

IMP — A WHILE-language and two semantics

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Abstract

The formalization of the denotational and operational semantics of a simple while-language together with an equivalence proof between the two semantics. The whole development essentially formalizes/transcribes chapters 2 and 5 of [1]. A much extended version of this development is found in HOL/IMP of the Isabelle distribution.

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1 Arithmetic expressions, boolean expressions, commands

```
theory Com imports Main begin
```

1.1 Arithmetic expressions

```
consts
```

```
  loc :: i  
  aexp :: i
```

```
datatype  $\subseteq$  "univ(loc  $\cup$  (nat  $\rightarrow$  nat)  $\cup$  ((nat  $\times$  nat)  $\rightarrow$  nat))"  
  aexp = N ("n  $\in$  nat")
```

```

| X ("x ∈ loc")
| Op1 ("f ∈ nat -> nat", "a ∈ aexp")
| Op2 ("f ∈ (nat × nat) -> nat", "a0 ∈ aexp", "a1 ∈ aexp")

consts evala :: i
syntax "_evala" :: "[i, i] => o"    (infixl "-a->" 50)
translations "p -a-> n" == "<p,n> ∈ evala"

```

inductive

```

domains "evala" ⊆ "(aexp × (loc -> nat)) × nat"
intros
  N: "[| n ∈ nat; sigma ∈ loc->nat |] ==> <N(n),sigma> -a-> n"
  X: "[| x ∈ loc; sigma ∈ loc->nat |] ==> <X(x),sigma> -a-> sigma`x"
  Op1: "[| <e,sigma> -a-> n; f ∈ nat -> nat |] ==> <Op1(f,e),sigma> -a-> f`n"
  Op2: "[| <e0,sigma> -a-> n0; <e1,sigma> -a-> n1; f ∈ (nat×nat) -> nat |]
    ==> <Op2(f,e0,e1),sigma> -a-> f`<n0,n1>"
type_intros aexp.intros apply_funtype

```

1.2 Boolean expressions

```

consts bexp :: i

```

```

datatype ⊆ "univ(aexp ∪ ((nat × nat)->bool))"
  bexp = true
    | false
    | ROp ("f ∈ (nat × nat)->bool", "a0 ∈ aexp", "a1 ∈ aexp")
    | noti ("b ∈ bexp")
    | andi ("b0 ∈ bexp", "b1 ∈ bexp")    (infixl 60)
    | ori  ("b0 ∈ bexp", "b1 ∈ bexp")    (infixl 60)

```

```

consts evalb :: i
syntax "_evalb" :: "[i,i] => o"    (infixl "-b->" 50)
translations "p -b-> b" == "<p,b> ∈ evalb"

```

inductive

```

domains "evalb" ⊆ "(bexp × (loc -> nat)) × bool"
intros
  true: "[| sigma ∈ loc -> nat |] ==> <true,sigma> -b-> 1"
  false: "[| sigma ∈ loc -> nat |] ==> <false,sigma> -b-> 0"
  ROp: "[| <a0,sigma> -a-> n0; <a1,sigma> -a-> n1; f ∈ (nat*nat)->bool |]
    ==> <ROp(f,a0,a1),sigma> -b-> f`<n0,n1> "
  noti: "[| <b,sigma> -b-> w |] ==> <noti(b),sigma> -b-> not(w)"
  andi: "[| <b0,sigma> -b-> w0; <b1,sigma> -b-> w1 |]
    ==> <b0 andi b1,sigma> -b-> (w0 and w1)"
  ori: "[| <b0,sigma> -b-> w0; <b1,sigma> -b-> w1 |]
    ==> <b0 ori b1,sigma> -b-> (w0 or w1)"
type_intros bexp.intros
  apply_funtype and_type or_type bool_1I bool_0I not_type

```

```
type_elims evala.dom_subset [THEN subsetD, elim_format]
```

1.3 Commands

```
consts com :: i
datatype com =
  skip                                ("skip" [])
  | assignment ("x ∈ loc", "a ∈ aexp") (infixl "[:=" 60)
  | semicolon ("c0 ∈ com", "c1 ∈ com") ("_;_" [60, 60] 10)
  | while ("b ∈ bexp", "c ∈ com") ("while _ do _" 60)
  | if ("b ∈ bexp", "c0 ∈ com", "c1 ∈ com") ("if _ then _ else _" 60)
```

```
consts evalc :: i
syntax "_evalc" :: "[i, i] => o" (infixl "-c->" 50)
translations "p -c-> s" == "<p,s> ∈ evalc"
```

inductive

```
domains "evalc" ⊆ "(com × (loc -> nat)) × (loc -> nat)"
```

intros

```
skip: "[| sigma ∈ loc -> nat |] ==> <skip,sigma> -c-> sigma"
```

```
assign: "[| m ∈ nat; x ∈ loc; <a,sigma> -a-> m |]
==> <x := a,sigma> -c-> sigma(x:=m)"
```

```
semi: "[| <c0,sigma> -c-> sigma2; <c1,sigma2> -c-> sigma1 |]
==> <c0; c1, sigma> -c-> sigma1"
```

```
if1: "[| b ∈ bexp; c1 ∈ com; sigma ∈ loc->nat;
<b,sigma> -b-> 1; <c0,sigma> -c-> sigma1 |]
==> <if b then c0 else c1, sigma> -c-> sigma1"
```

```
if0: "[| b ∈ bexp; c0 ∈ com; sigma ∈ loc->nat;
<b,sigma> -b-> 0; <c1,sigma> -c-> sigma1 |]
==> <if b then c0 else c1, sigma> -c-> sigma1"
```

```
while0: "[| c ∈ com; <b, sigma> -b-> 0 |]
==> <while b do c,sigma> -c-> sigma"
```

```
while1: "[| c ∈ com; <b,sigma> -b-> 1; <c,sigma> -c-> sigma2;
<while b do c, sigma2> -c-> sigma1 |]
==> <while b do c, sigma> -c-> sigma1"
```

```
type_intros com.intros update_type
```

```
type_elims evala.dom_subset [THEN subsetD, elim_format]
```

```
evalb.dom_subset [THEN subsetD, elim_format]
```

1.4 Misc lemmas

```
lemmas evala_1 [simp] = evala.dom_subset [THEN subsetD, THEN SigmaD1, THEN SigmaD1]
```

```

and evala_2 [simp] = evala.dom_subset [THEN subsetD, THEN SigmaD1, THEN SigmaD2]
and evala_3 [simp] = evala.dom_subset [THEN subsetD, THEN SigmaD2]

lemmas evalb_1 [simp] = evalb.dom_subset [THEN subsetD, THEN SigmaD1, THEN SigmaD1]
and evalb_2 [simp] = evalb.dom_subset [THEN subsetD, THEN SigmaD1, THEN SigmaD2]
and evalb_3 [simp] = evalb.dom_subset [THEN subsetD, THEN SigmaD2]

lemmas evalc_1 [simp] = evalc.dom_subset [THEN subsetD, THEN SigmaD1, THEN SigmaD1]
and evalc_2 [simp] = evalc.dom_subset [THEN subsetD, THEN SigmaD1, THEN SigmaD2]
and evalc_3 [simp] = evalc.dom_subset [THEN subsetD, THEN SigmaD2]

inductive_cases
  evala_N_E [elim!]: "<N(n),sigma> -a-> i"
  and evala_X_E [elim!]: "<X(x),sigma> -a-> i"
  and evala_Op1_E [elim!]: "<Op1(f,e),sigma> -a-> i"
  and evala_Op2_E [elim!]: "<Op2(f,a1,a2),sigma> -a-> i"

end

```

2 Denotational semantics of expressions and commands

theory Denotation imports Com begin

2.1 Definitions

consts

```

A    :: "i => i => i"
B    :: "i => i => i"
C    :: "i => i"

```

constdefs

```

Gamma :: "[i,i,i] => i"    ("Γ")
"Γ(b,cden) ==
  (λphi. {io ∈ (phi 0 cden). B(b,fst(io))=1} ∪
   {io ∈ id(loc->nat). B(b,fst(io))=0})"

```

primrec

```

"A(N(n), sigma) = n"
"A(X(x), sigma) = sigma`x"
"A(Op1(f,a), sigma) = f`A(a,sigma)"
"A(Op2(f,a0,a1), sigma) = f`<A(a0,sigma),A(a1,sigma)>"

```

primrec

```

"B(true, sigma) = 1"
"B(false, sigma) = 0"
"B(ROp(f,a0,a1), sigma) = f`<A(a0,sigma),A(a1,sigma)>"
"B(noti(b), sigma) = not(B(b,sigma))"
"B(b0 andi b1, sigma) = B(b0,sigma) and B(b1,sigma)"

```

"B(b0 ori b1, sigma) = B(b0,sigma) or B(b1,sigma)"

primrec

"C(skip) = id(loc->nat)"
 "C(x := a) =
 {io ∈ (loc->nat) × (loc->nat). snd(io) = fst(io)(x := A(a,fst(io)))}"
 "C(c0; c1) = C(c1) ∩ C(c0)"
 "C(if b then c0 else c1) =
 {io ∈ C(c0). B(b,fst(io)) = 1} ∪ {io ∈ C(c1). B(b,fst(io)) = 0}"
 "C(while b do c) = lfp((loc->nat) × (loc->nat), Γ(b,C(c)))"

2.2 Misc lemmas

lemma A_type [TC]: "[|a ∈ aexp; sigma ∈ loc->nat|] ==> A(a,sigma) ∈ nat"
 <proof>

lemma B_type [TC]: "[|b ∈ bexp; sigma ∈ loc->nat|] ==> B(b,sigma) ∈ bool"
 <proof>

lemma C_subset: "c ∈ com ==> C(c) ⊆ (loc->nat) × (loc->nat)"
 <proof>

lemma C_type_D [dest]:
 "[| <x,y> ∈ C(c); c ∈ com |] ==> x ∈ loc->nat & y ∈ loc->nat"
 <proof>

lemma C_type_fst [dest]: "[| x ∈ C(c); c ∈ com |] ==> fst(x) ∈ loc->nat"
 <proof>

lemma Gamma_bnd_mono:
 "cden ⊆ (loc->nat) × (loc->nat)
 ==> bnd_mono ((loc->nat) × (loc->nat), Γ(b,cden))"
 <proof>

end

3 Equivalence

theory Equiv imports Denotation Com **begin**

lemma aexp_iff [rule_format]:
 "[| a ∈ aexp; sigma: loc -> nat |]
 ==> ALL n. <a,sigma> -a-> n <-> A(a,sigma) = n"
 <proof>

declare aexp_iff [THEN iffD1, simp]
 aexp_iff [THEN iffD2, intro!]

```

inductive_cases [elim!]:
  "<true,sigma> -b-> x"
  "<false,sigma> -b-> x"
  "<ROp(f,a0,a1),sigma> -b-> x"
  "<noti(b),sigma> -b-> x"
  "<b0 andi b1,sigma> -b-> x"
  "<b0 ori b1,sigma> -b-> x"

lemma bexp_iff [rule_format]:
  "[| b ∈ bexp; sigma: loc -> nat |]
   ==> ALL w. <b,sigma> -b-> w <-> B(b,sigma) = w"
  ⟨proof⟩

declare bexp_iff [THEN iffD1, simp]
          bexp_iff [THEN iffD2, intro!]

lemma com1: "<c,sigma> -c-> sigma' ==> <sigma,sigma'> ∈ C(c)"
  ⟨proof⟩

declare B_type [intro!] A_type [intro!]
declare evalc.intros [intro]

lemma com2 [rule_format]: "c ∈ com ==> ∀x ∈ C(c). <c,fst(x)> -c-> snd(x)"
  ⟨proof⟩

3.1 Main theorem

theorem com_equivalence:
  "c ∈ com ==> C(c) = {io ∈ (loc->nat) × (loc->nat). <c,fst(io)> -c-> snd(io)}"
  ⟨proof⟩

end

```

References

- [1] Glynn Winskel. *The Formal Semantics of Programming Languages*. 1993.