UNIVERSITY OF ILLINOIS

DIGITAL COMPUTER

LIBRARY ROUTINE L 9 - 303

TITLE: Solution of Linear Equations by an Iterative Method (SADOI Only)

TYPE: Open

STORAGE: 3 words at 0, 1, 2

N words beginning at location S3

128 words beginning at location S5

N words beginning at drum location S4.

ACCURACY: Depends on condition of equations.

DURATION: Depends on initial approximation.

PARAMETERS:

S3: Address of first location in memory which is used for latest approximation.

S4: Address of first location on drum which is used as temporary storage.

S5: Address of first location in memory which is used as temporary storage. As the numbers are brought off the drum they are stored in these locations.

S6: N the number of equations to be solved.

S7: Address of subroutine for computing residuals.

S8: Location to which control is transferred when routine is finished.

NUMBER OF WORDS: 135 + (R1) = 146

We wish to solve the system of equation

$$A\vec{x} = \vec{y}$$

where A is symmetric and positive definite. It is assumed that

$$\sum_{j=0}^{N-1} |a_{ij}| < 1 \qquad i = 0, 1, \ldots, N-1.$$

This implies that the largest eigenvalue of A is < 1.

The algorithm which is used is the following:

$$(1)\quad \vec{\eta}_{k+1} = 2b_{k+1}(\alpha\vec{r}_k + \vec{\eta}_k - \vec{\eta}_{k-1}) + \vec{\eta}_{k-1}$$

$$\vec{\eta}_1 = \vec{\eta}_0 + \alpha\vec{r}_0$$

or

$$(2)\quad \Delta\vec{\eta}_k = 2b_{k+1}(\alpha\vec{r}_k + \Delta\vec{\eta}_{k-1}) - \Delta\vec{\eta}_{k-1}$$

$$\Delta\vec{\eta}_0 = \alpha\vec{r}_0$$

where

$\vec{r}_k = \vec{y} - A\vec{\eta}_k$, the residual vector

$\Delta\vec{\eta}_{k-1} = \vec{\eta}_k - \vec{\eta}_{k-1}$

$b_{k+1} = \dfrac{1}{2-\lambda^2 b_k}$, $b_1 = 1$

$\lambda = \dfrac{b-a}{b+a}$, $\qquad \alpha = \dfrac{2}{b+a}$

and a is a lower bound for the smallest eigenvalue and b is an upper bound for the largest eigenvalue of A. Equation (2) is actually used during the computation in this program.

Upon entry into this routine

0 contains a

2 contains b

A contains $s_0$

and S3, 1S3, ..., (N-1) S3

contain $s_0\eta_{0,0}$, $s_0\eta_{0,1}$, ..., $s_0\eta_{0,N-1}$.

$s_k$ will be made smaller during the iteration if overflow occurs.

Usually $s_k$ will be a power of 2, but this is not necessary and it may be more convenient to use some other base. If this is done, then (102) should contain the reciprocal of the base. The final scaling factor will be found in (506).

In the residual subroutine, the following must be computed.

$$s_k y_i - \sum_{j=0}^{N-1} a_{ij} (s_k \eta_{k,j}) \qquad i = 0, 1, \ldots, N-1$$

This should be done in a double precision fashion using 74 and 70 instructions. When the subroutine is entered, $s_k$ will be A, 0 will contain i, and 1 will contain in the R.H. address the address of $(s_k \eta_{k,i})$.

If there is overflow in the residual subroutine, then one must jump to (100).

This program will handle as a maximum about 600 equations.

For a more complete discussion of the mathematical method, see the write up of library routine L8 - 302.

DATE July 19, 1960

PROGRAMMED BY G. Golub

APPROVED BY J. N. Snyder

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| LOCATION | | ORDER | | NOTES PAGE 1 L 9 |
|---|---|---|---|---|
| | | 00K(L9) | | |
| 0 | | 40 (506) | | Store scale parameter |
| | | L5 0F | | |
| 1 | | 10 1F | | |
| | | 40 0F | | a/2 |
| 2 | | L5 1F | | |
| | | 10 1F | | |
| 3 | | 40 1F | | b/2 |
| | | L4 0F | | |
| 4 | | 40 (507) | | a/2 + b/2 |
| | | L5 1F | | |
| 5 | | L0 0F | | |
| | | 50 (508) | | Clear Q |
| 6 | | 66 (507) | | |
| | | S5 F | | |
| 7 | | 40 (509) | | $\lambda$ = b-a/b+a |
| | | 75 (509) | | |
| 8 | | 10 1F | | |
| | | 40 (510) | | $\lambda^2/2$ |
| 9 | | L5 (501) | | |
| | | L4 (505) | | |
| 10 | | 40 (503) | | |
| | | L5 (500) | | |
| 11 | | L4 (505) | | |
| | | 40 (502) | | |
| 12 | | L4 (511) | | |
| | | 40 (504) | | |
| 13 | | 50 (509) | | |
| | | 71 (509) | | |
| 14 | | L4 (511) | | |
| | | 40 1F | | |
| 15 | | S5 F | | |
| | | 40 0F | | |
| 16 | (0) | 00 1F | | |
| | | 50 (0) | | |

| LOCATION | | ORDER | | NOTES PAGE 2 L 9 |
|---|---|---|---|---|
| 17 | | 22 (R1) | | $\sqrt{1-\lambda^2}$ |
| | | 10 1F | | |
| 18 | | 40 0F | | |
| | | LJ 0F | | |
| 19 | | 40 0F | | |
| | | 50 (508) | | |
| 20 | | L5 (509) | | |
| | | 10 1F | | |
| 21 | | 66 0F | | $\rho = \lambda/1 + \sqrt{1-\lambda^2}$ |
| | | S5 F | | |
| 22 | | L4 (511) | | |
| | | 40 0F | | |
| 23 | | 50 0F | | |
| | | 7J 0F | | $(1-\rho)^2$ |
| 24 | | 40 0F | | |
| | | L5 (512) | | |
| 25 | | 50 (508) | | |
| | | 66 0F | | |
| 26 | | S7 F | | |
| | | 40 (513) | | Compute end constant |
| 27 | (25) | 49 0F | from 108 | |
| | | L1 0F | | |
| 28 | | 40 (514) | | |
| | | 41 (523) | | |
| 29 | (24) | 22 (1) | | |
| | | L5 (514) | from 90, 99 | |
| 30 | | 00 1F | | |
| | | 36 (2) | | Is $-b_k = -1/2$? |
| 31 | | 89 1F | | |
| | | 40 (514) | | |
| 32 | (2) | 50 (514) | from 30 | |
| | | 79 (510) | | |
| 33 | | L4 (511) | | |
| | | 40 0F | | |
| 34 | | 50 (508) | | |
| | | LJ (508) | | |

| LOCATION | | ORDER | NOTES PAGE 3 L 9 | |
|---|---|---|---|---|
| 35 | | 66 OF | | |
| | | S5 F | | |
| 36 | (1) | 40 (514) | | Compute - $b_{k+1}$ |
| | | 41 (517) | from 29 | |
| 37 | | 41 (515) | | |
| | | 41 (516) | | |
| 38 | | L5 (500) | | |
| | | 40 (3) | | |
| 39 | | 40 (19) | | |
| | | L5 (501) | | |
| 40 | | 40 (4) | | |
| | | L5 (518) | | |
| 41 | | 40 (5) | | |
| | | 42 (6) | | |
| 42 | (17) | L5 (520) | from 82 | |
| | | 42 (7) | | |
| 43 | | 42 (8) | | |
| | | 42 (9) | | |
| 44 | | 46 (6) | | |
| | | 42 (11) | | |
| 45 | (3) | 85 11F | | |
| | | 40 S4 | by 38,47; from 50 | Read in 128 components of $\Delta \vec{\eta}_{k-1}$ |
| 46 | (7) | 32 (7) | | |
| | | 40 S5 | by 42,49 | |
| 47 | | F5 (3) | | |
| | | 40 (3) | | |
| 48 | | L0 (502) | | |
| | | 36 (18) | | |
| 49 | | F5 (7) | | |
| | | 42 (7) | | |
| 50 | | L0 (521) | | |
| | | 36 (3) | | |
| 51 | (18) | L5 (6) | from 48,75 | |
| | | 40 1F | | |
| 52 | | L5 (517) | | |
| | | 40 OF | | |

| LOCATION | | ORDER | NOTES PAGE 4 L 9 |
|---|---|---|---|
| 53 | (10) | L5 (506) | |
| | | 50 (10) | |
| 54 | | 26 S7 | Jump to residual |
| | | 40 0F | |
| 55 | | L7 0F | |
| | | L2 (507) | |
| 56 | | 36 (100) | Rescale |
| | | L5 0F | |
| 57 | | 66 (507) | |
| | | S5 F | |
| 58 | (8) | 40 0F | |
| | | L4 S5 | by 43, 74 |
| 59 | | 40 1F | |
| | | 19 1F | |
| 60 | | 50 1F | |
| | | 70 (514) | |
| 61 | (9) | 00 1F | |
| | | L0 S5 | by 43, 74 |
| 62 | (6) | 40 S5 | $\Delta\vec{\eta}_k$; by 44, 73 |
| | | L4 S3 | by 41, 73 |
| 63 | | 40 1F | |
| | | LL 1F | |
| 64 | | 36 (12) | |
| | | 26 (100) | Rescale |
| 65 | (12) | L5 (515) | from 64 |
| | | 32 (13) | |
| 66 | (13) | 26 (14) | |
| | | 50 0F | from 65 |
| 67 | | L5 (516) | |
| | | 74 0F | |
| 68 | | L4 (515) | |
| | | 40 (515) | |
| 69 | | S5 F | |
| | | 40 (516) | |
| 70 | (14) | F5 (517) | from 66 |
| | | 40 (517) | |

| LOCATION | | ORDER | NOTES | PAGE 5 L 9 |
|---|---|---|---|---|
| 71 | | L0 (505) | | |
| | | 36 (15) | | |
| 72 | | L5 (6) | | |
| | | L4 (519) | | |
| 73 | | 40 (6) | | |
| | | F5 (8) | | |
| 74 | | 42 (8) | | |
| | | 42 (9) | | |
| 75 | | L0 (524) | | |
| | | 36 (18) | | |
| 76 | (11) | 00 1F | from 71 | |
| | (15) | L5 S5 | by 44,80; from 81 | |
| 77 | (4) | 86 11F | | |
| | | 00 S4 | by 40,78 | Record 128 components of $\Delta\vec{r}_k$ |
| 78 | | F5 (4) | | |
| | | 40 (4) | | |
| 79 | | L0 (503) | | |
| | | 36 (19) | | |
| 80 | | F5 (11) | | |
| | | 42 (11) | | |
| 81 | | L0 (525) | | |
| | | 32 (11) | | |
| 82 | | 26 (17) | | |
| | | 00 F | | |
| 83 | (19) | 85 11F | from 79,88 | |
| | | 00 S4 | by 39, 87 | |
| 84 | (5) | L4 S3 | | |
| | | 40 S3 | by 41,86 | |
| 85 | | L5 (5) | | |
| | | L4 (519) | | |
| 86 | | 40 (5) | | |
| | | F5 (19) | | |
| 87 | | 40 (19) | | |
| | | L0 (504) | | |
| 88 | | 36 (19) | | |
| | | L5 (516) | | |

| LOCATION | | ORDER | | NOTES PAGE 6 L 9 |
|---|---|---|---|---|
| 89 | | 40 0F | | |
| | | L5 (515) | | |
| 90 | | 32 (20) | | |
| | | 22 (24) | | |
| 91 | (20) | 00 1F | | |
| | | 50 (20) | from 90 | |
| 92 | | 26 (R1) | | |
| | | 40 0F | | |
| 93 | | L3 (523) | | |
| | | 36 (22) | | is $\vec{r}_k = 0$? |
| 94 | | L5 (523) | | |
| | | L0 (513) | | |
| 95 | | 36 (23) | | |
| | | L5 0F | | $\|\|\propto \vec{r}_k\|\|$ - end constant < 0? |
| 96 | | L0 (523) | | |
| | | 36 S8 | | ← C - θ! when done |
| 97 | (23) | L5 (523) | from 95 | |
| | | 40 (522) | | |
| 98 | (22) | L5 0F | from 93 | |
| | | 40 (523) | | |
| 99 | | 22 (24) | | |
| | | 00 F | | $\|\|\propto \vec{r}_{k+1}\|\| = \|\|\propto \vec{r}_k\|\|$ in acc. |
| 100 | (100) | L5 (526) | from 56,64 | |
| | | 40 (101) | | |
| 101 | (103) | 41 0F | | |
| | | 50 (102) | from 106 | |
| 102 | (101) | 7J S3 | | |
| | | 40 S3 | by 100, 104 | |
| 103 | | L5 (101) | | |
| | | L4 (519) | | |
| 104 | | 40 (101) | | |
| | | F5 0F | | |
| 105 | | 40 0F | | |
| | | L0 (505) | | |
| 106 | | 36 (104) | | |
| | | 22 (103) | | |

| LOCATION | | ORDER | | NOTES PAGE 7 L 9 |
|---|---|---|---|---|
| 107 | (104) | 50 (102) | from 106 | |
| | | 7J (506) | | |
| 108 | | 40 (506) | | |
| | | 26 (25) | | |
| 109 | (102) | 40 F | | |
| | | 00 F | | |
| 110 | (500) | 85 11F | | |
| | | 00 S4 | | |
| 111 | (501) | 86 11F | | |
| | | 00 S4 | | |
| 112 | (502) | 85 11F | | |
| | | 00 F | by 11 | 85 11F 00 NS4 |
| 113 | (503) | 86 11F | | |
| | | 00 F | by 10 | 86 11F 00 NS4 |
| 114 | (504) | 05 11F | | |
| | | 00 F | by 12 | 05 11F 00 NS4 |
| 115 | (505) | 00 F | | |
| | | 00 S6 | | 00 F 00 NF |
| 116 | (506) | 00 F | | |
| | | 00 F | by 0, 108 | Scaling factor |
| 117 | (507) | 00 F | | |
| | | 00 F | by 4 | $a/2 + b/2 = 1/\alpha$ |
| 118 | (508) | 00 F | | |
| | | 00 F | | zero |
| 119 | (509) | 00 F | | |
| | | 00 F | by 7; $\lambda$ | |
| 120 | (510) | 00 F | | |
| | | 00 F | by 8 | $\lambda^2/2$ |
| 121 | (511) | 80 F | | |
| | | 00 F | | -1 |
| 122 | (512) | 00 F | | |
| | | 00 70F | | |
| 123 | (513) | 00 F | | |
| | | 00 F | by 26 | End constant |

| LOCATION | | ORDER | NOTES | PAGE 8 L 9 |
|---|---|---|---|---|
| 124 | (514) | 00 F | | |
| | | 00 F | by 28,31,36 | $-b_k$ |
| 125 | (515) | 00 F | | |
| | | 00 F | by 68 | $\|\|\alpha \vec{r}_{k+1}\|\|^2_M$ |
| 126 | (516) | 00 F | | |
| | | 00 F | by 69 | $\|\|\alpha \vec{r}_{k+1}\|\|^2_L$ |
| 127 | (517) | 00 F | | |
| | | 00 F | by 36,70 | Equation counter |
| 128 | (518) | L4 S3 | | |
| | | 40 S3 | | |
| 129 | (519) | 00 1F | | |
| | | 00 1F | | |
| 130 | (520) | 00 S5 | | |
| | | 00 S5 | | |
| 131 | (521) | S2 (7) | | |
| | | 40 128S5 | | |
| 132 | (522) | 00 F | | |
| | | 00 F | by 99 | $\|\|\alpha r_{k-1}\|\|$ |
| 133 | (523) | 00 F | | |
| | | 00 F | by 28, 98 | $\|\|\alpha \mathbf{r}_k\|\|$ |
| 134 | (524) | NO F | | |
| | | L4 128S5 | | |
| 135 | (525) | 80 1F | | |
| | | L5 128S5 | | |
| 136 | (526) | 7J S3 | | |
| | | 40 S3 | | |
| | | 00K(R1) | | |