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Dynamic Scaling of Manipulator Trajectories

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Abstract. A fundamental time-scaling property of manipulator dynamics has been identified that allows modification of movement speed without complete dynamics recalculation. By exploiting this property, it can be determined whether a planned trajectory is dynamically realizable given actuator torque limits, and if not, how to modify the trajectory to bring it within dynamic and actuating constraints.

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1. Introduction

Trajectory planning algorithms seldom incorporate extensive knowledge of the interaction between inverse dynamics and actuator torque limits into the planning process. Past efforts have typically used fixed velocity limits of the joints as a way of determining how fast a trajectory may be executed [8]. Due to the complex relationship between joint velocities and dynamics, such a procedure is at best a very coarse approximation of the true influence of actuator limits on trajectory speed. An exact method for determining the optimal velocity distribution for a fixed path has been proposed in [13], where dynamic programming was straightforwardly applied to minimize energy under actuator and dynamic constraints. The computational cost of such optimization approaches however may prevent their useful application.

We develop a fundamental time-scaling property of manipulator dynamics that allows trajectory planning and inverse dynamics to be exactly and efficiently coupled. The dynamic realizability of a proposed trajectory can be readily determined, and a simple procedure to modify the movement speed can be applied to render proposed trajectories realizable.

We presume that a time sequence of joint angles $\underline{\theta}(t) = (\theta_1(t), \theta_2(t), \ldots, \theta_n(t))$ for an n-joint manipulator has been proposed by the trajectory planner, where t represents the time in the interval $0 \leq t \leq t_f$. Because of fast recursive formulations of inverse dynamics [4, 7, 11], for each sampling time t the joint torques $\mathbf{n}(t) = (n_1(t), n_2(t), \ldots, n_n(t))$ corresponding to $\underline{\theta}(t)$ can be efficiently found. The comparison of $\mathbf{n}(t)$ against motor torque limits is therefore readily accomplished, and it is straightforward to determine whether the proposed trajectory can be realized by the actuators.

A more difficult task is to ascertain how to change the trajectory in case motor torque limits are violated. Here we consider only changing the speed at which a manipulator follows a path, where by speed change is meant a constant scaling of the velocity profile so that the total movement duration is scaled without changing the actual path through space. It is not sufficient merely to slow down a trajectory, with the hope that a slower trajectory requires lower motor torques, because some trajectories can only be realized at higher speeds, and some trajectories may not be realizable at any speed. Moreover, unless one is careful to employ an algorithm such as is presented here, then modifying the movement speed requires that the inverse dynamics be recomputed from scratch.

The algorithm presented here determines what speed range is permissible for the proposed trajectory given actuator torque limits. At the same time the nominal dynamics for the proposed trajectory can be simply modified for the new trajectory, without dynamics recomputation.

2. Time Scaling and Trajectories

Suppose that some trajectory plan $\underline{\theta}(t)$ has been fashioned. A new trajectory $\underline{\tilde{\theta}}(t)$ will be defined such that $\underline{\tilde{\theta}}(t) = \underline{\theta}(r)$, where $r = r(t)$ is a monotonically increasing function of time with $r(0) = 0$ and $r(t_1) = t_f$ for some $t_1 > 0$. The function $r(t)$ can be considered a time warp which moves the arm along the same path but with a different time dependence, perhaps going slower along some points of the path and faster along others. $r(t)$ must increase monotonically because time cannot reverse itself, and $r(0) = 0$ because the movement must start at the same point.

To determine how the dynamics of the arm changes for the new trajectory, the time derivatives of the joint angles are required. From the chain rule,

$$\frac{d\underline{\tilde{\theta}}(t)}{dt} = \frac{d\underline{\theta}(r)}{dr}\frac{dr}{dt}$$

or, using the dot notation for time derivatives,

$$\dot{\underline{\tilde{\theta}}}(t) = \dot{\underline{\theta}}(r)\dot{r} \tag{1}$$

where $d\underline{\theta}(r)/dr$ has been written $\dot{\underline{\theta}}(r)$ because it takes the value $\dot{\underline{\theta}}$ evaluated at $r(t)$. Similarly,

$$\ddot{\underline{\tilde{\theta}}}(t) = \ddot{\underline{\theta}}(r)\dot{r}(t)^2 + \dot{\underline{\theta}}(r)\ddot{r}(t) \tag{2}$$

The dynamic equations of motion can be compactly written [4] as

$$\mathbf{n}(t) = \mathbf{I}(\underline{\theta}(t))\ddot{\underline{\theta}}(t) + \dot{\underline{\theta}}(t) \cdot \mathbf{C}(\underline{\theta}(t)) \cdot \dot{\underline{\theta}}(t) + \mathbf{g}(\underline{\theta}(t)) \tag{3}$$

where

- $\mathbf{n}(t)$ is the n-dimensional vector of net joint torques corresponding to the movement point,
- $\mathbf{I}(\underline{\theta}(t))$ is the $n \times n$ generalized inertia tensor of the manipulator,
- $\mathbf{C}(\theta(t))$ is the $n \times n \times n$ position-dependent tensor in the formulation of the Coriolis and centripetal torques, and
- $\mathbf{g}(\underline{\theta}(t))$ is the position-dependent n-dimensional vector of gravity torques.

The notation for the velocity product term $\dot{\underline{\theta}} \cdot \mathbf{C} \cdot \dot{\underline{\theta}}$ is slightly unconventional, but has been adopted for compactness. The product $\mathbf{C} \cdot \dot{\underline{\theta}}$ is an $n \times n$ matrix with element ij as $\sum_k C_{ijk}\dot{\theta}_k$, which in turn is multiplied against $\dot{\underline{\theta}}$ to yield an $n \times 1$ vector.

In the following derivations, the acceleration and velocity dependent torques are treated separately and are designated as $\mathbf{n}_a(t) = (n_{a1}(t), n_{a2}(t), \ldots, n_{an}(t))$, so that $\mathbf{n}(t) = \mathbf{n}_a(t) + \mathbf{g}(\theta(t))$. For the new trajectory $\underline{\tilde{\theta}}(t)$,

$$\tilde{\mathbf{n}}_a(t) = \mathbf{I}(\underline{\tilde{\theta}}(t))\ddot{\underline{\tilde{\theta}}}(t) + \dot{\underline{\tilde{\theta}}}(t) \cdot \mathbf{C}(\underline{\tilde{\theta}}(t)) \cdot \dot{\underline{\tilde{\theta}}}(t) \tag{4}$$

Substituting from (1) and (2),

$$\tilde{\mathbf{n}}_a(t) = \left[\mathbf{I}(\underline{\theta}(r))\ddot{\underline{\theta}}(r) + \dot{\underline{\theta}}(r) \cdot \mathbf{C}(\underline{\theta}(r)) \cdot \dot{\underline{\theta}}(r)\right]\dot{r}^2 + \mathbf{I}(\underline{\theta}(r))\dot{\underline{\theta}}(r)\ddot{r} \tag{5}$$

Rearranging and substituting from (3),

$$\tilde{\mathbf{n}}_a(t) = \dot{r}^2\mathbf{n}_a(r) + \ddot{r}\,\mathbf{I}(\underline{\theta}(r))\,\dot{\underline{\theta}}(r) \tag{6}$$

This is a potentially significant reformulation of dynamics, indicating how the underlying dynamics changes when the time dimension of a trajectory changes. The new torque $\tilde{\mathbf{n}}_a(t)$ is related to the old $\mathbf{n}_a(r)$ by the scaling factor $\dot{r}^2$ plus a term proportional to the generalized momentum $\mathrm{I}(\underline{\theta}(r))\underline{\dot{\theta}}(r)$ of the manipulator. Note that the gravity torque $\mathbf{g}(\underline{\tilde{\theta}}(t)) = \mathbf{g}(\underline{\theta}(r))$ is not scaled since it is position dependent only, which is the reason for the separation between $\mathbf{n}_a(t)$ and the gravity torques.

2.1 Constant Time Scaling

The simplest instance of (7) is when $\ddot{r}(t) = 0$, i.e., $r(t) = ct$ for some constant $c > 0$. If $c > 1$ the movement is sped up; if $c < 1$ the movement is slowed down. Then

$$\tilde{\mathbf{n}}_a(t) = c^2\mathbf{n}_a(ct) \tag{7}$$

Interestingly, movement speed can be proportionally changed without affecting the underlying dynamics very much, so long as the gravity contribution is separated from the acceleration and velocity term contributions. The relation was also noted by Bejzcy [1]. Humans apparently adopt such a strategy when changing movement speed, perhaps to simplify the dynamics computation [5].

This relation also shows that the velocity and acceleration terms of the dynamics would have the same significance relative to each other for all speeds of movement. For, the acceleration term $\mathrm{I}(\underline{\theta}(t))\underline{\ddot{\theta}}(t)$ is scaled by c^2 from (2), and the velocity term $\underline{\dot{\theta}}(t) \cdot \mathrm{C}(\underline{\theta}(t)) \cdot \underline{\dot{\theta}}(t)$ receives a c factor for each $\underline{\dot{\theta}}(t)$. Thus both terms change equally with differing movement speeds. This contradicts the normal assumption in the robotics literature, where in designing control systems workers typically throw out the velocity terms because they are a nonlinear product, with the presumption that they are significant only at higher speeds of movement [1,10]. For the slow movement speeds of most manipulators, and hence because of the predominance of frictional and gravitational effects, this is a reasonable assumption [2]. But for consistency the acceleration terms should be thrown out as well since they share the same significance as the velocity terms, yet this is not done. In any case, future generations of robots will contain examples of fast manipulators with low joint friction where dynamic effects, both acceleration and velocity terms, are highly significant [2].

In the remainder of this paper, we assume the special case (7) and use it to determine allowable speeds of movement for a given trajectory. By allowable speed it is meant that the trajectory is stretched or compressed uniformly to fit the allotted duration without changing the path or the velocity profile shape. Constant scaling of velocity is a simple but important method of bringing a trajectory within actuator constraints. Certainly there are many classes of manipulator trajectories where an exact path through space must be followed, as in straight-line Cartesian motions of the manipulator hand [9, 12], but where the time dependence along the path is not strongly restricted. While non-uniform time scaling may yield a realizable trajectory where a constant scaling would not, results for the general case (6) are not yet available while other approaches [13] may be too computationally inefficient for routine use. Even more difficult is path modification under actuator and dynamic constraints, for which no general results are yet available (see however [6] for an approximate time-optimal trajectory planning solution).

3. Time Scaling of Trajectories to Satisfy Torque Limitations

Torque limits of actuators restrict how fast a manipulator may move along a trajectory. In order to determine whether a proposed trajectory $\underline{\theta}(t)$ violates actuator limits, the inverse dynamics must be solved and the computed torques compared to these limits. Suppose we have computed the acceleration and velocity dependent torques $\mathbf{n}_a(t)$ separately from the gravitational torques $\mathbf{g}(\underline{\theta}(t))$. Suppose further that the maximum and minimum torque limits, $\mathbf{n}^+ = (n_1^+, n_2^+, \ldots, n_n^+)$ and $\mathbf{n}^- = (n_1^-, n_2^-, \ldots, n_n^-)$ respectively, are constant throughout a movement. (Ordinarily one would presume $\mathbf{n}^+ = -\mathbf{n}^-$.) Later we consider velocity dependencies as in electric torque motors.

At a given position $\underline{\theta}(t)$ of the manipulator, some of the actuator torque is required for postural support of the manipulator only. In terms of what torque capability is remaining to actually generate a movement, we formulate new effective torque limits by absorbing the gravitational torques into the torque limits, i.e.,

$$\begin{aligned} \mathbf{n}^+(t) &= \mathbf{n}^+ - \mathbf{g}(\underline{\theta}(t)) \\ \mathbf{n}^-(t) &= \mathbf{n}^- - \mathbf{g}(\underline{\theta}(t)) \end{aligned} \tag{8}$$

Note that the torque limits are now position dependent, and hence have been written as functions of time.

Because we are looking for a time scaling value c that brings the trajectory within the torque limits, a slight alteration of (7) is required. Since (7) holds for all times, we can write $\tilde{\mathbf{n}}_a(t/c) = c^2\mathbf{n}_a(t)$ and the torque limits for the new trajectory as $\tilde{\mathbf{n}}^{\pm}(t/c) = \mathbf{n}^{\pm}(t)$. We require that for the new trajectory $\tilde{\mathbf{n}}_a(t/c)$ be bounded by $\tilde{\mathbf{n}}^{\pm}(t/c)$, which is done by finding the c that bounds $c^2\mathbf{n}_a(t)$ by $\mathbf{n}^{\pm}(t)$ according to the following procedure.

For each time t and joint i, we find the minimum and maximum scaling values of c^2 that satisfy the torque limits by solving (7) together with the computed torques $n_{ai}(t)$ and the torque limits $n_i^-(t)$ and $n_i^+(t)$. The result will be denoted by the interval $[c_i^{2-}(t), c_i^{2+}(t)]$, where any scaling value within this interval is a permissible movement speed for this joint at this point in the trajectory. This scaling interval, however, may violate constraints at other joints and times, and the permissible range of c^2 values for the whole movement is found by intersecting all such intervals:

$$[c^{2-}, c^{2+}] = \bigcap_{i,t} [c_i^{2-}(t), c_i^{2+}(t)] \tag{9}$$

We can then choose any value in the final interval $[c^{2-}, c^{2+}]$ to generate a movement which satisfies the actuator constraints.

To determine $[c_i^{2-}(t), c_i^{2+}(t)]$, there are three cases.

Case 1: $n_i^+(t) > 0$, $n_i^-(t) < 0$.

Condition	$c_i^{2-}(t)$	$c_i^{2+}(t)$
$n_{ai}(t) > 0$	0	$n_i^+(t)/n_{ai}(t)$
$n_{ai}(t) = 0$	0	∞
$n_{ai}(t) < 0$	0	$n_i^-(t)/n_{ai}(t)$

Case 2: $n_i^+(t) > 0$, $n_i^-(t) > 0$.

Condition	$c_i^{2-}(t)$	$c_i^{2+}(t)$
$n_{ai}(t) > 0$	$n_i^-(t)/n_{ai}(t)$	$n_i^+(t)/n_{ai}(t)$
$n_{ai}(t) \leq 0$	unrealizable	

Case 3: $n_i^+(t) < 0$, $n_i^-(t) < 0$.

Condition	$c_i^{2-}(t)$	$c_i^{2+}(t)$
$n_{ai}(t) \geq 0$	unrealizable	
$n_{ai}(t) < 0$	$n_i^+(t)/n_{ai}(t)$	$n_i^-(t)/n_{ai}(t)$

To explain these cases, consider first case 1. The minimum value of c^2 is zero because $n_{ai}(t) = 0$ falls within actuator bounds and c^2 must be non-negative. If $n_{ai}(t) > 0$, then the appropriate torque limit for comparison is $n_i^+(t)$, because time scaling can change a torque magnitude but not a sign. The maximum value of c^2 is then determined by the ratio $n_i^+(t)/n_{ai}(t)$. Note that when $n_{ai}(t) < n_i^+(t)$, then $c^2 > 1$ and it is possible to speed the movement up and still satisfy actuator constraints. When $n_{ai}(t) > n_i^+(t)$, $c^2 < 1$ and the movement must be slowed down. To complete case 1, if $n_{ai}(t) < 0$, the appropriate torque limit is $n_i^-(t)$ and the maximum value of c^2 is $n_i^-(t)/n_{ai}(t)$.

In case 2, if $n_{ai}(t) \leq 0$, then this movement is unrealizable at any speed. The actuator can produce only a positive torque, but a non-positive torque is required by the movement. Put simply, the manipulator cannot even hold itself up at this position. Of course manipulator actuation is ordinarily designed to counteract gravity, but this actuation may become inadequate if too heavy a load is picked up. For $n_{ai}(t) > 0$, the maximum movement speed is determined by the ratio $n_i^+(t)/n_{ai}(t)$ and the minimum by $n_i^-(t)/n_{ai}(t)$. It is possible that $c_i^{2-}(t) > 0$, which says that there is a minimum non-zero speed at which the movement is realizable. Also it is possible that $c_i^{2-}(t) > 1$, so that the movement can be realized only by speeding up. Case 3 is analogous to case 2, except that the roles of $n_i^+(t)$ and $n_i^-(t)$ are reversed due to sign change.

The intersection of all the intervals $[c_i^{2-}(t), c_i^{2+}(t)]$ may be null, with incompatible scaling requirements at different parts of the trajectory. This movement is then unrealizable at any

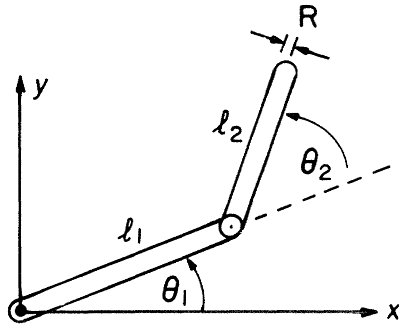

Figure 1
A planar two-link manipulator.

speed. If $c^{2-} > 1$, then the movement should be speeded up by at least a factor c^{2-}, while if $c^{2+} < 1$ the movement should be slowed by at least a factor c^{2+} in order to produce a realizable trajectory. Having chosen a c^2 value, the inverse dynamics can be simply recomputed from the old values of $\mathbf{n}_a(t)$ and $\mathbf{g}(\underline{\theta}(t))$ as follows:

$$\tilde{\mathbf{n}}(t) = c^2\mathbf{n}_a(ct) + \mathbf{g}(\underline{\theta}(ct)) \tag{10}$$

The acceleration and velocity torques are amplitude scaled, the gravity torque is added in separately, and both together are time scaled. Speed change can therefore be accomplished without dynamics recomputation.

4. Examples

This algorithm will be illustrated for straight-line movements by a two-link planar manipulator (Figure 1); the algorithm is quite easily applied to manipulators with more degrees of freedom. This manipulator has two rotary joints with joint angles θ_1 at the shoulder and θ_2 at the elbow. The axes of rotation are both directed along the z-axis, so that the manipulator only generates movement in the x-y plane. Gravity is presumed to act in the negative y direction with magnitude g. The length, mass, and moment of inertia about the proximal joint for each link are designated as l_i, m_i, and I_i respectively, where $i = 1$ refers to the upper arm link and $i = 2$ refers to the forearm link. Each link is a uniform cylinder with radius R.

The equations of motion are [2]:

$$n_2 = \ddot{\theta}_1\left(I_2 + \frac{m_2 l_1 l_2}{2}\cos\theta_2 + \frac{m_2 l_2^2}{4}\right) + \ddot{\theta}_2\left(I_2 + \frac{m_2 l_2^2}{4}\right)$$

$$+\frac{m_2 l_1 l_2}{2}\dot{\theta}_1^2 \sin\theta_2 + \frac{m_2 l_2 g}{2}\cos(\theta_1 + \theta_2) \tag{11}$$

$$n_1 = \ddot{\theta}_1\left(I_1 + I_2 + m_2 l_1 l_2 \cos\theta_2 + \frac{m_1 l_1^2 + m_2 l_2^2}{4} + m_2 l_1^2\right)$$
$$+\ddot{\theta}_2\left(I_2 + \frac{m_2 l_2^2}{4} + \frac{m_2 l_1 l_2}{2}\cos\theta_2\right) \tag{12}$$
$$-\frac{m_2 l_1 l_2}{2}\dot{\theta}_2^2 \sin\theta_2 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin\theta_2$$
$$+\left(\frac{m_2 l_2}{2}\cos(\theta_1+\theta_2) + l_1\left(\frac{m_1}{2} + m_2\right)\cos\theta_1\right)g$$

A common class of manipulator trajectories are straight line movements of the tip, i.e., $y - y_0 = (x - x_0)(y_1 - y_0)/(x_1 - x_0)$ for beginning and end positions of the tip (x_0, y_0) and (x_1, y_1) respectively. To solve the inverse dynamics, it is required to transform from the position, velocity, and acceleration of the tip to the position, velocity and acceleration for each joint angle. These inverse kinematic equations are presented below [2]:

$$\cos\theta_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2 l_1 l_2}$$
$$\theta_1 = \tan^{-1}\left(\frac{y}{x}\right) - \tan^{-1}\left(\frac{l_2 \sin\theta_2}{l_1 + l_2\cos\theta_2}\right) \tag{13}$$

$$\begin{bmatrix}\dot{\theta}_1 \\ \dot{\theta}_1 + \dot{\theta}_2\end{bmatrix} = \frac{1}{l_1 l_2 \sin\theta_2}\begin{bmatrix} l_2\cos(\theta_1+\theta_2) & l_2\sin(\theta_1+\theta_2) \\ -l_1\cos\theta_1 & -l_1\sin\theta_1\end{bmatrix}\begin{bmatrix}\dot{x} \\ \dot{y}\end{bmatrix} \tag{14}$$

$$\begin{bmatrix}\ddot{\theta}_1 \\ \ddot{\theta}_1 + \ddot{\theta}_2\end{bmatrix} = \frac{1}{l_1 l_2 \sin\theta_2}\begin{bmatrix} l_2\cos(\theta_1+\theta_2) & l_2\sin(\theta_1+\theta_2) \\ -l_1\cos\theta_1 & -l_1\sin\theta_1\end{bmatrix}\begin{bmatrix}\ddot{x} \\ \ddot{y}\end{bmatrix}$$
$$+\frac{1}{l_1 l_2 \sin\theta_2}\begin{bmatrix} l_1 l_2\cos\theta_2 & l_2^2 \\ -l_1^2 & -l_1 l_2\cos\theta_2\end{bmatrix}\begin{bmatrix}\dot{\theta}_1^2 \\ (\dot{\theta}_1 + \dot{\theta}_2)^2\end{bmatrix} \tag{15}$$

Three different movements are illustrated in the examples below: one that must be slowed down, one that must be sped up, and one that is unrealizable at any speed. For the link parameters, we have set $l_1 = l_2 = 0.5$ meters, $m_1 = m_2 = 1$ kg, $I_1 = I_2 = m_1 l_1^2/12 + m_1 R^2/4$, $R = 0.1 l_1$, and $g = 9.8$ m/sec^2.

4.1 A Movement Whose Speed Is Scaled Down

A straight line motion from $(x_0, y_0) = (0.5, -0.5)$ to $(x_1, y_1) = (0.5, 0)$ is to be generated at a constant velocity of 4 meters/second. The torque limits for the actuators are set at $n_1^+ = -n_1^- = 6.9$ kg-m, and $n_2^+ = -n_2^- = 2$ kg-m. A comparison between $\mathrm{n}^+(t)$, $\mathrm{n}^-(t)$, and $\mathrm{n}_a(t)$ is presented in Figure 2.

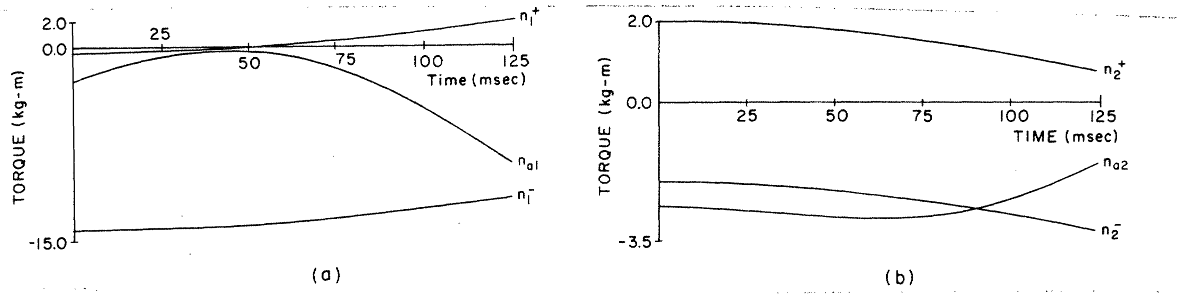

Figure 2
Torque profiles for a constant-velocity, straight-line trajectory of 4 m/sec from $(x, y) = (.5, -.5)$ to $(.5, 0)$ when $n_1^+ = -n_1^- = 6.9$ kg-m and $n_2^+ = -n_2^- = 2$ kg-m. (a) Joint 1 minimum and maximum torques, $n_1^-(t)$ and $n_1^+(t)$, and the velocity and acceleration torque $n_{a1}(t)$ are shown versus time; (b) corresponding torque profiles for joint 2.

Joint 1 is represented by Figure 2a, where the torque requirements $n_{a1}(t)$ for the complete movement fall within the modified actuator constraints $n_1^-(t)$ and $n_2^+(t)$. For joint 2 in Figure 2b, however, the required torque $n_{a2}(t)$ falls below the lower actuator bound $n_2^-(t)$ for the initial movement segment. This suggests that the movement must be slowed down. By scaling the torque $n_{a2}(t)$ by a factor $c^2 < 1$, the elements of the new torque $\tilde{n}_{a2}(t)$ become larger (i.e., less negative). The $\tilde{n}_{a2}(t)$ curve could then be made to lie completely above the $n_1^-(t)$ curve, as if it had been shifted upwards.

Carrying out the computations in (9), it is found that $[c^{2-}, c^{2+}] = [0.582, 0.745]$. The value $c^{2+} = 0.745$ arises from joint 2 at time $t = 0.035$ sec., while the value $c^{2-} = 0.582$ arises from joint 1 at the same time. Thus the fastest speed at which this movement can be executed is determined by $4\sqrt{0.745} = 3.45$ m/s. On the other hand, there is a non-zero lower speed limit, $4\sqrt{.582} = 3.05$ m/s. Examining Figure 2a, if the movement is slowed too much, then the $n_{a1}(t)$ curve is displaced upwards, intersecting the $n_1^+(t)$ curve and exceeding that upper torque limit.

4.2 A Movement Whose Speed Is Scaled Up

As shown above, if the movement speed falls under 3.05 m/s, then the actuator limits are exceeded. This condition is verified here by considering the same movement but executed at 2 m/s and by working through the algorithm. Figure 3a shows that the shoulder torque $n_{a1}(t)$ exceeds the upper actuator bound $n_1^+(t)$ at the beginning of the movement. Calculations show that for joint 1, $c^{2-} = 2.329$ at $t = 0.035$ s, so that the movement must be sped up by $2\sqrt{2.329} = 3.05$ m/s as predicted. This would push the $n_{a1}(t)$ curve down until it is completely beneath $n_1^+(t)$. The curve $n_1^-(t)$ is the same as in Figure 2a, but has been left out here to allow an expanded scale.

There is an upper limit of $c^{2+} = 2.981$ at $t = 0.035$ s as well, determined this time by

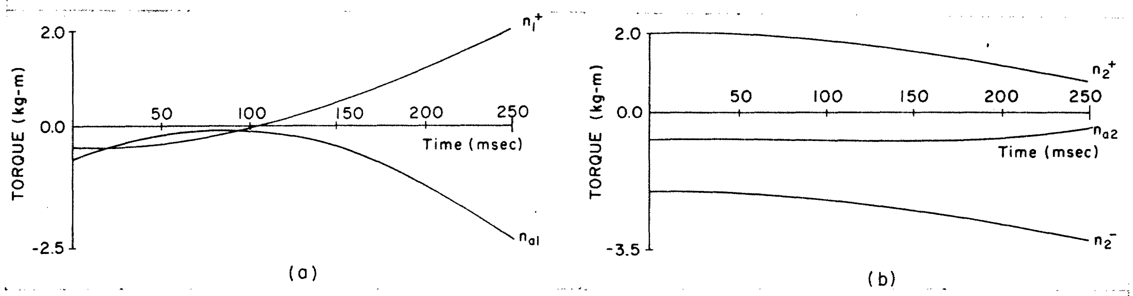

Figure 3
Torque profiles for a constant-velocity, straight-line trajectory of 2 m/sec with other conditions the same as in Figure 2.

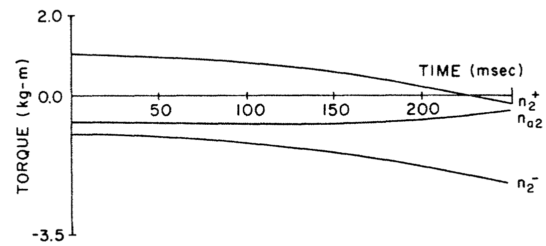

Figure 4
Joint 2 torque profiles for the same movement as in Figure 3 except with $n_2^+ = -n_2^- = 1$ kg-m.

joint 2 (Figure 3a). If the curve $n_{a2}(t)$ is pushed down too far, it will violate the lower bound $n_2^-(t)$. Thus the fastest this movement can be executed is $2\sqrt{2.981} = 3.45$ m/s, in agreement with the first movement analysis.

4.3 An Unrealizable Movement

For the third movement, the conditions are the same as for the second movement, but the second actuator limits are now changed to $n_2^+ = -n_2^-(t) = 1$ kg-m. As before, the actuator limits on joint 1 (Figure 3a) require that the minimum speed for this movement be determined by $c^{2-} = 2.329$. But Figure 4 shows that the joint 2 actuator limits prevent any higher speed scaling than $c^{2+} = 1.522$, because $n_{a2}(t)$ would fall below $n_2^-(t)$. Thus there are incompatible scaling requirements, and this movement cannot be realized at any speed.

5. Velocity-Dependent Motor Limits

We have assumed above that the actuator limits n^+ and n^- are constant throughout the motion.

In reality, the maximum actuator torque often does depend on velocity, such as for electric torque motors. For the case of a low-inductance motor with no dissipative effects [3],

$$V = IR + K_v\omega \tag{16}$$

where V is the motor voltage, I is the motor current, R is the motor resistance, ω is the rotational speed of the motor, and K_v is a constant of proportionality for the back-EMF term. The torque n produced by the motor is directly proportional to current:

$$n = K_n I \tag{17}$$

In voltage-control mode, for example in a chopping circuit where duty cycle is modulated, there is an upper voltage limit V_{max} which can be applied to the motor. Because of the back-EMF, the maximum current, and hence the maximum torque, is velocity dependent. Assuming a gear ratio of 1 (otherwise absorb the gear ratio into K_v), then $\omega = \dot{\theta}$, where $\dot{\theta}$ is the velocity of the joint actuated by the motor. Combining (16) and (17), and absorbing the gravity torque $g(\theta(t))$ into the motor limit,

$$n^+(t) = \frac{K_n}{R}(V_{max} - K_v\dot{\theta}(t)) - g(\theta(t)) \tag{18}$$

If the trajectory is to be time-scaled by a factor c, then

$$\tilde{n}^+(t) = \frac{K_n}{R}V_{max} - \frac{K_n K_v}{R}c\dot{\theta}(ct) - g(\theta(ct)) \tag{19}$$

where unlike the dynamic terms there is a linear dependence on the scale factor c. When solving for the scaling that satisfies the upper bound, again we need the relations $\tilde{n}_a(t/c) = c^2 n_a(t)$ and $\tilde{n}^+(t/c) = n^+(t)$. There is a quadratic equation in c when $\tilde{n}^+(t/c)$ is replaced by $c^2 n_a(t)$ in (19):

$$n_a(t)c^2 + \frac{K_n K_v}{R}c\dot{\theta}(t) - \frac{K_n}{R}V_{max} + g(\theta(t)) = 0 \tag{20}$$

When solved,

$$c = \frac{1}{2n_a(t)}\left(-\frac{K_n K_v}{R}\dot{\theta}(t) \pm \sqrt{\left(\frac{K_n K_v}{R}\dot{\theta}(t)\right)^2 + 4n_a(t)\left(\frac{K_n}{R}V_{max} - g(\theta(t))\right)}\right) \tag{21}$$

The root which gives the largest positive c should be chosen for c^{2+}. As before, it is possible that there is no positive (or even real) root, which indicates that the trajectory is unrealizable. We may also solve (21) with V_{min} to find c^{2-}. The procedure for determining the appropriate trajectory scaling factor then follows that indicated by (9).

The back-EMF can be considered a form of viscous friction, but if there were any additional viscous friction at a joint or actuator, it could be handled in the same manner. As regards Coulomb, or sliding, friction, it could be subtracted from the motor torque limits depending on the direction of movement, i.e., *sliding friction torque* $= -n_f \mathrm{sgn}(\dot{\theta}(t))$. Any actuator springiness, which is position-dependent, could also be readily absorbed into the torque limits. Motor inductance unfortunately seems to present an intractible problem, due to the need to find the time derivative of the dynamic equation (3).

Conclusion

Trajectory planning and inverse dynamics may be efficiently coupled to reflect the exact influence of actuator torque limits on execution capability. By factoring out gravity, a time-scaling property of manipulator dynamics readily allows a realizable speed of movement for the whole trajectory to be determined if there exists one. Rather than recomputing the dynamics corresponding to a new trajectory speed from scratch, the dynamics of the new trajectory is obtained by a simple linear combination of components of the original trajectory dynamics.

Velocity-dependent actuator limits, as well as various sources of joint friction, can be accommodated in this scheme. An important side effect of the dynamic time scaling property is that a ubiquitous assumption in manipulator control, namely that the velocity-product dynamic terms are significant only at high speeds of movement, is false: these terms have the same significance relative to the acceleration dynamic terms for all speeds of movement.

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