RECOMP II USERS' PROGRAM NO. 1068

PROGRAM TITLE:	PROGRAM FOR EIGENVALUES AND EIGENVECTORS OF A REAL SYMMETRIC MATRIX
PROGRAM CLASSIFICATION:	General
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PURPOSE:	To determine the Eigenvalues and Eigenvectors of a real symmetric matrix.
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PROGRAM FOR EIGENVALUES AND EIGENVECTORS OF A REAL SYMMETRIC MATRIX

<u>METHOD</u> - JACOBI METHOD ref. Goldstein, Murrary, and von Newmann, The Jacobi Method for Real Symmetric Matrices; Journal of the ACM, Vol. 6 (Jan. 1959), 59-96.

A sequence of orthonormal transformations, U^i, are found such that the matrix $A^i = U^i A^{i-1} U^{i*}$ is similar to A^{i-1} and the sum of the squares of the off-diagonal elements, τ^i, of A^i is less than τ^{i-1} of A^{i-1}. In this way the sequence of matrices A^i converges to a diagonal matrix with the eigenvalues on the diagonal. That is $U^i U^{i-1} \ldots\ldots U^o A^o U^{o*} U^{1*} \ldots\ldots U^{i*} \longrightarrow \Lambda$ and $U^i U^{i-1} \ldots\ldots U^o$ is the approximation to the matrix of eigenvectors.

For the i^{th} iteration we pick an off-diagonal element of A^{i-1}, say a^{i-1}_{jk}, whose magnitude is greater than or equal to the average magnitude of the off-diagonal elements, i. e.

$$\left(a^{i-1}_{jk}\right)^2 \geq \frac{\tau^{i-1}}{n(n-1)} \quad (\tau^{i-1} \text{ is the sum of the squares of the off-diagonal elements of } A^{i-1})$$

We then determine a matrix U^i, such that in the matrix $A^i = U^i U^{i-1} U^{i*}$, a^i_{jk} is zero and $\tau^i = \tau^{i-1} - 2\left(a^{i-1}_{jk}\right)^2$.

Let U^i be defined as follows:

$$u^i_{mn} = \delta mn \text{ for } m \neq j \text{ or } k \quad n \neq j \text{ or } k$$

$$u^i_{jj} = u^i_{kk} = \cos\phi$$

$$u^i_{jk} = -u^i_{kj} = \sin\phi$$

Then $$a_{jj}^{i} = \frac{1}{2}(a_{jj}^{i-1} + a_{kk}^{i-1}) + \frac{1}{2}(a_{jj}^{i-1} - a_{kk}^{i-1}) \cos 2\phi - a_{jk}^{i-1} \sin 2\phi$$

$$a_{kk}^{i} = \frac{1}{2}(a_{jj}^{i-1} + a_{kk}^{i-1}) - \frac{1}{2}(a_{jj}^{i-1} - a_{kk}^{i-1}) \cos 2\phi + a_{jk}^{i-1} \sin 2\phi$$

$$a_{mj}^{i} = a_{jm}^{i} = a_{mj}^{i-1} \cos\phi - a_{mk}^{i-1} \sin\phi \qquad m \neq j, k$$

$$a_{km}^{i} = a_{mk}^{i} = a_{mj}^{i-1} \sin\phi + a_{mk}^{i-1} \cos\phi \qquad m \neq j, k$$

and $$a_{jk}^{i} = a_{kj}^{i} = \frac{1}{2}(a_{jj}^{i-1} - a_{kk}^{i-1}) \sin 2\phi + a_{jk}^{i-1} \cos 2\phi$$

All other elements remain unchanged.

In order that $a_{jk}^{i} = 0$, ϕ must be defined as follows:

$$\tan 2\phi = \frac{2a_{jk}^{i-1}}{a_{kk}^{i-1} - a_{jj}^{i-1}}$$

Then $\cos\phi$ and $\sin\phi$ may be computed directly from:

$$R = \left[(a_{kk}^{i-1} - a_{jj}^{i-1})^2 + 4(a_{jk}^{i-1})^2\right]^{1/2}$$

$$\cos 2\phi = \frac{\left|a_{kk}^{i-1} - a_{jj}^{i-1}\right|}{R}$$

$$\sin 2\phi = \frac{2a_{jk}^{i-1}}{R}\left[\mathrm{Sign}\,(a_{kk}^{i-1} - a_{jj}^{i-1})\right]$$

$$\cos\phi = \left[\frac{1}{2}(1+\cos 2\phi)\right]^{1/2}$$

$$\sin\phi = \frac{1}{2}\,\frac{\sin 2\phi}{\cos\phi}$$

The i^{th} iterant of the matrix of eigenvectors $X^i = U^i X^{i-1}$ is also computed

$$x^i_{jm} = x^{i-1}_{jm}\cos\phi - x^{i-1}_{km}\sin\phi \qquad m = 1, 2, \ldots, n$$

$$x^i_{km} = x^{i-1}_{km}\cos\phi + x^{i-1}_{jm}\sin\phi$$

All other elements remain unchanged.

To show that the process converges:

Clearly $\tau^i = \tau^{i-1} - 2(a^{i-1}_{jk})^2$

and since $(a^{i-1}_{jk})^2 \geq \dfrac{\tau^{i-1}}{n(n-1)}$

$$\tau^i \leq \tau^{i-1}\left(1 - \frac{2}{n(n-1)}\right) \leq \tau^i\left[e^{-\frac{2}{n(n-1)}}\right] \leq \tau_o\, e^{-\frac{2i}{n(n-1)}}$$

STORAGE

The program occupies relative locations 0000-0367 (OCTAL) For an N x N matrix $4N^2$ additional consecutive locations are required.

INSTRUCTIONS FOR USE

a) INPUT

In rel. loc 0000 store A_o, the location at which the matrix begins in the left half address.

In rel. loc. 0001 store N, the size of the matrix, in binary at b18

In rel loc. 0062.1 store exit; the location to which you wish to return on completion of computation (do not destroy the rest of 0062)

Beginning in location A_o store the matrix row-wise in floating point. Although a square matrix is required as input, it is not actually necessary to store those elements which are below the principal diagonal. Only the upper triangle is used in computation. Start by transferring to relative location 0002.0.

b) OUTPUT

At the completion of the computation the eigenvalues are stored on the principal diagonal of the original input matrix. (The upper half of the input matrix is destroyed.) The eigenvectors are stored row-wise immediately following the diagonal matrix of eigenvalues. (in locations $A_o + 2N^2$ to $A_o + 4N^2 - 1$.) Control is transferred to the adress stored in 0062.1.

RELOCATION - The program is relocatable. The tape includes the relocation matrix and AN-004. The program may only be relocated to locations of the form XX00; XX10; XX50; XX60.

LIMITATIONS

The minimum matrix sixe is 2 x 2

The maximum size is 30 x 30, assuming optimum placement of program and matrix. There is no check to determine whether the capacity of the machine has been exceeded (when $A_o + 4N^2 > 7760$)

TIME - The computation takes approximately $.7N^3$ seconds for an N x N matrix.

ACCURACY:

Theoretically, if $\tau^i \leq \epsilon$, the maximum error in each of the eigenvalues is $\epsilon^{1/2}$ $\left[\left| a^i_{jj} - \lambda_k \right| < \epsilon^{1/2} \right]$. The error in the components of the eigenvectors is of the same order of magnitude.

Exit from the program takes place when $\tau^i < \epsilon$. $1/2\,\epsilon$ is stored in floating point in relative locations 0076 and 0077 and is presently $1/2 \times 2^{-38}$. Thus the error in the eigenvalues is $\pm\ .7 \times 2^{-19} = \pm\ 1.3 \times 10^{-6}$.

In practice, for large matrices the error may be greater than $\epsilon^{1/2}$ because of cumulative computational errors. To increase the accuracy it may be desirable to decrease ϵ . An estimate for the optimum value of ϵ is $\epsilon^{1/2} = 2(N^{1/2} + 7)N^2 2^{-s}$ where s is the number of significant binary digits in the input matrix. The error in the eigenvalues for this value of ϵ is

$$E < \epsilon^{1/2} \left[1 + .69 \left| \log_2 \epsilon^{1/2} \right| \right] + 13N^{5/2}\, 2^{-s}$$

(Experimentally it has been found that this overestimates the error by about a factor of 10).

COMMENTS:

1. The matrix may be singular.
2. There is no error halt
3. The program resets itself. N, A_o and the exit are not destroyed.
4. If only the eigenvalues are desired, the eigenvectors may be avoided by inserting the following changes.

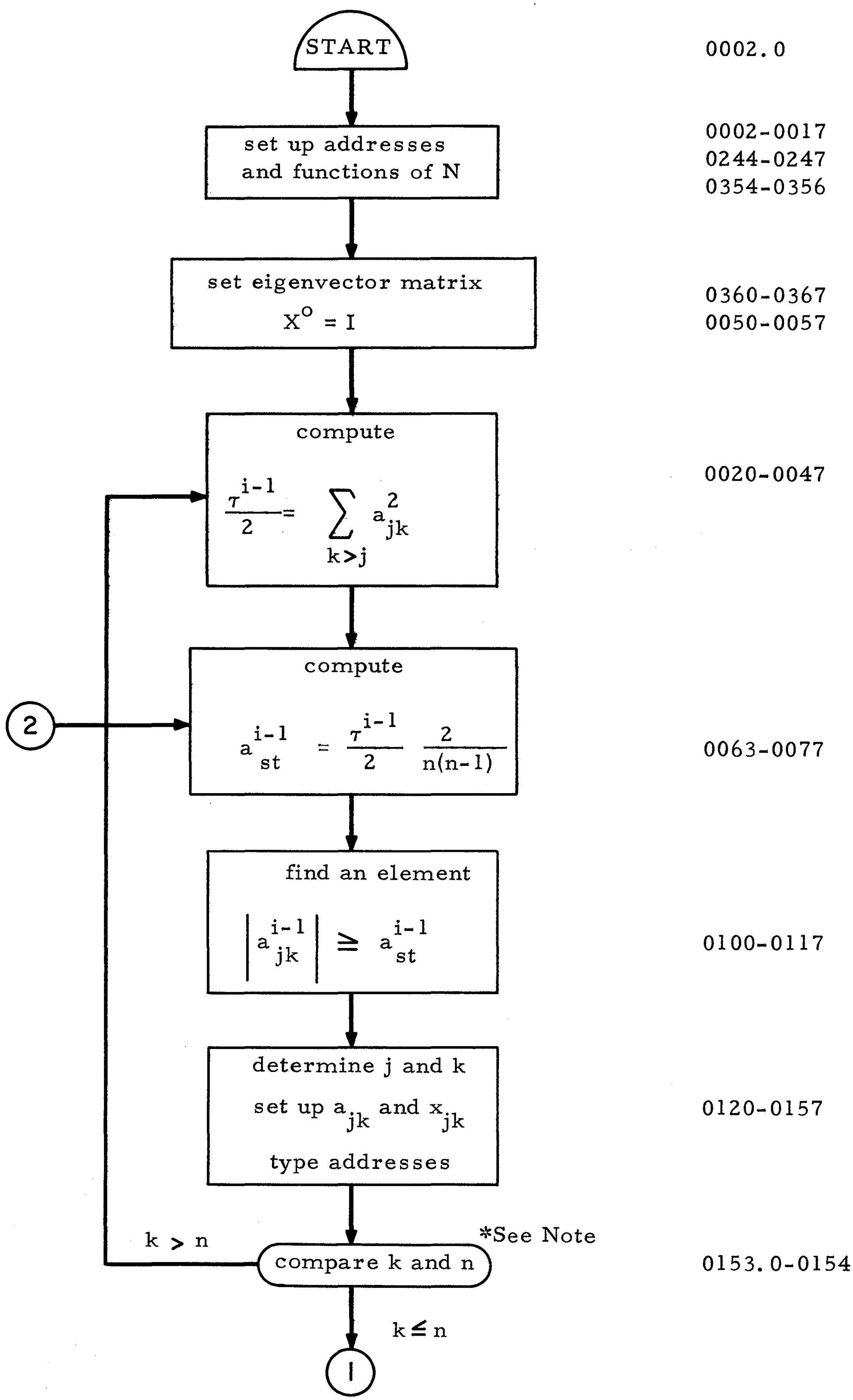
   in rel. loc. 0356 put +57 0054.0 + 40 0000.0

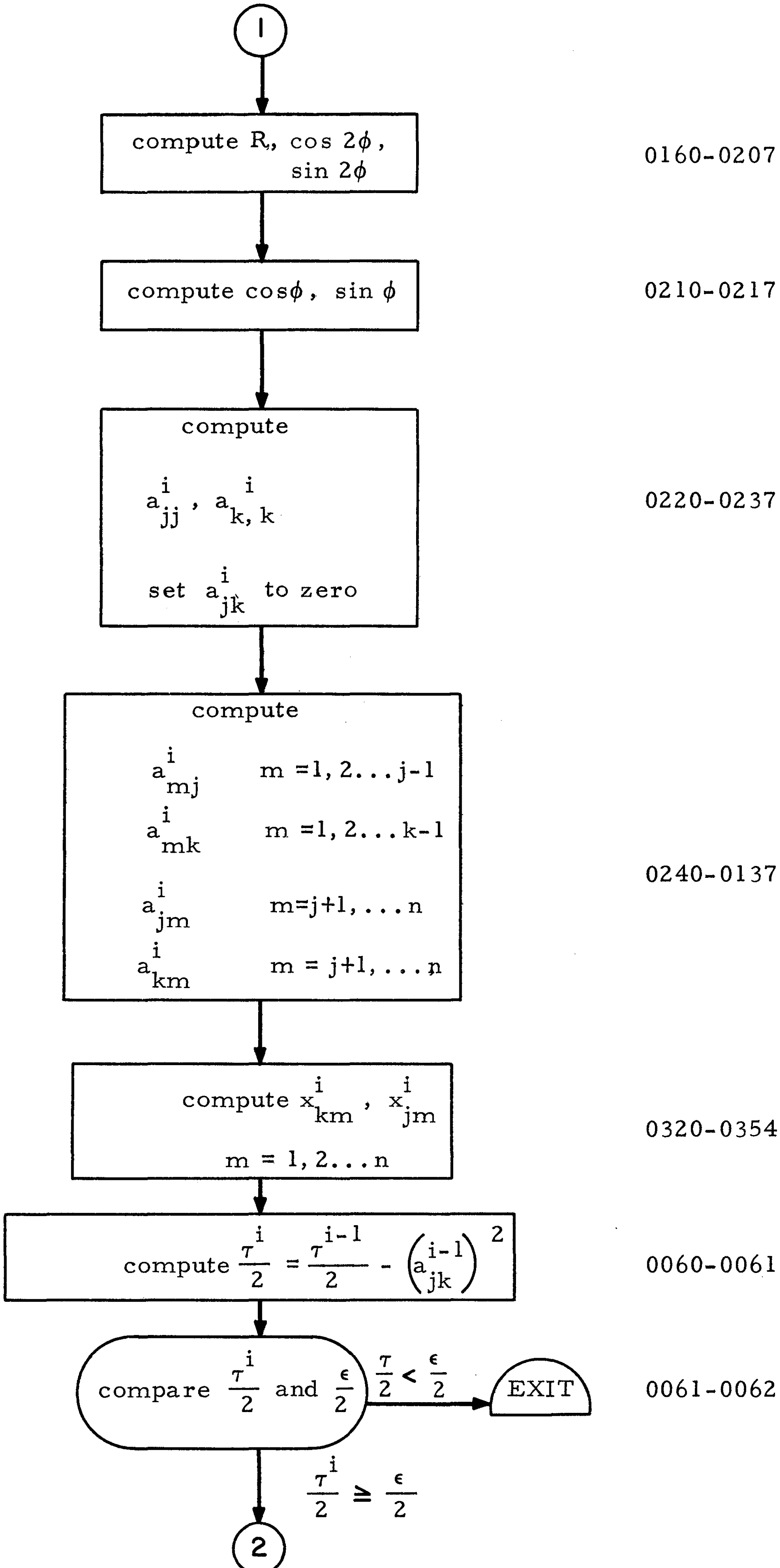
   in rel. loc. 0315 put +57 0345.0 + 40 0000.0

   Store some large floating point number of location $A_o + (2N^2)_8$ (large means greater than 1/2.)

5. A matrix consisting of all zeros will not be tolerated.

FLOW CHART


START
0002.0
set up addresses and functions of N
0002-0017
0244-0247
0354-0356
set eigenvector matrix
$X^{o} = I$
0360-0367
0050-0057
compute
$\frac{\tau^{i-1}}{2} = \sum_{k>j} a_{jk}^{2}$
0020-0047
compute
$a_{st}^{i-1} = \frac{\tau^{i-1}}{2} \frac{2}{n(n-1)}$
0063-0077
2
find an element
$\left| a_{jk}^{i-1} \right| \geqq a_{st}^{i-1}$
0100-0117
determine j and k
set up a_{jk} and x_{jk}
type addresses
0120-0157
$k > n$
*See Note
compare k and n
0153.0-0154
$k \leqq n$
1



* NOTE Since $\frac{\tau^i}{2}$ is computed from $\frac{\tau^{i-1}}{2} - \left(a^{i-1}_{jk}\right)^2$ it is possible that the

errors that accumulate because of the subtractions will be such that $\frac{\tau^i}{2}\left(\frac{2}{n(n-1)}\right)$ is greater than any off diagonal element. When this happens τ_i^i is recomputed from the actual off-diagonal elements. Notice that the estimate of $\frac{\tau^i}{2}$ will always be greater than it should be, which is actually an advantage until the above phenomenon occurs.

0000.0	+00.3000.0	---	A_0
	-00.0000.0	---	
0001.0	+00.0004.0	---	N
	-00.0000.0	---	
0002.0	+64.0000.0	CTL	START
	+66.0010.0	CTV	
0003.0	+57.7763.1	TRA	
	+00.7760.0	CLA	
0004.0	+42.0023.0	STA	
	+01.7777.0	ADD	
0005.0	+42.0115.0	STA	
	+00.7761.0	CLA	
0006.0	+41.0001.0	ALS	
	+60.0046.0	STO	2N
0007.0	+60.0071.0	STO	
	+60.0136.0	STO	
0010.0	+60.0147.0	STO	
	+01.7776.0	ADD	
0011.0	+60.0056.0	STO	2N + 2
	+00.7761.0	CLA	
0012.0	+13.7761.0	MPR	
	+41.0023.0	ALS	
0013.0	+60.0137.0	STO	$2N^2$
	+40.0000.0	ARS	
0014.0	+01.7760.0	ADD	
	+42.0051.0	STA	X_0
0015.0	+42.0045.0	STA	
	+57.0244.0	TRA	CONTINUE INITIAL
0016.0	+00.0002.0	---	
	-00.0000.0	---	
0017.0	+00.0000.1	---	
	-00.0000.0	---	

```
0020.0 +00.0000.0  ---        Σ
       -00.0000.0  ---

0021.0 +00.0000.0  ---
       -00.0000.0  ---

0022.0 +00.0000.0  ---        j
       -00.0000.0  ---

0023.0 +30.3000.0  FCA        fin k
       +35.7776.0  FST

0024.0 +30.0000.0  FCA        a_jk
       +35.7776.0  FST

0025.0 +07.7776.0  FMP
       +04.7760.0  FAD

0026.0 +35.7760.0  FST        Σ
       +00.7764.0  CLA        L(a_jk)

0027.0 +01.7775.0  ADD        2.0
       +60.7764.0  STO


0030.0 +03.7763.0  SUB        fin k
       +51.7764.0  TRN

0031.0 +00.7762.0  CLA        j
       +01.7775.0  ADD        2.0

0032.0 +60.7762.0  STO        j
       +01.7763.0  ADD        fin k

0033.0 +60.7764.0  STO        L(a_jK)
       +00.7763.0  CLA        fin k

0034.0 +66.0040.0  CTV
       +57.7770.0  TRA

0035.0 +00.0002.0  ---
       -00.0000.0  ---

0036.0 +00.0000.0  ---
       -00.0000.0  ---

0037.0 +00.0000.0  ---
       -00.0000.0  ---
```

0040.0	+01.7776.0	ADD	2N
	+60.7763.0	STO	
0041.0	+03.7775.0	SUB	$A_0 + 2N^2$
	+50.7773.0	TRZ	
0042.0	+66.0030.0	CTV	
	+57.7764.0	TRA	
0043.0	+30.7760.0	FCA	Σ
	+35.0074.0	FST	τ^i
0044.0	+64.0060.0	CTL	
	+57.7777.0	TRA	
0045.0	+30.3040.0	FCA	$A_0 + 2N^2$
	+35.7776.0	FST	
0046.0	+00.0010.0	---	2N
	-00.0000.0	---	
0047.0	+66.0070.0	CTV	
	+57.7763.0	TRA	COMPUTE $\lvert\overline{a_{jk}}\rvert$
0050.0	+40.0000.0	ARS	
	+30.0206.0	FCA	1 OR 0
0051.0	+35.3040.0	FST	x_{jk} OR x_{jk}
	+00.7761.0	CLA	
0052.0	+01.7766.0	ADD	2N+2 or 2
	+60.7761.0	STO	
0053.0	+03.7767.0	SUB	$X_0 + 2N^2$
	+51.7760.1	TRN	
0054.0	+40.0000.0	ARS	
	+66.0030.0	CTV	TO COMPUTE τ OR STORE 1's
0055.0	+64.0020.0	CTL	
	+57.7771.0	TRA	
0056.0	+00.0012.0	---	
	-00.0000.0	---	
0057.0	+35.3100.0	FST	
	+00.7761.0	CLA	

0060.0	+30.7774.0	FCA	τ^{i-1}
	+06.0264.0	FSB	$(a_{jk}^{i-1})^2$
0061.0	+35.0074.0	FST	τ^{i}
	+06.7776.0	FSB	ϵ
0062.0	+52.7763.0	TRP	CONTINUE
	+57.0000.0	TRA	EXIT
0063.0	+00.7771.0	CLA	2N
	+60.0276.0	STO	Δk
0064.0	+60.0277.0	STO	A_j
	+30.0074.0	FCA	τ^i
0065.0	+07.7772.0	FMP	2/N(N-1)
	+35.7760.0	FST	$\overline{\lvert a_{jk}\rvert}^2$
0066.0	+44.7760.0	FSQ	
	+35.0116.0	FST	$\overline{\lvert a_{jk}\rvert}$
0067.0	+40.0000.0	ARS	
	+64.0100.0	CTL	
0070.0	+66.0110.0	CTV	
	+57.7765.0	TRA	TO FIND a_{jk}^{i}
0071.0	+00.0010.0	---	
	-00.0000.0	---	
0072.0	+52.5252.1	---	
	-52.5253.0	---	
0073.0	-00.0000.0	---	
	-00.0001.0	---	
0074.0	+41.7462.1	---	
	-04.4000.0	---	
0075.0	-00.0000.0	---	
	-00.0021.1	---	
0076.0	+40.0000.0	---	
	-00.0000.0	---	
0077.0	-00.0000.0	---	
	-00.0023.0	---	

```
0100.0 +40.0000.0  ARS
       +30.7776.0  FCA

0101.0 -14.0000.1  ---        BECOMES  FSA  ajK
       +23.7763.0  DVR                 TM1  > ājk

0102.0 +50.7771.1  TRZ
       +00.7761.0  CLA

0103.0 +01.7773.0  ADD
       +60.7761.0  STO

0104.0 +03.7775.0  SUB
       +51.7760.1  TRN

0105.0 +00.7774.0  CLA
       +01.7773.0  ADD

0106.0 +60.7774.0  STO
       +01.7775.0  ADD

0107.0 +60.7761.0  STO
       +00.7775.0  CLA


0110.0 +01.0071.0  ADD
       +60.7775.0  STO

0111.0 +57.7760.1  TRA
       +00.7761.0  CLA

0112.0 +64.0120.0  CTL
       +57.7762.0  TRA        COMPUTE ADDRESSES

0113.0 +00.0002.0  ---
       -00.0000.0  ---

0114.0 +00.0000.0  ---        j
       -00.0000.0  ---

0115.0 +06.0000.0  FSB        fin k = (A0 + 2jn).1
       +51.7771.1  TRN

0116.0 +62.4223.1  ---
       +57.6416.1  ---        } ājk

0117.0 -00.0000.0  ---
       -00.0007.1  ---
```

0120.0	+00.7777.0	---		EXTRACT CONSTANT
	-00.0000.0	---		
0121.0	+00.0002.0	---		
	-00.0000.0	---		
0122.0	+33.7760.0	EXT		
	+42.0163.0	STA		$L(a_{jk})$
0123.0	+42.0232.0	STA		
	+60.7762.0	STO		
0124.0	+00.7774.0	CLA		j
	+03.7761.0	SUB		
0125.0	+60.7763.0	STO		$A_0 + 2(j-1)$
	+01.0000.0	ADD		
0126.0	+42.0241.0	STA		
	+42.0242.0	STA		
0127.0	+00.7775.0	CLA		
	+66.0130.0	CTV		
0130.0	+33.7760.0	EXT		
	+42.0306.0	STA		$A_0 + 2jN$
0131.0	+01.7777.0	ADD		
	+42.0327.0	STA		$X_0 + 2jN$
0132.0	+03.7776.0	SUB		
	+42.0320.0	STA		$X_0 + 2(j-1)N$
0133.0	+42.0322.0	STA		
	+03.7777.0	SUB		
0134.0	+60.7764.0	STO		
	+01.7763.0	ADD		
0135.0	+66.0140.0	CTV		
	+57.7770.0	TRA		
0136.0	+00.0010.0	---		$2N$
	-00.0000.0	---		
0137.0	+00.0040.0	---		
	-00.0000.0	---		$2N^2$

0140.0	+42.0161.0	STA	$L(a_{jj})$
	+42.0215.0	STA	
0141.0	+42.0223.0	STA	
	+42.0267.0	STA	
0142.0	+00.7762.0	CLA	
	+03.7764.0	SUB	
0143.0	+60.7765.0	STO	
	+01.0000.0	ADD	
0144.0	+42.0240.0	STA	$A_0 + 2(k-1)$
	+42.0243.0	STA	
0145.0	+00.7765.0	CLA	
	+13.7777.0	MPR	
0146.0	+66.0150.0	CTV	
	+57.7770.0	TRA	
0147.0	+00.0010.0	---	
	-00.0000.0	---	
0150.0	+41.0021.0	ALS	
	+01.0000.0	ADD	
0151.0	+60.7766.0	STO	
	+01.7765.0	ADD	
0152.0	+42.0160.0	STA	$L(a_{KK})$
	+42.0216.0	STA	
0153.0	+42.0231.0	STA	
	+03.0000.0	SUB	A_0
0154.0	+03.0137.0	SUB	$2N^2$
	+52.0054.0	TRP	TO RECOMPUTE r_i
0155.0	+00.7766.0	CLA	
	+01.0137.0	ADD	
0156.0	+42.0321.0	STA	$X_0 + 2(k-1)n$
	+42.0323.0	STA	
0157.0	+64.0160.0	CTL	
	+57.7760.0	TRA	TO COMPUTE $R, \cos^2\varphi, \sin 2\varphi$

```
0160.0 +30.3036.0  FCA        aKK
       +40.0000.0  ARS

0161.0 +06.3000.0  FSB        ajj
       +35.7760.0  FST

0162.0 +07.7760.0  FMP
       +35.7770.0  FST

0163.0 +30.3006.0  FCA        ajK
       +35.7762.0  FST

0164.0 +07.7762.0  FMP
       +35.0264.0  FST        (ajK)^2

0165.0 +43.0000.0  XAR
       +01.0177.0  ADD

0166.0 +43.0000.0  XAR        4 ajK^2
       +04.7770.0  FAD

0167.0 +35.7764.0  FST        R^2
       +66.0170.0  CTV


0170.0 +44.7764.0  FSQ
       +05.7764.0  FDV

0171.0 +35.7764.0  FST        1/R
       +07.7760.1  FMP

0172.0 +35.7766.0  FST        cos 2φ
       +07.7760.0  FMP

0173.0 +35.7760.0  FST
       +51.7775.0  TRN

0174.0 +30.7762.0  FCA
       +57.7775.1  TRA

0175.0 +34.7762.0  FCS
       +07.7764.0  FMP

0176.0 +66.0200.0  CTV
       +57.7770.0  TRA

0177.0 +00.0000.0  ---
       -00.0001.0  ---
```

```
0200.0 +35.7764.0  FST    ½ sin 2φ
       +07.7762.0  FMP

0201.0 +43.0000.0  XAR
       +01.7777.0  ADD

0202.0 +43.0000.0  XAR
       +35.7762.0  FST    a_jk sin 2φ

0203.0 +30.7766.0  FCA
       +04.7776.0  FAD

0204.0 +43.0000.0  XAR
       +03.7777.0  SUB

0205.0 +66.0210.0  CTV
       +57.7770.0  TRA

0206.0 +40.0000.0  ---
       -00.0000.0  ---

0207.0 +00.0000.0  ---
       -00.0000.1  ---


0210.0 +43.0000.0  XAR
       +35.7766.0  FST

0211.0 +44.7766.0  FSQ
       +35.7766.0  FST    cos φ

0212.0 +35.0256.0  FST
       +35.0334.0  FST

0213.0 +30.7764.0  FCA
       +05.7766.0  FDV

0214.0 +35.0254.0  FST    sin φ
       +35.0336.0  FST

0215.0 +30.3000.0  FCA    a_jj
       +40.0000.0  ARS

0216.0 +04.3036.0  FAD    a_kk
       +35.7766.0  FST

0217.0 +66.0220.0  CTV
       +57.7770.0  TRA
```

0220.0	+06.7760.0	FSB	
	+43.0000.0	XAR	
0221.0	+03.7777.0	SUB	
	+43.0000.0	XAR	
0222.0	+06.7762.0	FSB	
	+40.0000.0	ARS	
0223.0	+35.3000.0	FST	a^i_{jj}
	+30.7766.0	FCA	
0224.0	+04.7760.0	FAD	
	+43.0000.0	XAR	
0225.0	+03.7777.0	SUB	
	+43.0000.0	XAR	
0226.0	+66.0230.0	CTV	
	+57.7770.0	TRA	
0227.0	+00.0000.0	---	
	-00.0000.1	---	
0230.0	+04.7762.0	FAD	
	+40.0000.0	ARS	
0231.0	+35.3036.0	FST	a^i_{kk}
	+30.7776.0	FCA	
0232.0	+35.3006.0	FST	$a^i_{jk} = 0$
	+64.0240.0	CTL	
0233.0	+00.7761.0	CLA	
	+03.0267.0	SUB	
0234.0	+52.0266.0	TRP	
	+40.0000.0	ARS	
0235.0	+66.0250.0	CTV	V_1
	+57.7760.0	TRA	
0236.0	+00.0000.0	---	
	-00.0000.0	---	0
0237.0	+00.0000.0	---	
	-00.0000.0	---	

	0240.0	+30.3006.0	FCA	a_{mk}
		+35.7764.0	FST	
	0241.0	+30.3000.0	FCA	a_{mj}
		+57.7770.0	TRA	V_1
	0242.0	+35.3000.0	FST	a_{mj}
		+57.7770.0	TRA	V_2
	0243.0	+35.3006.0	FST	a_{mk}
		+57.7770.0	TRA	V_3
	0244.0	+01.0137.0	ADD	CONTINUATION OF INITIAL
		+42.0057.0	STA	
	0245.0	+30.0316.0	FCA	
		+00.0137.0	CLA	
	0246.0	+03.0046.0	SUB	
		+45.0000.0	FNM	
	0247.0	+35.7760.0	FST	
		+57.0354.0	TRA	CONTINUE INITIAL
V_1	0250.0	+35.7766.0	FST	
		+07.7776.0	FMP	$\cos\varphi$
	0251.0	+35.7776.0	FST	
		+34.7764.0	FCS	
	0252.0	+07.7774.0	FMP	$\sin\varphi$
		+04.7776.0	FAD	
	0253.0	+66.0254.0	CTV	V_2
		+57.7762.0	TRA	
	0254.0	-52.7057.0	---	
		-43.5043.0	---	$\sin\varphi$
	0255.0	-00.0000.0	---	
		-00.0005.1	---	
	0256.0	+77.7777.1	---	
		+74.3206.0	---	$\cos\varphi$
	0257.0	+00.0000.0	---	
		-00.0000.0	---	

V_2	0260.0	+30.7766.0	FCA	a_{mj}
		+07.7774.0	FMP	
	0261.0	+35.7774.0	FST	
		+30.7764.0	FCA	a_{mk}
	0262.0	+07.7776.0	FMP	
		+04.7774.0	FAD	
	0263.0	+66.0270.0	CTV	V_3
		+57.7763.0	TRA	
	0264.0	+73.4567.1	---	
		+54.5550.0	---	$(a_{jk}^{i-1})^2$
	0265.0	-00.0000.0	---	
		-00.0016.0	---	
	0266.0	+66.0300.0	CTV	V_4
		+57.7770.0	TRA	
	0267.0	+30.3010.0	FCA	f_{INj}
		+57.7770.0	TRA	
V_3	0270.0	+00.7760.0	CLA	
		+01.7776.0	ADD	
	0271.0	+60.7760.0	STO	
		+42.7763.0	STA	
	0272.0	+00.7761.0	CLA	
		+01.7777.0	ADD	
	0273.0	+60.7761.0	STO	
		+42.7762.0	STA	
	0274.0	+03.0267.0	SUB	
		+52.0266.0	TRP	
	0275.0	+66.0250.0	CTV	V_1
		+57.7760.0	TRA	
	0276.0	+00.0010.0	---	Δk
		-00.0000.0	---	
	0277.0	+00.0010.0	---	
		-00.0000.0	---	Δj

V_4	0300.0	+00.7775.0	CLA	
		+03.0277.0	SUB	Δj
	0301.0	+51.7774.0	TRN	CHANGE Δj
		+00.0276.0	CLA	Δk
	0302.0	+03.7775.0	SUB	
		+50.0315.0	TRZ	
	0303.0	+66.0305.0	CTV	V_5
		+57.7770.0	TRA	CHANGE Δk
	0304.0	+66.0305.0	CTV	V_5
		+57.7772.0	TRA	CHANGE Δj
	0305.0	+00.0002.0	---	
		-00.0000.0	---	
	0306.0	+00.3010.0	---	$A_0 + 2jN$
		-00.0000.0	---	
	0307.0	+00.0000.0	---	~~BLANK~~ NOT USED
		-00.0000.0	---	
V_5	0310.0	+00.7775.0	CLA	
		+60.0276.0	STO	
	0311.0	+00.7776.0	CLA	
		+57.7773.1	TRA	
	0312.0	+00.7775.0	CLA	
		+60.0277.0	STO	
	0313.0	+00.0232.0	CLA	
		+42.0267.0	STA	
	0314.0	+66.0270.0	CTV	V_3
		+57.7770.0	TRA	
	0315.0	+64.0320.0	CTL	
		+66.0330.0	CTV	
	0316.0	+57.7760.0	TRA	TO COMPUTE X_{km}, x_{jm}
		-00.0000.0	---	
	0317.0	+00.0000.0	---	EXPONENT
		-00.0010.0	---	

	0320.0	+30.3040.0	FCA	x_{jm}
		+35.7764.0	FST	
	0321.0	+30.3070.0	FCA	x_{km}
		+57.7770.0	TRA	V_1
	0322.0	+35.3040.0	FST	x_{jm}
		+57.7770.0	TRA	V_2
	0323.0	+35.3070.0	FST	x_{km}
		+57.7770.0	TRA	V_3
	0324.0	+00.0000.0	---	TEMPORARY STORAGE
		-00.0000.0	---	
	0325.0	+00.0000.0	---	
		-00.0000.0	---	
	0326.0	+00.0000.0	---	
		-00.0000.0	---	
	0327.0	+30.3050.0	FCA	fin j
		+35.7764.0	FST	
V_1	0330.0	+35.7766.0	FST	
		+07.7774.0	FMP	$\cos\varphi$
	0331.0	+35.7774.0	FST	
		+30.7764.0	FCA	x_{jn}
	0332.0	+07.7776.0	FMP	$\sin\varphi$
		+04.7774.0	FAD	
	0333.0	+66.0334.0	CTV	V_2
		+57.7763.0	TRA	
	0334.0	+77.7777.1	---	$\cos\varphi$
		+74.3206.0	---	
	0335.0	+00.0000.0	---	
		-00.0000.0	---	
	0336.0	-52.7057.0	---	$\sin\varphi$
		-43.5043.0	---	
	0337.0	-00.0000.0	---	
		-00.0005.1	---	

V_2	0340.0	+34.7766.0	FCS	x_{km}
		+07.7776.0	FMP	$\sin\varphi$
	0341.0	+35.7776.0	FST	
		+30.7764.0	FCA	x_{jm}
	0342.0	+07.7774.0	FMP	$\cos\varphi$
		+04.7776.0	FAD	
	0343.0	+66.0344.0	CTV	V_3
		+57.7762.0	TRA	
V_3	0344.0	+03.0327.0	SUB	
		+51.7776.0	TRN	
	0345.0	+66.0070.0	CTV	
		+57.0357.0	TRA	NEXT ITERATION
	0346.0	+66.0330.0	CTV	V_1
		+57.7760.0	TRA	
	0347.0	+00.0002.0	---	
		-00.0000.0	---	
	0350.0	+00.7761.0	CLA	$L(x_{km})$
		+01.7777.0	ADD	
	0351.0	+60.7761.0	STO	
		+42.7763.0	STA	
	0352.0	+00.7760.0	CLA	$L(x_{jm})$
		+01.7777.0	ADD	
	0353.0	+60.7760.0	STO	
		+42.7762.0	STA	
	0354.0	+30.0206.0	FCA	CONTINUATION OF INITIAL
		+05.7760.0	FDV	
	0355.0	+35.0072.0	FST	
		+66.0360.0	CTV	
	0356.0	+57.7770.0	TRA	TO SET $X^0 = I$
		-00.0000.0	---	
	0357.0	+64.0060.0	CTL	
		+57.7760.0	TRA	COMPUTE $\tau^i = \tau^{i-1} - (a_{jk}^{i-1})^2$

```
0360.0 +64.0050.0  CTL
       +00.7776.0  CLA
0361.0 +42.7760.1  STA
       +42.7766.0  STA
0362.0 +00.7777.0  CLA
       +60.7764.0  STO
0363.0 +57.7760.1  TRA
       -00.0000.0  ---
0364.0 +00.0000.0  ---
       -00.0000.0  ---
0365.0 +00.0000.0  ---
       -00.0000.0  ---
0366.0 +00.0002.0  ---
       +00.7774.0  ---
0367.0 +64.0050.0  CTL
       +57.7760.0  TRA
```

SET UP 0050 TO STORE ZEROS IN X