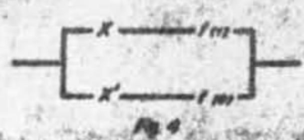

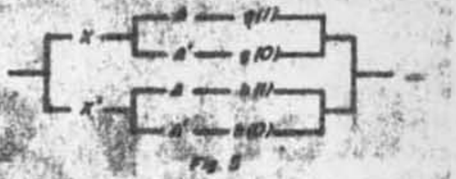

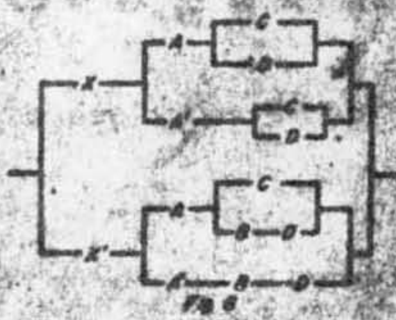

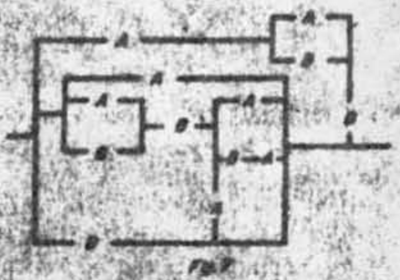

Fig. 4 — Circuit represented by Theorem 1 in the article

Fig. 5—Circuit for the expansion about two switches

Fig. 6—Successive expansion of circuit such as shown in Fig. 5

Fig. 7—Circuit made up of several positions of switches A and B, which is equivalent to A in parallel with B

being an even number in conventional algebra, becomes identified as 0 in odd-even algebra:

$0 + 1 = 1$; an odd number added to an even number is an odd number

$1 + 0 = 1$; an even number added to an odd number is an odd number

$0 + 0 = 0$; an even number added to an even number is an even number

$1 + 1 = 0$; an odd number added to an odd number is an even number

From the last two laws of multiplication, it will be observed that any power of a quantity is identical with the quantity:

$$X^n = X \qquad (1)$$

where n is a positive integer.

From the last two laws of addition, it should be noted that any quantity added to itself equals zero and thus in any expression two like terms can be cancelled:

$$X + X = 0 \qquad (2)$$

Because of Equations 1 and 2, exponents or coefficients are not needed in odd-even algebra.

It will be observed that odd-even algebra obeys the laws of ordinary algebra if we withhold the identification of even numbers with zero and odd numbers with unity until the very end of the calculations, which is permissible. The only necessary change in thinking is the concept of identifying a whole class of numbers with a single number.

Complementary Switches: Two switches are complementary if one is always in the opposite position from the other. Designating the complement of X by X', X is open when X' is closed and X is closed when X' is open. The formula for X' is:

$$X' = X + 1 \qquad (3)$$

The three-way switch circuit is in effect two complementary switches, the circuit employing two three-way switches being shown in *Fig.* 2. The formula for this circuit is derived as follows: The formula for the upper branch is XY. The formula for the lower branch is $(X + 1)(Y + 1) = XY + X + Y + 1$. Combining these two formulas by use of definition 4 under Basic Vocabulary, the formula for the circuit is:

$$XY + (XY + X + Y + 1) + XY(XY + X + Y + 1)$$

Removing parentheses:

$$XY + XY + X + Y + 1 + X^2Y^2 + X^2Y + XY^2 + XY$$

Applying Equation 1,

$$XY + XY + X + Y + 1 + XY + XY + XY + XY$$

and applying Equation 2, $XY + XY = 0$, therefore the formula for the circuit is

$$X + Y + 1 \qquad (4)$$

It will be noted from Equation 4 that a change of either X or Y will change the entire circuit. This is the property for which the circuit is designed.

Function Theorems in Odd-even Algebra: A function of X is an algebraic expression containing the letter X. Even if the expression contains other letters, it can be considered as a function of X. Thus for example, the expression $X + Y + XY + 1$ is a function of X, which is written

$$f(X) = X + Y + XY + 1$$

If this is the function of X that is meant, then by $f(0)$ is meant the result found by letting $X = 0$ in $f(X)$, thus:

$$f(0) = 0 + Y + 0 \times Y + 1 = Y + 1$$

Similarly, $f(1)$ is found by letting $X = 1$ in $f(X)$:

$$f(1) = 1 + Y + 1 \times Y + 1 = 0$$

where the reduction to 0 has been accomplished by noting that $1 + 1 = 0$ and $Y + Y = 0$. The three foregoing equations are simply examples to illustrate the function idea and are not to be taken too seriously; thus, $f(1)$ does not always equal 0, because $f(X)$ is not always $X + Y + XY + 1$.

Any function of X in odd-even algebra can be expanded by the following theorem:

Theorem 1: $f(X) = Xf(1) + (X + 1)\,f(0)$

This expansion theorem is verified by substitution of the only two possible values of X, viz. 0 and 1:

$$f(0) = 0 \times f(1) + (0 + 1)\,f(0) \quad \ldots\ldots\ldots (5)$$

$$f(1) = 1 \times f(1) + (1 + 1)\,f(0) \quad \ldots\ldots\ldots (6)$$

Using the laws of multiplication and addition, i.e., zero times any quantity is zero, one times any quantity is the quantity itself, zero added to any quantity is the quantity itself, and the sum of two like quantities is zero, the truth of Equations 5 and 6 is shown.

By multiplying both members of Theorem 1 by X and noting that $X^2 = X$ and that $X(X + 1) = X^2 + X = X + X = 0$ the following theorem may be derived:

Theorem 2: $Xf(X) = Xf(1)$

An inspection of Theorem 2 might tempt one to divide both members by X and thus conclude that all switches are closed. However, the fact that some switches are open, ($X = 0$) prohibits division by X because, as in ordinary algebra, division by zero is to be avoided. Inasmuch as X has a good chance of being zero, division by X is to be avoided.

Application of Function Theorems: The expansion theorem may be applied to circuit analysis as follows: Assume a circuit resembling the customary bridge circuit, as shown in *Fig.* 3. Let the formula for the circuit be $f(X)$. By considering X closed, there results a circuit in which A and B in parallel are in series with C and D in parallel, so that by definitions 3 and 4 under Basic Vocabulary:

$$f(1) = (A + B + AB)\,(C + D + CD) \ldots . (7)$$

By considering X open, A and C in series are in parallel with B and D in series, so that

$$f(0) = AC + BD + ABCD \quad \ldots\ldots\ldots (8)$$

Putting Equations 7 and 8 into *Theorem* 1 the formula for the circuit is

$$f(X) = X(A + B + AB)\,(C + D + CD) + (X + 1)\,(AC + BD + ABCD) \ldots\ldots\ldots (9)$$

After expanding and cancelling, the formula for the circuit can be written

$$BCX + ABCX + ADX + ABDX + ACDX + BCDX + AC + BD + ABCD \quad \ldots\ldots (10)$$

By successive application of the expansion theorem, the formula for any circuit can be found by considering one switch at a time.

Theorem 2 has a circuit interpretation by noting that if a switch X is in series with a circuit which contains other switches X, then the openness or closure of the entire system is unchanged by permanently closing all switches X except the original one. The truth of the foregoing is apparent when it is realized that if X is open the entire circuit is open and if X is closed, the permanently closed X switches agree with the "outside" X. Having discovered a circuit fact by algebraic manipulation, its validity can be seen by elementary considerations. The algebraic manipulation, then, is an aid in ferreting out simple relationships which might otherwise be overlooked.

Table 1. Circuits Using Two Relays

Circuit*	Formula	Condition of Circuit with			
		Y on Z on	Y on Z off	Y off Z on	Y off Z off
—— ——	0	Off	Off	Off	Off
—Y'——Z'—	1 + Y + Z + YZ	Off	Off	Off	On
—Y'——Z—	Z + YZ	Off	Off	On	Off
——Y'——	1 + Y	Off	Off	On	On
—Y——Z'—	Y + YZ	Off	On	Off	Off
——Z'——	1 + Z	Off	On	Off	On
[illegible]	Y + Z	Off	On	On	Off
[illegible]	1 + YZ	Off	On	On	On
—Y——Z—	YZ	On	Off	Off	Off
[illegible]	1+Y+Z	On	Off	Off	On
——Z——	Z	On	Off	On	Off
[illegible]	1+Y+YZ	On	Off	On	On
——Y——	Y	On	On	Off	Off
Y, Z'	1+ Z + YZ	On	On	Off	On
Y, Z	Y+ Z + YZ	On	On	On	Off
————	1	On	On	On	On

* The prime (') denotes a complementary switch which is open when its complement is closed and closed when its complement is open. Thus Y' is open when Y is closed, and vice versa.

Example: The notation being 1 = On and 0 = Off, the circuit for which the formula is $Y + YZ$ is On because $1 + 1 \times 0 = 1$ when $Y = 1$ (On) and $Z = 0$ (Off).

Deriving a Circuit from a Given Formula: The final step in applying mathematical methods is to translate the symbolic result back into the physical system. In order to derive a circuit for a given formula, note that the formula in Theorem 1 is given by the circuit in *Fig.* 4, because the formula for *Fig.* 4 is obtained by applying definitions 3 and 4 under Basic Vocabulary:

$$X \times f(1) + (X + 1)\,f(0) + X \times f(1) \times (X + 1)\,f(0) \quad \ldots\ldots (11)$$

$$X \times f(1) + (X + 1)\,f(0) + Xf(1) \times f(0) + Xf(1)f(0) \quad \ldots\ldots (12)$$

$$Xf(1) + (X + 1)f(0) \quad \ldots\ldots (13)$$

For any given formula the foregoing process can be applied successively for each switch by calculating $f(1)$ and $f(0)$ and substituting into the circuit of *Fig.* 4.

As an example, consider the formula $X + Y + 1$

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as $f(X)$. Then $f(1) = Y$ and $f(0) = Y + 1 = Y'$. Putting these results into the circuit of ***Fig.*** **4**, there results the circuit of ***Fig.*** 2.

Consider also Equation 10 as $f(X)$. Then by letting $X = 1$

$$f(1) = BC + ABC + AD + ABD + ACD + BCD + AC + BD + ABCD \quad \ldots\ldots (14)$$

By letting $X = 0$

$$f(0) = AC + BD + ABCD \quad \ldots\ldots (15)$$

Now expanding each of these about, say, switch A: For $A = 1$, Equation 14 becomes

$$g(1) = BC + BC + D + BD + CD + BCD + C + BD + BCD = C + D + CD \ldots (16)$$

and for $A = 0$, Equation 14 becomes

$$g(0) = BC + BCD + BD = B(C + D + CD) \quad \ldots\ldots (17)$$

for $A = 1$, Equation 15 becomes

$$h(1) = C + BD + BCD = C + BD + C(BD) \quad \ldots\ldots (18)$$

and for $A = 0$, Equation 15 becomes

$$h(0) = BD \quad \ldots\ldots (19)$$

The general form of the circuit for the expansion about two switches, as illustrated in the foregoing, is given in *Fig.* 5.

By comparison with definitions 3 and 4 it can be seen that $g(1)$ is C and D in parallel, that $g(0)$ is B in series with C and D in parallel, that $h(1)$ is C in parallel with B and D in series and that $h(0)$ is B in series with D. Putting these values into ***Fig.*** **5** a circuit can be derived whose openness and closure is identical with the bridge circuit of ***Fig.*** 3.

By successive expansion as begun already for X and A, it can be seen, ***Fig.*** **6**, that any circuit having N switches can be expressed as an equivalent circuit having at most S switches where

$$S = 2 + 4 + 8 + 16 + \ldots + 2^n = 2^{n+1} - 2 \quad \ldots\ldots (20)$$

or by using three-way switches for pairs of complementary switches, any circuit can be expressed by using not more than $2_n - 1$ three-way switches. In general, of course, all the possible switches will not be needed to express an equivalent circuit.

CIRCUIT SIMPLIFICATIONS: It is obvious that, if X factors out of a formula, the circuit can be expressed as the switch X in series with the circuit represented by the factor by which X is multiplied. This fact is expressed in the following theorem:

> ***Theorem*** **3**: If $f(0) = 0$, then $f(X) = Xf(1)$ and the circuit can be expressed by switch X in series with $f(1)$

The first part of Theorem 4 follows from Theorem 1, and the second conclusion of Theorem 4 is a consequence of definition 3. In a like manner it is possible to state a relationship by which the entire circuit can be expressed as a circuit in parallel with switch X, viz:

> ***Theorem*** **4**: If $f(1) = 1$, then $f(X) = X + (X + 1)f(0) = X + f(0) + Xf(0)$, and the circuit can be expressed by switch X in parallel with $f(0)$

To illustrate Theorem 4, consider a circuit made up of several positions of switches A and B, as given in ***Fig.*** 7. For $A = 1$, it is seen that the circuit would be closed, so that $f(1) = 1$. For $A = 0$, it is seen that the circuit depends on the condition of the lowest B switch, so that $f(0) = B$. Thus Theorem 4 states that the circuit is equivalent to A in parallel with B.

CONCLUSION: The foregoing is admittedly a mere beginning in the analysis of relay circuits by means of odd-even algebra. The primary interest has been in determining whether an entire circuit is open or closed. TABLE 1 illustrates circuits using two relays, showing the condition of the circuit for various combinations of relay position. These conditions are established by the use of the formulas included in the table.

Other applications of odd-even algebra appear possible. The system can be extended to symbolic logic by identifying odd numbers with true propositions, even numbers with false propositions, multiplication with the "and" function and, finally, identifying "or" with the $X + Y + XY$ relationship defined in this article for two switches in parallel. The study of the foregoing theory has been found useful for solving circuit problems.

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