The Isabelle/HOL Algebra Library

Clemens Ballarin
Florian Kammüller
Lawrence C Paulson

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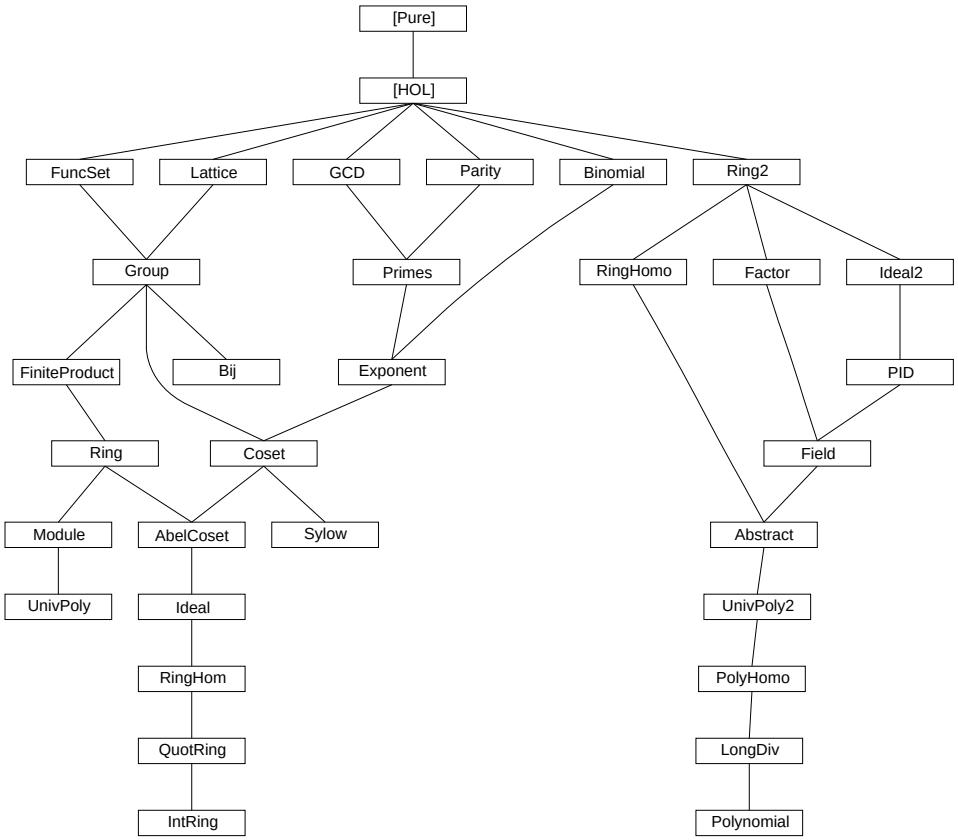
Contents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theory *Lattice* **imports** *Main* **begin**

1 Orders and Lattices

Object with a carrier set.

record *'a partial-object* =
  carrier :: *'a set*

1.1 Partial Orders

record *'a order* = *'a partial-object* +
  le :: [*'a*, *'a*] => *bool* (**infixl** $\sqsubseteq_1$ *50*)

locale *partial-order* =
  fixes *L* (**structure**)
  assumes *refl* [*intro*, *simp*]:
    $x \in$ *carrier L* ==> $x \sqsubseteq x$
  and *anti-sym* [*intro*]:
    [| $x \sqsubseteq y$; $y \sqsubseteq x$; $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x = y$
  and *trans* [*trans*]:
    [| $x \sqsubseteq y$; $y \sqsubseteq z$;
     $x \in$ *carrier L*; $y \in$ *carrier L*; $z \in$ *carrier L* |] ==> $x \sqsubseteq z$

constdefs (**structure** *L*)
  lless :: [-, *'a*, *'a*] => *bool* (**infixl** $\sqsubset_1$ *50*)
  $x \sqsubset y$ == $x \sqsubseteq y$ & $x \sim= y$

  — Upper and lower bounds of a set.
  Upper :: [-, *'a set*] => *'a set*
  Upper L A == $\{u.\ (ALL\ x.\ x \in A \cap carrier\ L \dashrightarrow x \sqsubseteq u)\} \cap$
    carrier L

  Lower :: [-, *'a set*] => *'a set*
  Lower L A == $\{l.\ (ALL\ x.\ x \in A \cap carrier\ L \dashrightarrow l \sqsubseteq x)\} \cap$
    carrier L

  — Least and greatest, as predicate.
  least :: [-, *'a*, *'a set*] => *bool*
  least L l A == $A \subseteq$ *carrier L* & $l \in A$ & $(ALL\ x : A.\ l \sqsubseteq x)$

  greatest :: [-, *'a*, *'a set*] => *bool*
  greatest L g A == $A \subseteq$ *carrier L* & $g \in A$ & $(ALL\ x : A.\ x \sqsubseteq g)$

  — Supremum and infimum
  sup :: [-, *'a set*] => *'a* ($\bigsqcup_1$- [*90*] *90*)
  $\bigsqcup A$ == *THE x. least L x* (*Upper L A*)

inf :: [-, *'a set*] => *'a* (⨅₁- [*90*] *90*)
⨅ *A* == *THE x*. *greatest L x* (*Lower L A*)

join :: [-, *'a*, *'a*] => *'a* (**infixl** ⊔₁ *65*)
x ⊔ *y* == *sup L* {*x*, *y*}

meet :: [-, *'a*, *'a*] => *'a* (**infixl** ⊓₁ *70*)
x ⊓ *y* == *inf L* {*x*, *y*}

1.1.1 Upper

lemma *Upper-closed* [*intro*, *simp*]:
Upper L A ⊆ *carrier L*
by (*unfold Upper-def*) *clarify*

lemma *UpperD* [*dest*]:
fixes *L* (**structure**)
shows [| *u* ∈ *Upper L A*; *x* ∈ *A*; *A* ⊆ *carrier L* |] ==> *x* ⊑ *u*
by (*unfold Upper-def*) *blast*

lemma *Upper-memI*:
fixes *L* (**structure**)
shows [| !! *y*. *y* ∈ *A* ==> *y* ⊑ *x*; *x* ∈ *carrier L* |] ==> *x* ∈ *Upper L A*
by (*unfold Upper-def*) *blast*

lemma *Upper-antimono*:
A ⊆ *B* ==> *Upper L B* ⊆ *Upper L A*
by (*unfold Upper-def*) *blast*

1.1.2 Lower

lemma *Lower-closed* [*intro*, *simp*]:
Lower L A ⊆ *carrier L*
by (*unfold Lower-def*) *clarify*

lemma *LowerD* [*dest*]:
fixes *L* (**structure**)
shows [| *l* ∈ *Lower L A*; *x* ∈ *A*; *A* ⊆ *carrier L* |] ==> *l* ⊑ *x*
by (*unfold Lower-def*) *blast*

lemma *Lower-memI*:
fixes *L* (**structure**)
shows [| !! *y*. *y* ∈ *A* ==> *x* ⊑ *y*; *x* ∈ *carrier L* |] ==> *x* ∈ *Lower L A*
by (*unfold Lower-def*) *blast*

lemma *Lower-antimono*:
A ⊆ *B* ==> *Lower L B* ⊆ *Lower L A*
by (*unfold Lower-def*) *blast*

1.1.3 least

lemma *least-carrier* [*intro*, *simp*]:
shows *least L l A* ==> *l* ∈ *carrier L*
by (*unfold least-def*) *fast*

lemma *least-mem*:
least L l A ==> *l* ∈ *A*
by (*unfold least-def*) *fast*

lemma (**in** *partial-order*) *least-unique*:
[| *least L x A*; *least L y A* |] ==> *x* = *y*
by (*unfold least-def*) *blast*

lemma *least-le*:
fixes *L* (**structure**)
shows [| *least L x A*; *a* ∈ *A* |] ==> *x* ⊑ *a*
by (*unfold least-def*) *fast*

lemma *least-UpperI*:
fixes *L* (**structure**)
assumes *above*: !! *x*. *x* ∈ *A* ==> *x* ⊑ *s*
and *below*: !! *y*. *y* ∈ *Upper L A* ==> *s* ⊑ *y*
and *L*: *A* ⊆ *carrier L* *s* ∈ *carrier L*
shows *least L s* (*Upper L A*)
proof −
have *Upper L A* ⊆ *carrier L* **by** *simp*
moreover from *above L* **have** *s* ∈ *Upper L A* **by** (*simp add*: *Upper-def*)
moreover from *below* **have** *ALL x* : *Upper L A*. *s* ⊑ *x* **by** *fast*
ultimately show *?thesis* **by** (*simp add*: *least-def*)
qed

1.1.4 greatest

lemma *greatest-carrier* [*intro*, *simp*]:
shows *greatest L l A* ==> *l* ∈ *carrier L*
by (*unfold greatest-def*) *fast*

lemma *greatest-mem*:
greatest L l A ==> *l* ∈ *A*
by (*unfold greatest-def*) *fast*

lemma (**in** *partial-order*) *greatest-unique*:
[| *greatest L x A*; *greatest L y A* |] ==> *x* = *y*
by (*unfold greatest-def*) *blast*

lemma *greatest-le*:
fixes *L* (**structure**)
shows [| *greatest L x A*; *a* ∈ *A* |] ==> *a* ⊑ *x*
by (*unfold greatest-def*) *fast*

lemma *greatest-LowerI*:
fixes *L* (**structure**)
assumes *below*: !! *x*. *x* ∈ *A* ==> *i* ⊑ *x*
and *above*: !! *y*. *y* ∈ *Lower L A* ==> *y* ⊑ *i*
and *L*: *A* ⊆ *carrier L* *i* ∈ *carrier L*
shows *greatest L i* (*Lower L A*)
proof −
have *Lower L A* ⊆ *carrier L* **by** *simp*
moreover from *below L* **have** *i* ∈ *Lower L A* **by** (*simp add*: *Lower-def*)
moreover from *above* **have** *ALL x* : *Lower L A*. *x* ⊑ *i* **by** *fast*
ultimately show *?thesis* **by** (*simp add*: *greatest-def*)
qed

1.2 Lattices

locale *lattice* = *partial-order* +
assumes *sup-of-two-exists*:
[| *x* ∈ *carrier L*; *y* ∈ *carrier L* |] ==> *EX s*. *least L s* (*Upper L* {*x*, *y*})
and *inf-of-two-exists*:
[| *x* ∈ *carrier L*; *y* ∈ *carrier L* |] ==> *EX s*. *greatest L s* (*Lower L* {*x*, *y*})

lemma *least-Upper-above*:
fixes *L* (**structure**)
shows [| *least L s* (*Upper L A*); *x* ∈ *A*; *A* ⊆ *carrier L* |] ==> *x* ⊑ *s*
by (*unfold least-def*) *blast*

lemma *greatest-Lower-above*:
fixes *L* (**structure**)
shows [| *greatest L i* (*Lower L A*); *x* ∈ *A*; *A* ⊆ *carrier L* |] ==> *i* ⊑ *x*
by (*unfold greatest-def*) *blast*

1.2.1 Supremum

lemma (**in** *lattice*) *joinI*:
[| !!*l*. *least L l* (*Upper L* {*x*, *y*}) ==> *P l*; *x* ∈ *carrier L*; *y* ∈ *carrier L* |]
==> *P* (*x* ⊔ *y*)
proof (*unfold join-def sup-def*)
assume *L*: *x* ∈ *carrier L* *y* ∈ *carrier L*
and *P*: !!*l*. *least L l* (*Upper L* {*x*, *y*}) ==> *P l*
with *sup-of-two-exists* **obtain** *s* **where** *least L s* (*Upper L* {*x*, *y*}) **by** *fast*
with *L* **show** *P* (*THE l*. *least L l* (*Upper L* {*x*, *y*}))
by (*fast intro*: *theI2 least-unique P*)
qed

lemma (**in** *lattice*) *join-closed* [*simp*]:
[| *x* ∈ *carrier L*; *y* ∈ *carrier L* |] ==> *x* ⊔ *y* ∈ *carrier L*
by (*rule joinI*) (*rule least-carrier*)

lemma (**in** *partial-order*) *sup-of-singletonI*:

```
  x ∈ carrier L ==> least L x (Upper L {x})
  by (rule least-UpperI) fast+

lemma (in partial-order) sup-of-singleton [simp]:
  x ∈ carrier L ==> ⨆{x} = x
  by (unfold sup-def) (blast intro: least-unique least-UpperI sup-of-singletonI)
```

Condition on A: supremum exists.

```
lemma (in lattice) sup-insertI:
  [| !!s. least L s (Upper L (insert x A)) ==> P s;
  least L a (Upper L A); x ∈ carrier L; A ⊆ carrier L |]
  ==> P (⨆(insert x A))
proof (unfold sup-def)
  assume L: x ∈ carrier L  A ⊆ carrier L
    and P: !!l. least L l (Upper L (insert x A)) ==> P l
    and least-a: least L a (Upper L A)
  from L least-a have La: a ∈ carrier L by simp
  from L sup-of-two-exists least-a
  obtain s where least-s: least L s (Upper L {a, x}) by blast
  show P (THE l. least L l (Upper L (insert x A)))
  proof (rule theI2)
    show least L s (Upper L (insert x A))
    proof (rule least-UpperI)
      fix z
      assume z ∈ insert x A
      then show z ⊑ s
      proof
        assume z = x then show ?thesis
          by (simp add: least-Upper-above [OF least-s] L La)
      next
        assume z ∈ A
        with L least-s least-a show ?thesis
          by (rule-tac trans [where y = a]) (auto dest: least-Upper-above)
      qed
    next
      fix y
      assume y: y ∈ Upper L (insert x A)
      show s ⊑ y
      proof (rule least-le [OF least-s], rule Upper-memI)
        fix z
        assume z: z ∈ {a, x}
        then show z ⊑ y
        proof
          have y': y ∈ Upper L A
            apply (rule subsetD [where A = Upper L (insert x A)])
             apply (rule Upper-antimono)
             apply blast
            apply (rule y)
            done
```

```
      assume z = a
      with y' least-a show ?thesis by (fast dest: least-le)
    next
      assume z ∈ {x}
      with y L show ?thesis by blast
    qed
  qed (rule Upper-closed [THEN subsetD, OF y])
 next
  from L show insert x A ⊆ carrier L by simp
  from least-s show s ∈ carrier L by simp
 qed
next
 fix l
 assume least-l: least L l (Upper L (insert x A))
 show l = s
 proof (rule least-unique)
  show least L s (Upper L (insert x A))
  proof (rule least-UpperI)
   fix z
   assume z ∈ insert x A
   then show z ⊑ s
   proof
    assume z = x then show ?thesis
     by (simp add: least-Upper-above [OF least-s] L La)
   next
    assume z ∈ A
    with L least-s least-a show ?thesis
     by (rule-tac trans [where y = a]) (auto dest: least-Upper-above)
   qed
  next
   fix y
   assume y: y ∈ Upper L (insert x A)
   show s ⊑ y
   proof (rule least-le [OF least-s], rule Upper-memI)
    fix z
    assume z: z ∈ {a, x}
    then show z ⊑ y
    proof
     have y': y ∈ Upper L A
      apply (rule subsetD [where A = Upper L (insert x A)])
      apply (rule Upper-antimono)
       apply blast
      apply (rule y)
      done
     assume z = a
     with y' least-a show ?thesis by (fast dest: least-le)
    next
     assume z ∈ {x}
     with y L show ?thesis by blast
```

```
        qed
      qed (rule Upper-closed [THEN subsetD, OF y])
    next
      from L show insert x A ⊆ carrier L by simp
      from least-s show s ∈ carrier L by simp
    qed
  qed (rule least-l)
 qed (rule P)
qed

lemma (in lattice) finite-sup-least:
  [| finite A; A ⊆ carrier L; A ~= {} |] ==> least L (⨆ A) (Upper L A)
proof (induct set: finite)
 case empty
 then show ?case by simp
next
 case (insert x A)
 show ?case
 proof (cases A = {})
   case True
   with insert show ?thesis by (simp add: sup-of-singletonI)
 next
   case False
   with insert have least L (⨆ A) (Upper L A) by simp
   with - show ?thesis
     by (rule sup-insertI) (simp-all add: insert [simplified])
 qed
qed

lemma (in lattice) finite-sup-insertI:
 assumes P: !!l. least L l (Upper L (insert x A)) ==> P l
   and xA: finite A  x ∈ carrier L  A ⊆ carrier L
 shows P (⨆ (insert x A))
proof (cases A = {})
 case True with P and xA show ?thesis
   by (simp add: sup-of-singletonI)
next
 case False with P and xA show ?thesis
   by (simp add: sup-insertI finite-sup-least)
qed

lemma (in lattice) finite-sup-closed:
 [| finite A; A ⊆ carrier L; A ~= {} |] ==> ⨆ A ∈ carrier L
proof (induct set: finite)
 case empty then show ?case by simp
next
 case insert then show ?case
   by - (rule finite-sup-insertI, simp-all)
qed
```

lemma (**in** *lattice*) *join-left*:
[| $x \in$ *carrier* L; $y \in$ *carrier* L |] ==> $x \sqsubseteq x \sqcup y$
by (*rule joinI* [*folded join-def*]) (*blast dest*: *least-mem*)

lemma (**in** *lattice*) *join-right*:
[| $x \in$ *carrier* L; $y \in$ *carrier* L |] ==> $y \sqsubseteq x \sqcup y$
by (*rule joinI* [*folded join-def*]) (*blast dest*: *least-mem*)

lemma (**in** *lattice*) *sup-of-two-least*:
[| $x \in$ *carrier* L; $y \in$ *carrier* L |] ==> *least* L ($\bigsqcup\{x, y\}$) (*Upper* L $\{x, y\}$)
proof (*unfold sup-def*)
assume L: $x \in$ *carrier* L $y \in$ *carrier* L
with *sup-of-two-exists* **obtain** s **where** *least* L s (*Upper* L $\{x, y\}$) **by** *fast*
with L **show** *least* L (*THE xa*. *least* L *xa* (*Upper* L $\{x, y\}$)) (*Upper* L $\{x, y\}$)
by (*fast intro*: *theI2 least-unique*)
qed

lemma (**in** *lattice*) *join-le*:
assumes *sub*: $x \sqsubseteq z$ $y \sqsubseteq z$
and x: $x \in$ *carrier* L **and** y: $y \in$ *carrier* L **and** z: $z \in$ *carrier* L
shows $x \sqcup y \sqsubseteq z$
proof (*rule joinI* [*OF - x y*])
fix s
assume *least* L s (*Upper* L $\{x, y\}$)
with *sub* z **show** $s \sqsubseteq z$ **by** (*fast elim*: *least-le intro*: *Upper-memI*)
qed

lemma (**in** *lattice*) *join-assoc-lemma*:
assumes L: $x \in$ *carrier* L $y \in$ *carrier* L $z \in$ *carrier* L
shows $x \sqcup (y \sqcup z) = \bigsqcup\{x, y, z\}$
proof (*rule finite-sup-insertI*)
— The textbook argument in Jacobson I, p 457
fix s
assume *sup*: *least* L s (*Upper* L $\{x, y, z\}$)
show $x \sqcup (y \sqcup z) = s$
proof (*rule anti-sym*)
from *sup* L **show** $x \sqcup (y \sqcup z) \sqsubseteq s$
by (*fastsimp intro*!: *join-le elim*: *least-Upper-above*)
next
from *sup* L **show** $s \sqsubseteq x \sqcup (y \sqcup z)$
by (*erule-tac least-le*)
(*blast intro*!: *Upper-memI intro*: *trans join-left join-right join-closed*)
qed (*simp-all add*: L *least-carrier* [*OF sup*])
qed (*simp-all add*: L)

lemma *join-comm*:
fixes L (**structure**)
shows $x \sqcup y = y \sqcup x$

by (*unfold join-def*) (*simp add*: *insert-commute*)

lemma (**in** *lattice*) *join-assoc*:
assumes *L*: $x \in$ *carrier L* $y \in$ *carrier L* $z \in$ *carrier L*
shows $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
proof −
have $(x \sqcup y) \sqcup z = z \sqcup (x \sqcup y)$ **by** (*simp only*: *join-comm*)
also from *L* **have** ... $= \bigsqcup\{z, x, y\}$ **by** (*simp add*: *join-assoc-lemma*)
also from *L* **have** ... $= \bigsqcup\{x, y, z\}$ **by** (*simp add*: *insert-commute*)
also from *L* **have** ... $= x \sqcup (y \sqcup z)$ **by** (*simp add*: *join-assoc-lemma*)
finally show *?thesis* **.**
qed

1.2.2 Infimum

lemma (**in** *lattice*) *meetI*:
[| !!*i*. *greatest L i* (*Lower L* $\{x, y\}$) ==> *P i*;
$x \in$ *carrier L*; $y \in$ *carrier L* |]
==> *P* $(x \sqcap y)$
proof (*unfold meet-def inf-def*)
assume *L*: $x \in$ *carrier L* $y \in$ *carrier L*
and *P*: !!*g*. *greatest L g* (*Lower L* $\{x, y\}$) ==> *P g*
with *inf-of-two-exists* **obtain** *i* **where** *greatest L i* (*Lower L* $\{x, y\}$) **by** *fast*
with *L* **show** *P* (*THE g*. *greatest L g* (*Lower L* $\{x, y\}$))
by (*fast intro*: *theI2 greatest-unique P*)
qed

lemma (**in** *lattice*) *meet-closed* [*simp*]:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqcap y \in$ *carrier L*
by (*rule meetI*) (*rule greatest-carrier*)

lemma (**in** *partial-order*) *inf-of-singletonI*:
$x \in$ *carrier L* ==> *greatest L x* (*Lower L* $\{x\}$)
by (*rule greatest-LowerI*) *fast*+

lemma (**in** *partial-order*) *inf-of-singleton* [*simp*]:
$x \in$ *carrier L* ==> $\bigsqcap\ \{x\} = x$
by (*unfold inf-def*) (*blast intro*: *greatest-unique greatest-LowerI inf-of-singletonI*)

Condition on A: infimum exists.

lemma (**in** *lattice*) *inf-insertI*:
[| !!*i*. *greatest L i* (*Lower L* (*insert x A*)) ==> *P i*;
greatest L a (*Lower L A*); $x \in$ *carrier L*; $A \subseteq$ *carrier L* |]
==> *P* $(\bigsqcap(\mathit{insert}\ x\ A))$
proof (*unfold inf-def*)
assume *L*: $x \in$ *carrier L* $A \subseteq$ *carrier L*
and *P*: !!*g*. *greatest L g* (*Lower L* (*insert x A*)) ==> *P g*
and *greatest-a*: *greatest L a* (*Lower L A*)
from *L greatest-a* **have** *La*: $a \in$ *carrier L* **by** *simp*

```
from L inf-of-two-exists greatest-a
obtain i where greatest-i: greatest L i (Lower L {a, x}) by blast
show P (THE g. greatest L g (Lower L (insert x A)))
proof (rule theI2)
  show greatest L i (Lower L (insert x A))
  proof (rule greatest-LowerI)
    fix z
    assume z ∈ insert x A
    then show i ⊑ z
    proof
      assume z = x then show ?thesis
        by (simp add: greatest-Lower-above [OF greatest-i] L La)
    next
      assume z ∈ A
      with L greatest-i greatest-a show ?thesis
        by (rule-tac trans [where y = a]) (auto dest: greatest-Lower-above)
    qed
  next
    fix y
    assume y: y ∈ Lower L (insert x A)
    show y ⊑ i
    proof (rule greatest-le [OF greatest-i], rule Lower-memI)
      fix z
      assume z: z ∈ {a, x}
      then show y ⊑ z
      proof
        have y': y ∈ Lower L A
          apply (rule subsetD [where A = Lower L (insert x A)])
          apply (rule Lower-antimono)
           apply blast
          apply (rule y)
          done
        assume z = a
        with y' greatest-a show ?thesis by (fast dest: greatest-le)
      next
        assume z ∈ {x}
        with y L show ?thesis by blast
      qed
    qed (rule Lower-closed [THEN subsetD, OF y])
  next
    from L show insert x A ⊆ carrier L by simp
    from greatest-i show i ∈ carrier L by simp
  qed
next
  fix g
  assume greatest-g: greatest L g (Lower L (insert x A))
  show g = i
  proof (rule greatest-unique)
    show greatest L i (Lower L (insert x A))
```

```
    proof (rule greatest-LowerI)
      fix z
      assume z ∈ insert x A
      then show i ⊑ z
      proof
        assume z = x then show ?thesis
          by (simp add: greatest-Lower-above [OF greatest-i] L La)
      next
        assume z ∈ A
        with L greatest-i greatest-a show ?thesis
          by (rule-tac trans [where y = a]) (auto dest: greatest-Lower-above)
      qed
    next
      fix y
      assume y: y ∈ Lower L (insert x A)
      show y ⊑ i
      proof (rule greatest-le [OF greatest-i], rule Lower-memI)
        fix z
        assume z: z ∈ {a, x}
        then show y ⊑ z
        proof
          have y': y ∈ Lower L A
            apply (rule subsetD [where A = Lower L (insert x A)])
            apply (rule Lower-antimono)
             apply blast
            apply (rule y)
            done
          assume z = a
          with y' greatest-a show ?thesis by (fast dest: greatest-le)
        next
          assume z ∈ {x}
          with y L show ?thesis by blast
        qed
      qed (rule Lower-closed [THEN subsetD, OF y])
    next
      from L show insert x A ⊆ carrier L by simp
      from greatest-i show i ∈ carrier L by simp
    qed
  qed (rule greatest-g)
 qed (rule P)
qed

lemma (in lattice) finite-inf-greatest:
  [| finite A; A ⊆ carrier L; A ~= {} |] ==> greatest L (⨅ A) (Lower L A)
proof (induct set: finite)
  case empty then show ?case by simp
next
  case (insert x A)
  show ?case
```

```
  proof (cases A = {})
    case True
    with insert show ?thesis by (simp add: inf-of-singletonI)
  next
    case False
    from insert show ?thesis
    proof (rule-tac inf-insertI)
      from False insert show greatest L (⨅ A) (Lower L A) by simp
    qed simp-all
  qed
qed

lemma (in lattice) finite-inf-insertI:
  assumes P: !!i. greatest L i (Lower L (insert x A)) ==> P i
    and xA: finite A  x ∈ carrier L  A ⊆ carrier L
  shows P (⨅ (insert x A))
proof (cases A = {})
  case True with P and xA show ?thesis
    by (simp add: inf-of-singletonI)
next
  case False with P and xA show ?thesis
    by (simp add: inf-insertI finite-inf-greatest)
qed

lemma (in lattice) finite-inf-closed:
  [| finite A; A ⊆ carrier L; A ~= {} |] ==> ⨅ A ∈ carrier L
proof (induct set: finite)
  case empty then show ?case by simp
next
  case insert then show ?case
    by (rule-tac finite-inf-insertI) (simp-all)
qed

lemma (in lattice) meet-left:
  [| x ∈ carrier L; y ∈ carrier L |] ==> x ⊓ y ⊑ x
  by (rule meetI [folded meet-def]) (blast dest: greatest-mem)

lemma (in lattice) meet-right:
  [| x ∈ carrier L; y ∈ carrier L |] ==> x ⊓ y ⊑ y
  by (rule meetI [folded meet-def]) (blast dest: greatest-mem)

lemma (in lattice) inf-of-two-greatest:
  [| x ∈ carrier L; y ∈ carrier L |] ==>
  greatest L (⨅ {x, y}) (Lower L {x, y})
proof (unfold inf-def)
  assume L: x ∈ carrier L  y ∈ carrier L
  with inf-of-two-exists obtain s where greatest L s (Lower L {x, y}) by fast
  with L
  show greatest L (THE xa. greatest L xa (Lower L {x, y})) (Lower L {x, y})
```

by (*fast intro*: *theI2 greatest-unique*)
qed

lemma (**in** *lattice*) *meet-le*:
assumes *sub*: $z \sqsubseteq x \;\; z \sqsubseteq y$
and x: $x \in$ *carrier* L **and** y: $y \in$ *carrier* L **and** z: $z \in$ *carrier* L
shows $z \sqsubseteq x \sqcap y$
proof (*rule meetI* [*OF* - *x y*])
fix *i*
assume *greatest L i* (*Lower L* $\{x, y\}$)
with *sub z* **show** $z \sqsubseteq i$ **by** (*fast elim*: *greatest-le intro*: *Lower-memI*)
qed

lemma (**in** *lattice*) *meet-assoc-lemma*:
assumes *L*: $x \in$ *carrier* $L \;\; y \in$ *carrier* $L \;\; z \in$ *carrier* L
shows $x \sqcap (y \sqcap z) = \bigsqcap \{x, y, z\}$
proof (*rule finite-inf-insertI*)

The textbook argument in Jacobson I, p 457

fix *i*
assume *inf*: *greatest L i* (*Lower L* $\{x, y, z\}$)
show $x \sqcap (y \sqcap z) = i$
proof (*rule anti-sym*)
from *inf L* **show** $i \sqsubseteq x \sqcap (y \sqcap z)$
by (*fastsimp intro*!: *meet-le elim*: *greatest-Lower-above*)
next
from *inf L* **show** $x \sqcap (y \sqcap z) \sqsubseteq i$
by (*erule-tac greatest-le*)
(*blast intro*!: *Lower-memI intro*: *trans meet-left meet-right meet-closed*)
qed (*simp-all add*: *L greatest-carrier* [*OF inf*])
qed (*simp-all add*: *L*)

lemma *meet-comm*:
fixes *L* (**structure**)
shows $x \sqcap y = y \sqcap x$
by (*unfold meet-def*) (*simp add*: *insert-commute*)

lemma (**in** *lattice*) *meet-assoc*:
assumes *L*: $x \in$ *carrier* $L \;\; y \in$ *carrier* $L \;\; z \in$ *carrier* L
shows $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
proof −
have $(x \sqcap y) \sqcap z = z \sqcap (x \sqcap y)$ **by** (*simp only*: *meet-comm*)
also from *L* **have** ... $= \bigsqcap \{z, x, y\}$ **by** (*simp add*: *meet-assoc-lemma*)
also from *L* **have** ... $= \bigsqcap \{x, y, z\}$ **by** (*simp add*: *insert-commute*)
also from *L* **have** ... $= x \sqcap (y \sqcap z)$ **by** (*simp add*: *meet-assoc-lemma*)
finally show *?thesis* **.**
qed

1.3 Total Orders

locale *total-order* = *partial-order* +
  assumes *total*: [| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqsubseteq y \mid y \sqsubseteq x$

Introduction rule: the usual definition of total order

lemma (**in** *partial-order*) *total-orderI*:
  assumes *total*: !!$x\ y$. [| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqsubseteq y \mid y \sqsubseteq x$
  shows *total-order L*
  by *unfold-locales* (*rule total*)

Total orders are lattices.

interpretation *total-order* < *lattice*
proof *unfold-locales*
  fix $x\ y$
  assume *L*: $x \in$ *carrier L* $y \in$ *carrier L*
  show *EX s. least L s* (*Upper L* {x, y})
  proof −
    note *total L*
    moreover
    {
      assume $x \sqsubseteq y$
      with *L* **have** *least L y* (*Upper L* {x, y})
        by (*rule-tac least-UpperI*) *auto*
    }
    moreover
    {
      assume $y \sqsubseteq x$
      with *L* **have** *least L x* (*Upper L* {x, y})
        by (*rule-tac least-UpperI*) *auto*
    }
    ultimately show *?thesis* **by** *blast*
  qed
next
  fix $x\ y$
  assume *L*: $x \in$ *carrier L* $y \in$ *carrier L*
  show *EX i. greatest L i* (*Lower L* {x, y})
  proof −
    note *total L*
    moreover
    {
      assume $y \sqsubseteq x$
      with *L* **have** *greatest L y* (*Lower L* {x, y})
        by (*rule-tac greatest-LowerI*) *auto*
    }
    moreover
    {
      assume $x \sqsubseteq y$
      with *L* **have** *greatest L x* (*Lower L* {x, y})
        by (*rule-tac greatest-LowerI*) *auto*

```
    }
    ultimately show ?thesis by blast
  qed
qed
```

1.4 Complete lattices

```
locale complete-lattice = lattice +
  assumes sup-exists:
    [| A ⊆ carrier L |] ==> EX s. least L s (Upper L A)
    and inf-exists:
    [| A ⊆ carrier L |] ==> EX i. greatest L i (Lower L A)
```

Introduction rule: the usual definition of complete lattice

```
lemma (in partial-order) complete-latticeI:
  assumes sup-exists:
    !!A. [| A ⊆ carrier L |] ==> EX s. least L s (Upper L A)
    and inf-exists:
    !!A. [| A ⊆ carrier L |] ==> EX i. greatest L i (Lower L A)
  shows complete-lattice L
proof intro-locales
  show lattice-axioms L
    by (rule lattice-axioms.intro) (blast intro: sup-exists inf-exists)+
qed (rule complete-lattice-axioms.intro sup-exists inf-exists | assumption)+

constdefs (structure L)
  top :: - => 'a (⊤₁)
  ⊤ == sup L (carrier L)

  bottom :: - => 'a (⊥₁)
  ⊥ == inf L (carrier L)


lemma (in complete-lattice) supI:
  [| !!l. least L l (Upper L A) ==> P l; A ⊆ carrier L |]
  ==> P (⨆ A)
proof (unfold sup-def)
  assume L: A ⊆ carrier L
    and P: !!l. least L l (Upper L A) ==> P l
  with sup-exists obtain s where least L s (Upper L A) by blast
  with L show P (THE l. least L l (Upper L A))
  by (fast intro: theI2 least-unique P)
qed

lemma (in complete-lattice) sup-closed [simp]:
  A ⊆ carrier L ==> ⨆ A ∈ carrier L
  by (rule supI) simp-all

lemma (in complete-lattice) top-closed [simp, intro]:
```

```
  ⊤ ∈ carrier L
  by (unfold top-def) simp

lemma (in complete-lattice) infI:
  [| !!i. greatest L i (Lower L A) ==> P i; A ⊆ carrier L |]
  ==> P (⨅ A)
proof (unfold inf-def)
  assume L: A ⊆ carrier L
    and P: !!l. greatest L l (Lower L A) ==> P l
  with inf-exists obtain s where greatest L s (Lower L A) by blast
  with L show P (THE l. greatest L l (Lower L A))
  by (fast intro: theI2 greatest-unique P)
qed

lemma (in complete-lattice) inf-closed [simp]:
  A ⊆ carrier L ==> ⨅ A ∈ carrier L
  by (rule infI) simp-all

lemma (in complete-lattice) bottom-closed [simp, intro]:
  ⊥ ∈ carrier L
  by (unfold bottom-def) simp
```

Jacobson: Theorem 8.1

```
lemma Lower-empty [simp]:
  Lower L {} = carrier L
  by (unfold Lower-def) simp

lemma Upper-empty [simp]:
  Upper L {} = carrier L
  by (unfold Upper-def) simp

theorem (in partial-order) complete-lattice-criterion1:
  assumes top-exists: EX g. greatest L g (carrier L)
    and inf-exists:
      !!A. [| A ⊆ carrier L; A ~= {} |] ==> EX i. greatest L i (Lower L A)
  shows complete-lattice L
proof (rule complete-latticeI)
  from top-exists obtain top where top: greatest L top (carrier L) ..
  fix A
  assume L: A ⊆ carrier L
  let ?B = Upper L A
  from L top have top ∈ ?B by (fast intro!: Upper-memI intro: greatest-le)
  then have B-non-empty: ?B ~= {} by fast
  have B-L: ?B ⊆ carrier L by simp
  from inf-exists [OF B-L B-non-empty]
  obtain b where b-inf-B: greatest L b (Lower L ?B) ..
  have least L b (Upper L A)
apply (rule least-UpperI)
   apply (rule greatest-le [where A = Lower L ?B])
```

```
    apply (rule b-inf-B)
   apply (rule Lower-memI)
    apply (erule UpperD)
     apply assumption
    apply (rule L)
   apply (fast intro: L [THEN subsetD])
  apply (erule greatest-Lower-above [OF b-inf-B])
  apply simp
 apply (rule L)
apply (rule greatest-carrier [OF b-inf-B])
done
  then show EX s. least L s (Upper L A) ..
next
  fix A
  assume L: A ⊆ carrier L
  show EX i. greatest L i (Lower L A)
  proof (cases A = {})
    case True then show ?thesis
      by (simp add: top-exists)
  next
    case False with L show ?thesis
      by (rule inf-exists)
  qed
qed
```

1.5 Examples

1.5.1 Powerset of a Set is a Complete Lattice

```
theorem powerset-is-complete-lattice:
  complete-lattice (| carrier = Pow A, le = op ⊆ |)
  (is complete-lattice ?L)
proof (rule partial-order.complete-latticeI)
  show partial-order ?L
    by (rule partial-order.intro) auto
next
  fix B
  assume B: B ⊆ carrier ?L
  show EX s. least ?L s (Upper ?L B)
  proof
    from B show least ?L (⋃ B) (Upper ?L B)
      by (fastsimp intro!: least-UpperI simp: Upper-def)
  qed
next
  fix B
  assume B: B ⊆ carrier ?L
  show EX i. greatest ?L i (Lower ?L B)
  proof
    from B show greatest ?L (⋂ B ∩ A) (Lower ?L B)
```

$\bigcap B$ is not the infimum of B: $\bigcap \{\} = UNIV$ which is in general bigger than A!

by (*fastsimp intro*!: *greatest-LowerI simp*: *Lower-def*)
qed
qed

An other example, that of the lattice of subgroups of a group, can be found in Group theory (Section 2.7).

end

theory *Group* **imports** *FuncSet Lattice* **begin**

2 Monoids and Groups

2.1 Definitions

Definitions follow [2].

record $'a$ *monoid* = $'a$ *partial-object* +
mult :: $['a, 'a] \Rightarrow 'a$ (**infixl** $\otimes_1$ *70*)
one :: $'a$ ($\mathbf{1}_1$)

constdefs (**structure** *G*)
m-inv :: ($'a$, $'b$) *monoid-scheme* => $'a$ => $'a$ (*inv*$_1$ - [*81*] *80*)
inv x == (*THE y*. $y \in$ *carrier G* & $x \otimes y = \mathbf{1}$ & $y \otimes x = \mathbf{1}$)

Units :: - => $'a$ *set*
— The set of invertible elements
Units G == $\{y.\ y \in$ *carrier G* & $(\exists x \in$ *carrier G*. $x \otimes y = \mathbf{1}$ & $y \otimes x = \mathbf{1})\}$

consts
pow :: [($'a$, $'m$) *monoid-scheme*, $'a$, $'b$::*number*] => $'a$ (**infixr** $'(\hat{}')_1$ *75*)

defs (**overloaded**)
nat-pow-def: *pow G a n* == *nat-rec* $\mathbf{1}_G$ (%*u b*. $b \otimes_G a$) *n*
int-pow-def: *pow G a z* ==
let p = *nat-rec* $\mathbf{1}_G$ (%*u b*. $b \otimes_G a$)
in if neg z then inv$_G$ (*p* (*nat* $(-z)$)) *else p* (*nat z*)

locale *monoid* =
fixes *G* (**structure**)
assumes *m-closed* [*intro*, *simp*]:
$[\![x \in$ *carrier G*; $y \in$ *carrier G*$]\!] \Longrightarrow x \otimes y \in$ *carrier G*
and *m-assoc*:
$[\![x \in$ *carrier G*; $y \in$ *carrier G*; $z \in$ *carrier G*$]\!]$
$\Longrightarrow (x \otimes y) \otimes z = x \otimes (y \otimes z)$
and *one-closed* [*intro*, *simp*]: $\mathbf{1} \in$ *carrier G*
and *l-one* [*simp*]: $x \in$ *carrier G* $\Longrightarrow \mathbf{1} \otimes x = x$

and *r-one* [*simp*]: $x \in \mathit{carrier}\ G \Longrightarrow x \otimes \mathbf{1} = x$

lemma *monoidI*:
fixes G (**structure**)
assumes *m-closed*:
!!$x\ y$. [| $x \in \mathit{carrier}\ G$; $y \in \mathit{carrier}\ G$ |] ==> $x \otimes y \in \mathit{carrier}\ G$
and *one-closed*: $\mathbf{1} \in \mathit{carrier}\ G$
and *m-assoc*:
!!$x\ y\ z$. [| $x \in \mathit{carrier}\ G$; $y \in \mathit{carrier}\ G$; $z \in \mathit{carrier}\ G$ |] ==>
$(x \otimes y) \otimes z = x \otimes (y \otimes z)$
and *l-one*: !!x. $x \in \mathit{carrier}\ G$ ==> $\mathbf{1} \otimes x = x$
and *r-one*: !!x. $x \in \mathit{carrier}\ G$ ==> $x \otimes \mathbf{1} = x$
shows *monoid* G
by (*fast intro*!: *monoid.intro intro*: *prems*)

lemma (**in** *monoid*) *Units-closed* [*dest*]:
$x \in \mathit{Units}\ G$ ==> $x \in \mathit{carrier}\ G$
by (*unfold Units-def*) *fast*

lemma (**in** *monoid*) *inv-unique*:
assumes *eq*: $y \otimes x = \mathbf{1}$ $x \otimes y' = \mathbf{1}$
and G: $x \in \mathit{carrier}\ G$ $y \in \mathit{carrier}\ G$ $y' \in \mathit{carrier}\ G$
shows $y = y'$
proof −
from G *eq* **have** $y = y \otimes (x \otimes y')$ **by** *simp*
also from G **have** ... $= (y \otimes x) \otimes y'$ **by** (*simp add*: *m-assoc*)
also from G *eq* **have** ... $= y'$ **by** *simp*
finally show *?thesis* **.**
qed

lemma (**in** *monoid*) *Units-one-closed* [*intro*, *simp*]:
$\mathbf{1} \in \mathit{Units}\ G$
by (*unfold Units-def*) *auto*

lemma (**in** *monoid*) *Units-inv-closed* [*intro*, *simp*]:
$x \in \mathit{Units}\ G$ ==> $\mathit{inv}\ x \in \mathit{carrier}\ G$
apply (*unfold Units-def m-inv-def*, *auto*)
apply (*rule theI2*, *fast*)
apply (*fast intro*: *inv-unique*, *fast*)
done

lemma (**in** *monoid*) *Units-l-inv-ex*:
$x \in \mathit{Units}\ G$ ==> $\exists y \in \mathit{carrier}\ G.\ y \otimes x = \mathbf{1}$
by (*unfold Units-def*) *auto*

lemma (**in** *monoid*) *Units-r-inv-ex*:
$x \in \mathit{Units}\ G$ ==> $\exists y \in \mathit{carrier}\ G.\ x \otimes y = \mathbf{1}$
by (*unfold Units-def*) *auto*

lemma (**in** *monoid*) *Units-l-inv*:
  $x \in$ *Units* G ==> *inv* $x \otimes x = \mathbf{1}$
  apply (*unfold Units-def m-inv-def*, *auto*)
  apply (*rule theI2*, *fast*)
   apply (*fast intro*: *inv-unique*, *fast*)
  done

lemma (**in** *monoid*) *Units-r-inv*:
  $x \in$ *Units* G ==> $x \otimes$ *inv* $x = \mathbf{1}$
  apply (*unfold Units-def m-inv-def*, *auto*)
  apply (*rule theI2*, *fast*)
   apply (*fast intro*: *inv-unique*, *fast*)
  done

lemma (**in** *monoid*) *Units-inv-Units* [*intro*, *simp*]:
  $x \in$ *Units* G ==> *inv* $x \in$ *Units* G
proof −
  assume x: $x \in$ *Units* G
  show *inv* $x \in$ *Units* G
    by (*auto simp add*: *Units-def*
      intro: *Units-l-inv Units-r-inv x Units-closed* [*OF* x])
qed

lemma (**in** *monoid*) *Units-l-cancel* [*simp*]:
  [| $x \in$ *Units* G; $y \in$ *carrier* G; $z \in$ *carrier* G |] ==>
   $(x \otimes y = x \otimes z) = (y = z)$
proof
  assume *eq*: $x \otimes y = x \otimes z$
    and G: $x \in$ *Units* G $y \in$ *carrier* G $z \in$ *carrier* G
  then have (*inv* $x \otimes x$) $\otimes y =$ (*inv* $x \otimes x$) $\otimes z$
    by (*simp add*: *m-assoc Units-closed*)
  with G **show** $y = z$ **by** (*simp add*: *Units-l-inv*)
next
  assume *eq*: $y = z$
    and G: $x \in$ *Units* G $y \in$ *carrier* G $z \in$ *carrier* G
  then show $x \otimes y = x \otimes z$ **by** *simp*
qed

lemma (**in** *monoid*) *Units-inv-inv* [*simp*]:
  $x \in$ *Units* G ==> *inv* (*inv* x) = x
proof −
  assume x: $x \in$ *Units* G
  then have *inv* $x \otimes$ *inv* (*inv* x) = *inv* $x \otimes x$
    by (*simp add*: *Units-l-inv Units-r-inv*)
  with x **show** *?thesis* **by** (*simp add*: *Units-closed*)
qed

lemma (**in** *monoid*) *inv-inj-on-Units*:
  inj-on (*m-inv* G) (*Units* G)

proof (*rule inj-onI*)
fix *x y*
assume *G*: *x* ∈ *Units G* *y* ∈ *Units G* **and** *eq*: *inv x* = *inv y*
then have *inv* (*inv x*) = *inv* (*inv y*) **by** *simp*
with *G* **show** *x* = *y* **by** *simp*
qed

lemma (**in** *monoid*) *Units-inv-comm*:
assumes *inv*: *x* ⊗ *y* = **1**
and *G*: *x* ∈ *Units G* *y* ∈ *Units G*
shows *y* ⊗ *x* = **1**
proof −
from *G* **have** *x* ⊗ *y* ⊗ *x* = *x* ⊗ **1** **by** (*auto simp add*: *inv Units-closed*)
with *G* **show** *?thesis* **by** (*simp del*: *r-one add*: *m-assoc Units-closed*)
qed

Power

lemma (**in** *monoid*) *nat-pow-closed* [*intro*, *simp*]:
x ∈ *carrier G* ==> *x* (^) (*n*::*nat*) ∈ *carrier G*
by (*induct n*) (*simp-all add*: *nat-pow-def*)

lemma (**in** *monoid*) *nat-pow-0* [*simp*]:
x (^) (*0*::*nat*) = **1**
by (*simp add*: *nat-pow-def*)

lemma (**in** *monoid*) *nat-pow-Suc* [*simp*]:
x (^) (*Suc n*) = *x* (^) *n* ⊗ *x*
by (*simp add*: *nat-pow-def*)

lemma (**in** *monoid*) *nat-pow-one* [*simp*]:
1 (^) (*n*::*nat*) = **1**
by (*induct n*) *simp-all*

lemma (**in** *monoid*) *nat-pow-mult*:
x ∈ *carrier G* ==> *x* (^) (*n*::*nat*) ⊗ *x* (^) *m* = *x* (^) (*n* + *m*)
by (*induct m*) (*simp-all add*: *m-assoc* [*THEN sym*])

lemma (**in** *monoid*) *nat-pow-pow*:
x ∈ *carrier G* ==> (*x* (^) *n*) (^) *m* = *x* (^) (*n* ∗ *m*::*nat*)
by (*induct m*) (*simp*, *simp add*: *nat-pow-mult add-commute*)

A group is a monoid all of whose elements are invertible.

locale *group* = *monoid* +
assumes *Units*: *carrier G* <= *Units G*

lemma (**in** *group*) *is-group*: *group G* **by** (*rule group-axioms*)

theorem *groupI*:

```
  fixes G (structure)
  assumes m-closed [simp]:
      !!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊗ y ∈ carrier G
    and one-closed [simp]: 1 ∈ carrier G
    and m-assoc:
      !!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
      (x ⊗ y) ⊗ z = x ⊗ (y ⊗ z)
    and l-one [simp]: !!x. x ∈ carrier G ==> 1 ⊗ x = x
    and l-inv-ex: !!x. x ∈ carrier G ==> ∃ y ∈ carrier G. y ⊗ x = 1
  shows group G
proof −
  have l-cancel [simp]:
    !!x y z. [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
    (x ⊗ y = x ⊗ z) = (y = z)
  proof
    fix x y z
    assume eq: x ⊗ y = x ⊗ z
      and G: x ∈ carrier G  y ∈ carrier G  z ∈ carrier G
    with l-inv-ex obtain x-inv where xG: x-inv ∈ carrier G
      and l-inv: x-inv ⊗ x = 1 by fast
    from G eq xG have (x-inv ⊗ x) ⊗ y = (x-inv ⊗ x) ⊗ z
      by (simp add: m-assoc)
    with G show y = z by (simp add: l-inv)
  next
    fix x y z
    assume eq: y = z
      and G: x ∈ carrier G  y ∈ carrier G  z ∈ carrier G
    then show x ⊗ y = x ⊗ z by simp
  qed
  have r-one:
    !!x. x ∈ carrier G ==> x ⊗ 1 = x
  proof −
    fix x
    assume x: x ∈ carrier G
    with l-inv-ex obtain x-inv where xG: x-inv ∈ carrier G
      and l-inv: x-inv ⊗ x = 1 by fast
    from x xG have x-inv ⊗ (x ⊗ 1) = x-inv ⊗ x
      by (simp add: m-assoc [symmetric] l-inv)
    with x xG show x ⊗ 1 = x by simp
  qed
  have inv-ex:
    !!x. x ∈ carrier G ==> ∃ y ∈ carrier G. y ⊗ x = 1 & x ⊗ y = 1
  proof −
    fix x
    assume x: x ∈ carrier G
    with l-inv-ex obtain y where y: y ∈ carrier G
      and l-inv: y ⊗ x = 1 by fast
    from x y have y ⊗ (x ⊗ y) = y ⊗ 1
      by (simp add: m-assoc [symmetric] l-inv r-one)
```

```
    with x y have r-inv: x ⊗ y = 1
      by simp
    from x y show ∃ y ∈ carrier G. y ⊗ x = 1 & x ⊗ y = 1
      by (fast intro: l-inv r-inv)
  qed
  then have carrier-subset-Units: carrier G <= Units G
    by (unfold Units-def) fast
  show ?thesis
    by (fast intro!: group.intro monoid.intro group-axioms.intro
      carrier-subset-Units intro: prems r-one)
qed

lemma (in monoid) monoid-groupI:
  assumes l-inv-ex:
    !!x. x ∈ carrier G ==> ∃ y ∈ carrier G. y ⊗ x = 1
  shows group G
  by (rule groupI) (auto intro: m-assoc l-inv-ex)

lemma (in group) Units-eq [simp]:
  Units G = carrier G
proof
  show Units G <= carrier G by fast
next
  show carrier G <= Units G by (rule Units)
qed

lemma (in group) inv-closed [intro, simp]:
  x ∈ carrier G ==> inv x ∈ carrier G
  using Units-inv-closed by simp

lemma (in group) l-inv-ex [simp]:
  x ∈ carrier G ==> ∃ y ∈ carrier G. y ⊗ x = 1
  using Units-l-inv-ex by simp

lemma (in group) r-inv-ex [simp]:
  x ∈ carrier G ==> ∃ y ∈ carrier G. x ⊗ y = 1
  using Units-r-inv-ex by simp

lemma (in group) l-inv [simp]:
  x ∈ carrier G ==> inv x ⊗ x = 1
  using Units-l-inv by simp
```

2.2 Cancellation Laws and Basic Properties

```
lemma (in group) l-cancel [simp]:
  [| x ∈ carrier G; y ∈ carrier G; z ∈ carrier G |] ==>
   (x ⊗ y = x ⊗ z) = (y = z)
  using Units-l-inv by simp
```

lemma (**in** *group*) *r-inv* [*simp*]:
  $x \in carrier\ G \Longrightarrow x \otimes inv\ x = \mathbf{1}$
proof −
  assume *x*: $x \in carrier\ G$
  then have $inv\ x \otimes (x \otimes inv\ x) = inv\ x \otimes \mathbf{1}$
    by (*simp add*: *m-assoc* [*symmetric*] *l-inv*)
  with *x* **show** *?thesis* **by** (*simp del*: *r-one*)
qed

lemma (**in** *group*) *r-cancel* [*simp*]:
  $[\!|\ x \in carrier\ G; y \in carrier\ G; z \in carrier\ G\ |\!] \Longrightarrow$
   $(y \otimes x = z \otimes x) = (y = z)$
proof
  assume *eq*: $y \otimes x = z \otimes x$
    and *G*: $x \in carrier\ G\ \ y \in carrier\ G\ \ z \in carrier\ G$
  then have $y \otimes (x \otimes inv\ x) = z \otimes (x \otimes inv\ x)$
    by (*simp add*: *m-assoc* [*symmetric*] *del*: *r-inv*)
  with *G* **show** $y = z$ **by** *simp*
next
  assume *eq*: $y = z$
    and *G*: $x \in carrier\ G\ \ y \in carrier\ G\ \ z \in carrier\ G$
  then show $y \otimes x = z \otimes x$ **by** *simp*
qed

lemma (**in** *group*) *inv-one* [*simp*]:
  $inv\ \mathbf{1} = \mathbf{1}$
proof −
  have $inv\ \mathbf{1} = \mathbf{1} \otimes (inv\ \mathbf{1})$ **by** (*simp del*: *r-inv*)
  moreover have $... = \mathbf{1}$ **by** *simp*
  finally show *?thesis* **.**
qed

lemma (**in** *group*) *inv-inv* [*simp*]:
  $x \in carrier\ G \Longrightarrow inv\ (inv\ x) = x$
  using *Units-inv-inv* **by** *simp*

lemma (**in** *group*) *inv-inj*:
  inj-on (*m-inv G*) (*carrier G*)
  using *inv-inj-on-Units* **by** *simp*

lemma (**in** *group*) *inv-mult-group*:
  $[\!|\ x \in carrier\ G; y \in carrier\ G\ |\!] \Longrightarrow inv\ (x \otimes y) = inv\ y \otimes inv\ x$
proof −
  assume *G*: $x \in carrier\ G\ \ y \in carrier\ G$
  then have $inv\ (x \otimes y) \otimes (x \otimes y) = (inv\ y \otimes inv\ x) \otimes (x \otimes y)$
    by (*simp add*: *m-assoc l-inv*) (*simp add*: *m-assoc* [*symmetric*])
  with *G* **show** *?thesis* **by** (*simp del*: *l-inv*)
qed

lemma (**in** *group*) *inv-comm*:
[| $x \otimes y = \mathbf{1}$; $x \in$ *carrier* G; $y \in$ *carrier* G |] ==> $y \otimes x = \mathbf{1}$
by (*rule Units-inv-comm*) *auto*

lemma (**in** *group*) *inv-equality*:
[|$y \otimes x = \mathbf{1}$; $x \in$ *carrier* G; $y \in$ *carrier* G|] ==> *inv* $x = y$
apply (*simp add*: *m-inv-def*)
apply (*rule the-equality*)
apply (*simp add*: *inv-comm* [*of y x*])
apply (*rule r-cancel* [*THEN iffD1*], *auto*)
done

Power

lemma (**in** *group*) *int-pow-def2*:
a (^) (z::*int*) = (*if neg z then inv* (a (^) (*nat* ($-z$))) *else* a (^) (*nat* z))
by (*simp add*: *int-pow-def nat-pow-def Let-def*)

lemma (**in** *group*) *int-pow-0* [*simp*]:
x (^) (*0*::*int*) = $\mathbf{1}$
by (*simp add*: *int-pow-def2*)

lemma (**in** *group*) *int-pow-one* [*simp*]:
$\mathbf{1}$ (^) (z::*int*) = $\mathbf{1}$
by (*simp add*: *int-pow-def2*)

2.3 Subgroups

locale *subgroup* =
fixes H **and** G (**structure**)
assumes *subset*: $H \subseteq$ *carrier* G
and *m-closed* [*intro*, *simp*]: $[\![x \in H; y \in H]\!] \Longrightarrow x \otimes y \in H$
and *one-closed* [*simp*]: $\mathbf{1} \in H$
and *m-inv-closed* [*intro*,*simp*]: $x \in H \Longrightarrow$ *inv* $x \in H$

lemma (**in** *subgroup*) *is-subgroup*:
subgroup H G **by** (*rule subgroup-axioms*)

declare (**in** *subgroup*) *group.intro* [*intro*]

lemma (**in** *subgroup*) *mem-carrier* [*simp*]:
$x \in H \Longrightarrow x \in$ *carrier* G
using *subset* **by** *blast*

lemma *subgroup-imp-subset*:
subgroup H $G \Longrightarrow H \subseteq$ *carrier* G
by (*rule subgroup.subset*)

lemma (**in** *subgroup*) *subgroup-is-group* [*intro*]:
includes *group* G

shows *group* (*G*⦇*carrier* := *H*⦈)
by (*rule groupI*) (*auto intro*: *m-assoc l-inv mem-carrier*)

Since H is nonempty, it contains some element x. Since it is closed under inverse, it contains $inv\ x$. Since it is closed under product, it contains $x \otimes inv\ x = \mathbf{1}$.

lemma (**in** *group*) *one-in-subset*:
[| $H \subseteq carrier\ G$; $H \neq \{\}$; $\forall\, a \in H.\ inv\ a \in H$; $\forall\, a{\in}H.\ \forall\, b{\in}H.\ a \otimes b \in H$ |]
==> $\mathbf{1} \in H$
by (*force simp add*: *l-inv*)

A characterization of subgroups: closed, non-empty subset.

lemma (**in** *group*) *subgroupI*:
assumes *subset*: $H \subseteq carrier\ G$ **and** *non-empty*: $H \neq \{\}$
and *inv*: !!*a*. $a \in H \Longrightarrow inv\ a \in H$
and *mult*: !!*a b*. $[\![a \in H;\ b \in H]\!] \Longrightarrow a \otimes b \in H$
shows *subgroup H G*
proof (*simp add*: *subgroup-def prems*)
show $\mathbf{1} \in H$ **by** (*rule one-in-subset*) (*auto simp only*: *prems*)
qed

declare *monoid.one-closed* [*iff*] *group.inv-closed* [*simp*]
monoid.l-one [*simp*] *monoid.r-one* [*simp*] *group.inv-inv* [*simp*]

lemma *subgroup-nonempty*:
~ *subgroup* {} *G*
by (*blast dest*: *subgroup.one-closed*)

lemma (**in** *subgroup*) *finite-imp-card-positive*:
finite (*carrier G*) ==> *0* < *card H*
proof (*rule classical*)
assume *finite* (*carrier G*) ~ *0* < *card H*
then have *finite H* **by** (*blast intro*: *finite-subset* [*OF subset*])
with *prems* **have** *subgroup* {} *G* **by** *simp*
with *subgroup-nonempty* **show** *?thesis* **by** *contradiction*
qed

2.4 Direct Products

constdefs
DirProd :: - ⇒ - ⇒ ('*a* × '*b*) *monoid* (**infixr** ×× *80*)
G ×× *H* ≡ ⦇*carrier* = *carrier G* × *carrier H*,
mult = (λ(*g*, *h*) (*g'*, *h'*). (*g* ⊗$_G$ *g'*, *h* ⊗$_H$ *h'*)),
one = (**1**$_G$, **1**$_H$)⦈

lemma *DirProd-monoid*:
includes *monoid G* + *monoid H*
shows *monoid* (*G* ×× *H*)
proof −

from *prems*
show *?thesis* **by** (*unfold monoid-def DirProd-def*, *auto*)
qed

Does not use the previous result because it's easier just to use auto.

lemma *DirProd-group*:
includes *group G* + *group H*
shows *group* (*G* ×× *H*)
by (*rule groupI*)
(*auto intro*: *G.m-assoc H.m-assoc G.l-inv H.l-inv*
simp add: *DirProd-def*)

lemma *carrier-DirProd* [*simp*]:
carrier (*G* ×× *H*) = *carrier G* × *carrier H*
by (*simp add*: *DirProd-def*)

lemma *one-DirProd* [*simp*]:
$\mathbf{1}_{G \times\times H} = (\mathbf{1}_G, \mathbf{1}_H)$
by (*simp add*: *DirProd-def*)

lemma *mult-DirProd* [*simp*]:
$(g, h) \otimes_{(G \times\times H)} (g', h') = (g \otimes_G g', h \otimes_H h')$
by (*simp add*: *DirProd-def*)

lemma *inv-DirProd* [*simp*]:
includes *group G* + *group H*
assumes *g*: *g* ∈ *carrier G*
and *h*: *h* ∈ *carrier H*
shows *m-inv* (*G* ×× *H*) (*g*, *h*) = (inv_G *g*, inv_H *h*)
apply (*rule group.inv-equality* [*OF DirProd-group*])
apply (*simp-all add*: *prems group.l-inv*)
done

This alternative proof of the previous result demonstrates interpret. It uses *Prod.inv-equality* (available after *interpret*) instead of *group.inv-equality* [*OF DirProd-group*].

lemma
includes *group G* + *group H*
assumes *g*: *g* ∈ *carrier G*
and *h*: *h* ∈ *carrier H*
shows *m-inv* (*G* ×× *H*) (*g*, *h*) = (inv_G *g*, inv_H *h*)
proof −
interpret *Prod*: *group* [*G* ×× *H*]
by (*auto intro*: *DirProd-group group.intro group.axioms prems*)
show *?thesis* **by** (*simp add*: *Prod.inv-equality g h*)
qed

2.5 Homomorphisms and Isomorphisms

constdefs (**structure** G **and** H)
  $hom :: \text{-} \Rightarrow \text{-} \Rightarrow ('a \Rightarrow 'b)\ set$
  $hom\ G\ H ==$
    $\{h.\ h \in carrier\ G \rightarrow carrier\ H\ \&$
      $(\forall x \in carrier\ G.\ \forall y \in carrier\ G.\ h\ (x \otimes_G y) = h\ x \otimes_H h\ y)\}$

lemma *hom-mult*:
  $[|\ h \in hom\ G\ H;\ x \in carrier\ G;\ y \in carrier\ G\ |]$
   $\Longrightarrow h\ (x \otimes_G y) = h\ x \otimes_H h\ y$
  by (*simp add*: *hom-def*)

lemma *hom-closed*:
  $[|\ h \in hom\ G\ H;\ x \in carrier\ G\ |] \Longrightarrow h\ x \in carrier\ H$
  by (*auto simp add*: *hom-def funcset-mem*)

lemma (**in** *group*) *hom-compose*:
    $[|h \in hom\ G\ H;\ i \in hom\ H\ I|] \Longrightarrow compose\ (carrier\ G)\ i\ h \in hom\ G\ I$
apply (*auto simp add*: *hom-def funcset-compose*)
apply (*simp add*: *compose-def funcset-mem*)
done

constdefs
  $iso :: \text{-} \Rightarrow \text{-} \Rightarrow ('a \Rightarrow 'b)\ set$ (**infixr** $\cong$ *60*)
  $G \cong H == \{h.\ h \in hom\ G\ H\ \&\ bij\text{-}betw\ h\ (carrier\ G)\ (carrier\ H)\}$

lemma *iso-refl*: $(\%x.\ x) \in G \cong G$
by (*simp add*: *iso-def hom-def inj-on-def bij-betw-def Pi-def*)

lemma (**in** *group*) *iso-sym*:
    $h \in G \cong H \Longrightarrow Inv\ (carrier\ G)\ h \in H \cong G$
apply (*simp add*: *iso-def bij-betw-Inv*)
apply (*subgoal-tac Inv* (*carrier G*) $h \in carrier\ H \rightarrow carrier\ G$)
 prefer *2* **apply** (*simp add*: *bij-betw-imp-funcset* [*OF bij-betw-Inv*])
apply (*simp add*: *hom-def bij-betw-def Inv-f-eq funcset-mem f-Inv-f*)
done

lemma (**in** *group*) *iso-trans*:
    $[|h \in G \cong H;\ i \in H \cong I|] \Longrightarrow (compose\ (carrier\ G)\ i\ h) \in G \cong I$
by (*auto simp add*: *iso-def hom-compose bij-betw-compose*)

lemma *DirProd-commute-iso*:
  shows $(\lambda(x,y).\ (y,x)) \in (G \times\times H) \cong (H \times\times G)$
by (*auto simp add*: *iso-def hom-def inj-on-def bij-betw-def Pi-def*)

lemma *DirProd-assoc-iso*:
  shows $(\lambda(x,y,z).\ (x,(y,z))) \in (G \times\times H \times\times I) \cong (G \times\times (H \times\times I))$
by (*auto simp add*: *iso-def hom-def inj-on-def bij-betw-def Pi-def*)

Basis for homomorphism proofs: we assume two groups G and H, with a homomorphism h between them

locale *group-hom* = *group* G + *group* H + *var* h +
  assumes *homh*: $h \in hom\ G\ H$
  notes *hom-mult* [*simp*] = *hom-mult* [*OF homh*]
    and *hom-closed* [*simp*] = *hom-closed* [*OF homh*]

lemma (**in** *group-hom*) *one-closed* [*simp*]:
  $h\ \mathbf{1} \in carrier\ H$
  by *simp*

lemma (**in** *group-hom*) *hom-one* [*simp*]:
  $h\ \mathbf{1} = \mathbf{1}_H$
proof −
  have $h\ \mathbf{1} \otimes_H \mathbf{1}_H = h\ \mathbf{1} \otimes_H h\ \mathbf{1}$
    by (*simp add*: *hom-mult* [*symmetric*] *del*: *hom-mult*)
  then show *?thesis* **by** (*simp del*: *r-one*)
qed

lemma (**in** *group-hom*) *inv-closed* [*simp*]:
  $x \in carrier\ G \Longrightarrow h\ (inv\ x) \in carrier\ H$
  by *simp*

lemma (**in** *group-hom*) *hom-inv* [*simp*]:
  $x \in carrier\ G \Longrightarrow h\ (inv\ x) = inv_H\ (h\ x)$
proof −
  assume *x*: $x \in carrier\ G$
  then have $h\ x \otimes_H h\ (inv\ x) = \mathbf{1}_H$
    by (*simp add*: *hom-mult* [*symmetric*] *del*: *hom-mult*)
  also from *x* **have** $... = h\ x \otimes_H inv_H\ (h\ x)$
    by (*simp add*: *hom-mult* [*symmetric*] *del*: *hom-mult*)
  finally have $h\ x \otimes_H h\ (inv\ x) = h\ x \otimes_H inv_H\ (h\ x)$ **.**
  with *x* **show** *?thesis* **by** (*simp del*: *H.r-inv*)
qed

2.6 Commutative Structures

Naming convention: multiplicative structures that are commutative are called *commutative*, additive structures are called *Abelian*.

locale *comm-monoid* = *monoid* +
  assumes *m-comm*: $[\![x \in carrier\ G;\ y \in carrier\ G]\!] \Longrightarrow x \otimes y = y \otimes x$

lemma (**in** *comm-monoid*) *m-lcomm*:
  $[\![x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G]\!] \Longrightarrow$
   $x \otimes (y \otimes z) = y \otimes (x \otimes z)$
proof −
  assume *xyz*: $x \in carrier\ G\ \ y \in carrier\ G\ \ z \in carrier\ G$
  from *xyz* **have** $x \otimes (y \otimes z) = (x \otimes y) \otimes z$ **by** (*simp add*: *m-assoc*)

also from *xyz* **have** *...* $= (y \otimes x) \otimes z$ **by** (*simp add*: *m-comm*)
also from *xyz* **have** *...* $= y \otimes (x \otimes z)$ **by** (*simp add*: *m-assoc*)
finally show *?thesis* **.**
qed

lemmas (**in** *comm-monoid*) *m-ac* = *m-assoc m-comm m-lcomm*

lemma *comm-monoidI*:
fixes *G* (**structure**)
assumes *m-closed*:
!!*x y*. [| *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==> *x* ⊗ *y* ∈ *carrier G*
and *one-closed*: **1** ∈ *carrier G*
and *m-assoc*:
!!*x y z*. [| *x* ∈ *carrier G*; *y* ∈ *carrier G*; *z* ∈ *carrier G* |] ==>
(*x* ⊗ *y*) ⊗ *z* = *x* ⊗ (*y* ⊗ *z*)
and *l-one*: !!*x*. *x* ∈ *carrier G* ==> **1** ⊗ *x* = *x*
and *m-comm*:
!!*x y*. [| *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==> *x* ⊗ *y* = *y* ⊗ *x*
shows *comm-monoid G*
using *l-one*
by (*auto intro*!: *comm-monoid.intro comm-monoid-axioms.intro monoid.intro*
intro: *prems simp*: *m-closed one-closed m-comm*)

lemma (**in** *monoid*) *monoid-comm-monoidI*:
assumes *m-comm*:
!!*x y*. [| *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==> *x* ⊗ *y* = *y* ⊗ *x*
shows *comm-monoid G*
by (*rule comm-monoidI*) (*auto intro*: *m-assoc m-comm*)

lemma (**in** *comm-monoid*) *nat-pow-distr*:
[| *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==>
(*x* ⊗ *y*) (^) (*n*::*nat*) = *x* (^) *n* ⊗ *y* (^) *n*
by (*induct n*) (*simp*, *simp add*: *m-ac*)

locale *comm-group* = *comm-monoid* + *group*

lemma (**in** *group*) *group-comm-groupI*:
assumes *m-comm*: !!*x y*. [| *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==>
x ⊗ *y* = *y* ⊗ *x*
shows *comm-group G*
by *unfold-locales* (*simp-all add*: *m-comm*)

lemma *comm-groupI*:
fixes *G* (**structure**)
assumes *m-closed*:
!!*x y*. [| *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==> *x* ⊗ *y* ∈ *carrier G*
and *one-closed*: **1** ∈ *carrier G*

and *m-assoc*:
!!$x\ y\ z$. [| $x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* G |] ==>
$(x \otimes y) \otimes z = x \otimes (y \otimes z)$
and *m-comm*:
!!$x\ y$. [| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==> $x \otimes y = y \otimes x$
and *l-one*: !!x. $x \in$ *carrier* G ==> $\mathbf{1} \otimes x = x$
and *l-inv-ex*: !!x. $x \in$ *carrier* G ==> $\exists\, y \in$ *carrier* G. $y \otimes x = \mathbf{1}$
shows *comm-group* G
by (*fast intro*: *group.group-comm-groupI groupI prems*)

lemma (**in** *comm-group*) *inv-mult*:
[| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==> *inv* $(x \otimes y) =$ *inv* $x \otimes$ *inv* y
by (*simp add*: *m-ac inv-mult-group*)

2.7 The Lattice of Subgroups of a Group

theorem (**in** *group*) *subgroups-partial-order*:
partial-order (| *carrier* = {H. *subgroup* H G}, *le* = *op* $\subseteq$ |)
by (*rule partial-order.intro*) *simp-all*

lemma (**in** *group*) *subgroup-self*:
subgroup (*carrier* G) G
by (*rule subgroupI*) *auto*

lemma (**in** *group*) *subgroup-imp-group*:
subgroup H G ==> *group* (G(| *carrier* := H |))
by (*erule subgroup.subgroup-is-group*) (*rule group-axioms*)

lemma (**in** *group*) *is-monoid* [*intro*, *simp*]:
monoid G
by (*auto intro*: *monoid.intro m-assoc*)

lemma (**in** *group*) *subgroup-inv-equality*:
[| *subgroup* H G; $x \in H$ |] ==> *m-inv* (G (| *carrier* := H |)) x = *inv* x
apply (*rule-tac inv-equality* [*THEN sym*])
apply (*rule group.l-inv* [*OF subgroup-imp-group*, *simplified*], *assumption*+)
apply (*rule subsetD* [*OF subgroup.subset*], *assumption*+)
apply (*rule subsetD* [*OF subgroup.subset*], *assumption*)
apply (*rule-tac group.inv-closed* [*OF subgroup-imp-group*, *simplified*], *assumption*+)
done

theorem (**in** *group*) *subgroups-Inter*:
assumes *subgr*: (!!H. $H \in A$ ==> *subgroup* H G)
and *not-empty*: $A \neq \{\}$
shows *subgroup* ($\bigcap A$) G
proof (*rule subgroupI*)
from *subgr* [*THEN subgroup.subset*] **and** *not-empty*
show $\bigcap A \subseteq$ *carrier* G **by** *blast*
next

```
  from subgr [THEN subgroup.one-closed]
  show ⋂A ~= {} by blast
next
  fix x assume x ∈ ⋂A
  with subgr [THEN subgroup.m-inv-closed]
  show inv x ∈ ⋂A by blast
next
  fix x y assume x ∈ ⋂A y ∈ ⋂A
  with subgr [THEN subgroup.m-closed]
  show x ⊗ y ∈ ⋂A by blast
qed

theorem (in group) subgroups-complete-lattice:
  complete-lattice (| carrier = {H. subgroup H G}, le = op ⊆ |)
    (is complete-lattice ?L)
proof (rule partial-order.complete-lattice-criterion1)
  show partial-order ?L by (rule subgroups-partial-order)
next
  show ∃G. greatest ?L G (carrier ?L)
  proof
    show greatest ?L (carrier G) (carrier ?L)
      by (unfold greatest-def)
        (simp add: subgroup.subset subgroup-self)
  qed
next
  fix A
  assume L: A ⊆ carrier ?L and non-empty: A ~= {}
  then have Int-subgroup: subgroup (⋂A) G
    by (fastsimp intro: subgroups-Inter)
  show ∃I. greatest ?L I (Lower ?L A)
  proof
    show greatest ?L (⋂A) (Lower ?L A)
      (is greatest - ?Int -)
    proof (rule greatest-LowerI)
      fix H
      assume H: H ∈ A
      with L have subgroupH: subgroup H G by auto
      from subgroupH have groupH: group (G (| carrier := H |)) (is group ?H)
        by (rule subgroup-imp-group)
      from groupH have monoidH: monoid ?H
        by (rule group.is-monoid)
      from H have Int-subset: ?Int ⊆ H by fastsimp
      then show le ?L ?Int H by simp
    next
      fix H
      assume H: H ∈ Lower ?L A
      with L Int-subgroup show le ?L H ?Int
        by (fastsimp simp: Lower-def intro: Inter-greatest)
    next
```

show $A \subseteq$ *carrier ?L* **by** (*rule L*)
next
show *?Int* $\in$ *carrier ?L* **by** *simp* (*rule Int-subgroup*)
qed
qed
qed

end

theory *FiniteProduct* **imports** *Group* **begin**

3 Product Operator for Commutative Monoids

3.1 Inductive Definition of a Relation for Products over Sets

Instantiation of locale *LC* of theory *Finite-Set* is not possible, because here we have explicit typing rules like $x \in$ *carrier G*. We introduce an explicit argument for the domain *D*.

inductive-set
foldSetD :: [*'a set*, *'b* => *'a* => *'a*, *'a*] => (*'b set* * *'a*) *set*
for *D* :: *'a set* **and** *f* :: *'b* => *'a* => *'a* **and** *e* :: *'a*
where
emptyI [*intro*]: *e* $\in$ *D* ==> ({}, *e*) $\in$ *foldSetD D f e*
| *insertI* [*intro*]: [| *x* ~: *A*; *f x y* $\in$ *D*; (*A*, *y*) $\in$ *foldSetD D f e* |] ==>
(*insert x A*, *f x y*) $\in$ *foldSetD D f e*

inductive-cases *empty-foldSetDE* [*elim*!]: ({}, *x*) $\in$ *foldSetD D f e*

constdefs
foldD :: [*'a set*, *'b* => *'a* => *'a*, *'a*, *'b set*] => *'a*
foldD D f e A == *THE x*. (*A*, *x*) $\in$ *foldSetD D f e*

lemma *foldSetD-closed*:
[| (*A*, *z*) $\in$ *foldSetD D f e* ; *e* $\in$ *D*; !!*x y*. [| *x* $\in$ *A*; *y* $\in$ *D* |] ==> *f x y* $\in$ *D*
|] ==> *z* $\in$ *D*
by (*erule foldSetD.cases*) *auto*

lemma *Diff1-foldSetD*:
[| (*A* - {*x*}, *y*) $\in$ *foldSetD D f e*; *x* $\in$ *A*; *f x y* $\in$ *D* |] ==>
(*A*, *f x y*) $\in$ *foldSetD D f e*
apply (*erule insert-Diff* [*THEN subst*], *rule foldSetD.intros*)
apply *auto*
done

lemma *foldSetD-imp-finite* [*simp*]: (*A*, *x*) $\in$ *foldSetD D f e* ==> *finite A*
by (*induct set*: *foldSetD*) *auto*

lemma *finite-imp-foldSetD*:
[| *finite A*; *e* ∈ *D*; !!*x y*. [| *x* ∈ *A*; *y* ∈ *D* |] ==> *f x y* ∈ *D* |] ==>
EX x. (*A*, *x*) ∈ *foldSetD D f e*
proof (*induct set*: *finite*)
case *empty* **then show** *?case* **by** *auto*
next
case (*insert x F*)
then obtain *y* **where** *y*: (*F*, *y*) ∈ *foldSetD D f e* **by** *auto*
with *insert* **have** *y* ∈ *D* **by** (*auto dest*: *foldSetD-closed*)
with *y* **and** *insert* **have** (*insert x F*, *f x y*) ∈ *foldSetD D f e*
by (*intro foldSetD.intros*) *auto*
then show *?case* **..**
qed

3.2 Left-Commutative Operations

locale *LCD* =
fixes *B* :: *'b set*
and *D* :: *'a set*
and *f* :: *'b* => *'a* => *'a* (**infixl** · *70*)
assumes *left-commute*:
[| *x* ∈ *B*; *y* ∈ *B*; *z* ∈ *D* |] ==> *x* · (*y* · *z*) = *y* · (*x* · *z*)
and *f-closed* [*simp*, *intro*!]: !!*x y*. [| *x* ∈ *B*; *y* ∈ *D* |] ==> *f x y* ∈ *D*

lemma (**in** *LCD*) *foldSetD-closed* [*dest*]:
(*A*, *z*) ∈ *foldSetD D f e* ==> *z* ∈ *D*
by (*erule foldSetD.cases*) *auto*

lemma (**in** *LCD*) *Diff1-foldSetD*:
[| (*A* − {*x*}, *y*) ∈ *foldSetD D f e*; *x* ∈ *A*; *A* ⊆ *B* |] ==>
(*A*, *f x y*) ∈ *foldSetD D f e*
apply (*subgoal-tac x* ∈ *B*)
prefer *2* **apply** *fast*
apply (*erule insert-Diff* [*THEN subst*], *rule foldSetD.intros*)
apply *auto*
done

lemma (**in** *LCD*) *foldSetD-imp-finite* [*simp*]:
(*A*, *x*) ∈ *foldSetD D f e* ==> *finite A*
by (*induct set*: *foldSetD*) *auto*

lemma (**in** *LCD*) *finite-imp-foldSetD*:
[| *finite A*; *A* ⊆ *B*; *e* ∈ *D* |] ==> *EX x*. (*A*, *x*) ∈ *foldSetD D f e*
proof (*induct set*: *finite*)
case *empty* **then show** *?case* **by** *auto*
next
case (*insert x F*)
then obtain *y* **where** *y*: (*F*, *y*) ∈ *foldSetD D f e* **by** *auto*

with *insert* **have** $y \in D$ **by** *auto*
with y **and** *insert* **have** (*insert x F*, *f x y*) $\in$ *foldSetD D f e*
by (*intro foldSetD.intros*) *auto*
then show *?case* **..**
qed

lemma (**in** *LCD*) *foldSetD-determ-aux*:
$e \in D$ ==> $\forall A\ x.\ A \subseteq B$ & *card* $A < n$ --> $(A, x) \in$ *foldSetD D f e* -->
$(\forall y.\ (A, y) \in$ *foldSetD D f e* --> $y = x)$
apply (*induct n*)
apply (*auto simp add*: *less-Suc-eq*)
apply (*erule foldSetD.cases*)
apply *blast*
apply (*erule foldSetD.cases*)
apply *blast*
apply *clarify*

force simplification of *card A* < *card* (*insert* ...).

apply (*erule rev-mp*)
apply (*simp add*: *less-Suc-eq-le*)
apply (*rule impI*)
apply (*rename-tac xa Aa ya xb Ab yb*, *case-tac xa* = *xb*)
apply (*subgoal-tac Aa* = *Ab*)
prefer *2* **apply** (*blast elim*!: *equalityE*)
apply *blast*

case $xa \notin xb$.

apply (*subgoal-tac Aa* − {*xb*} = *Ab* − {*xa*} & *xb* ∈ *Aa* & *xa* ∈ *Ab*)
prefer *2* **apply** (*blast elim*!: *equalityE*)
apply *clarify*
apply (*subgoal-tac Aa* = *insert xb Ab* − {*xa*})
prefer *2* **apply** *blast*
apply (*subgoal-tac card Aa* ≤ *card Ab*)
prefer *2*
apply (*rule Suc-le-mono* [*THEN subst*])
apply (*simp add*: *card-Suc-Diff1*)
apply (*rule-tac A1* = *Aa* − {*xb*} **in** *finite-imp-foldSetD* [*THEN exE*])
apply (*blast intro*: *foldSetD-imp-finite finite-Diff*)
apply *best*
apply *assumption*
apply (*frule* (*1*) *Diff1-foldSetD*)
apply *best*
apply (*subgoal-tac ya* = *f xb x*)
prefer *2*
apply (*subgoal-tac Aa* ⊆ *B*)
prefer *2* **apply** *best*
apply (*blast del*: *equalityCE*)
apply (*subgoal-tac* (*Ab* − {*xa*}, *x*) ∈ *foldSetD D f e*)
prefer *2* **apply** *simp*

```
  apply (subgoal-tac yb = f xa x)
   prefer 2
   apply (blast del: equalityCE dest: Diff1-foldSetD)
  apply (simp (no-asm-simp))
  apply (rule left-commute)
    apply assumption
   apply best
  apply best
  done

lemma (in LCD) foldSetD-determ:
  [| (A, x) ∈ foldSetD D f e; (A, y) ∈ foldSetD D f e; e ∈ D; A ⊆ B |]
  ==> y = x
  by (blast intro: foldSetD-determ-aux [rule-format])

lemma (in LCD) foldD-equality:
  [| (A, y) ∈ foldSetD D f e; e ∈ D; A ⊆ B |] ==> foldD D f e A = y
  by (unfold foldD-def) (blast intro: foldSetD-determ)

lemma foldD-empty [simp]:
  e ∈ D ==> foldD D f e {} = e
  by (unfold foldD-def) blast

lemma (in LCD) foldD-insert-aux:
  [| x ~: A; x ∈ B; e ∈ D; A ⊆ B |] ==>
    ((insert x A, v) ∈ foldSetD D f e) =
    (EX y. (A, y) ∈ foldSetD D f e & v = f x y)
  apply auto
  apply (rule-tac A1 = A in finite-imp-foldSetD [THEN exE])
     apply (fastsimp dest: foldSetD-imp-finite)
    apply assumption
   apply assumption
  apply (blast intro: foldSetD-determ)
  done

lemma (in LCD) foldD-insert:
    [| finite A; x ~: A; x ∈ B; e ∈ D; A ⊆ B |] ==>
     foldD D f e (insert x A) = f x (foldD D f e A)
  apply (unfold foldD-def)
  apply (simp add: foldD-insert-aux)
  apply (rule the-equality)
   apply (auto intro: finite-imp-foldSetD
     cong add: conj-cong simp add: foldD-def [symmetric] foldD-equality)
  done

lemma (in LCD) foldD-closed [simp]:
  [| finite A; e ∈ D; A ⊆ B |] ==> foldD D f e A ∈ D
proof (induct set: finite)
  case empty then show ?case by (simp add: foldD-empty)
```

next
case *insert* **then show** *?case* **by** (*simp add*: *foldD-insert*)
qed

lemma (**in** *LCD*) *foldD-commute*:
[| *finite A*; $x \in B$; $e \in D$; $A \subseteq B$ |] ==>
f x (*foldD D f e A*) = *foldD D f* (*f x e*) *A*
apply (*induct set*: *finite*)
apply *simp*
apply *simp*
done

lemma *Int-mono2*:
[| $A \subseteq C$; $B \subseteq C$ |] ==> *A Int B* $\subseteq$ *C*
by *blast*

lemma (**in** *LCD*) *foldD-nest-Un-Int*:
[| *finite A*; *finite C*; $e \in D$; $A \subseteq B$; $C \subseteq B$ |] ==>
foldD D f (*foldD D f e C*) *A* = *foldD D f* (*foldD D f e* (*A Int C*)) (*A Un C*)
apply (*induct set*: *finite*)
apply *simp*
apply (*simp add*: *foldD-insert foldD-commute Int-insert-left insert-absorb*
Int-mono2 Un-subset-iff)
done

lemma (**in** *LCD*) *foldD-nest-Un-disjoint*:
[| *finite A*; *finite B*; *A Int B* = {}; $e \in D$; $A \subseteq B$; $C \subseteq B$ |]
==> *foldD D f e* (*A Un B*) = *foldD D f* (*foldD D f e B*) *A*
by (*simp add*: *foldD-nest-Un-Int*)

— Delete rules to do with *foldSetD* relation.

declare *foldSetD-imp-finite* [*simp del*]
empty-foldSetDE [*rule del*]
foldSetD.intros [*rule del*]
declare (**in** *LCD*)
foldSetD-closed [*rule del*]

3.3 Commutative Monoids

We enter a more restrictive context, with $f :: 'a => 'a => 'a$ instead of $'b => 'a => 'a$.

locale *ACeD* =
fixes *D* :: *'a set*
and *f* :: *'a => 'a => 'a* (**infixl** · *70*)
and *e* :: *'a*
assumes *ident* [*simp*]: $x \in D$ ==> $x \cdot e = x$
and *commute*: [| $x \in D$; $y \in D$ |] ==> $x \cdot y = y \cdot x$
and *assoc*: [| $x \in D$; $y \in D$; $z \in D$ |] ==> $(x \cdot y) \cdot z = x \cdot (y \cdot z)$

and *e-closed* [*simp*]: $e \in D$
and *f-closed* [*simp*]: [| $x \in D$; $y \in D$ |] ==> $x \cdot y \in D$

lemma (**in** *ACeD*) *left-commute*:
[| $x \in D$; $y \in D$; $z \in D$ |] ==> $x \cdot (y \cdot z) = y \cdot (x \cdot z)$
proof −
assume *D*: $x \in D$ $y \in D$ $z \in D$
then have $x \cdot (y \cdot z) = (y \cdot z) \cdot x$ **by** (*simp add*: *commute*)
also from *D* **have** ... $= y \cdot (z \cdot x)$ **by** (*simp add*: *assoc*)
also from *D* **have** $z \cdot x = x \cdot z$ **by** (*simp add*: *commute*)
finally show *?thesis* **.**
qed

lemmas (**in** *ACeD*) *AC* = *assoc commute left-commute*

lemma (**in** *ACeD*) *left-ident* [*simp*]: $x \in D$ ==> $e \cdot x = x$
proof −
assume $x \in D$
then have $x \cdot e = x$ **by** (*rule ident*)
with ‹$x \in D$› **show** *?thesis* **by** (*simp add*: *commute*)
qed

lemma (**in** *ACeD*) *foldD-Un-Int*:
[| *finite A*; *finite B*; $A \subseteq D$; $B \subseteq D$ |] ==>
foldD D f e A · *foldD D f e B* =
foldD D f e (*A Un B*) · *foldD D f e* (*A Int B*)
apply (*induct set*: *finite*)
apply (*simp add*: *left-commute LCD.foldD-closed* [*OF LCD.intro* [*of D*]])
apply (*simp add*: *AC insert-absorb Int-insert-left*
LCD.foldD-insert [*OF LCD.intro* [*of D*]]
LCD.foldD-closed [*OF LCD.intro* [*of D*]]
Int-mono2 Un-subset-iff)
done

lemma (**in** *ACeD*) *foldD-Un-disjoint*:
[| *finite A*; *finite B*; *A Int B* = {}; $A \subseteq D$; $B \subseteq D$ |] ==>
foldD D f e (*A Un B*) = *foldD D f e A* · *foldD D f e B*
by (*simp add*: *foldD-Un-Int*
left-commute LCD.foldD-closed [*OF LCD.intro* [*of D*]] *Un-subset-iff*)

3.4 Products over Finite Sets

constdefs (**structure** *G*)
finprod :: [('*b*, '*m*) *monoid-scheme*, '*a* => '*b*, '*a set*] => '*b*
finprod G f A == *if finite A*
then foldD (*carrier G*) (*mult G o f*) **1** *A*
else arbitrary

syntax

```
  -finprod :: index => idt => 'a set => 'b => 'b
      ((3⨂ --:-. -) [1000, 0, 51, 10] 10)
syntax (xsymbols)
  -finprod :: index => idt => 'a set => 'b => 'b
      ((3⨂ --∈-. -) [1000, 0, 51, 10] 10)
syntax (HTML output)
  -finprod :: index => idt => 'a set => 'b => 'b
      ((3⨂ --∈-. -) [1000, 0, 51, 10] 10)
translations
  ⨂i:A. b == finprod ◇ (%i. b) A
  — Beware of argument permutation!

lemma (in comm-monoid) finprod-empty [simp]:
  finprod G f {} = 1
  by (simp add: finprod-def)

declare funcsetI [intro]
  funcset-mem [dest]

lemma (in comm-monoid) finprod-insert [simp]:
  [| finite F; a ∉ F; f ∈ F -> carrier G; f a ∈ carrier G |] ==>
   finprod G f (insert a F) = f a ⊗ finprod G f F
  apply (rule trans)
   apply (simp add: finprod-def)
  apply (rule trans)
   apply (rule LCD.foldD-insert [OF LCD.intro [of insert a F]])
        apply simp
        apply (rule m-lcomm)
          apply fast
         apply fast
        apply assumption
       apply (fastsimp intro: m-closed)
      apply simp+
   apply fast
  apply (auto simp add: finprod-def)
  done

lemma (in comm-monoid) finprod-one [simp]:
  finite A ==> (⨂ i:A. 1) = 1
proof (induct set: finite)
  case empty show ?case by simp
next
  case (insert a A)
  have (%i. 1) ∈ A -> carrier G by auto
  with insert show ?case by simp
qed

lemma (in comm-monoid) finprod-closed [simp]:
  fixes A
```

assumes *fin*: *finite A* **and** *f*: *f ∈ A −> carrier G*
shows *finprod G f A ∈ carrier G*
using *fin f*
proof *induct*
case *empty* **show** *?case* **by** *simp*
next
case (*insert a A*)
then have *a*: *f a ∈ carrier G* **by** *fast*
from *insert* **have** *A*: *f ∈ A −> carrier G* **by** *fast*
from *insert A a* **show** *?case* **by** *simp*
qed

lemma *funcset-Int-left* [*simp*, *intro*]:
[| *f ∈ A −> C*; *f ∈ B −> C* |] ==> *f ∈ A Int B −> C*
by *fast*

lemma *funcset-Un-left* [*iff*]:
(*f ∈ A Un B −> C*) = (*f ∈ A −> C* & *f ∈ B −> C*)
by *fast*

lemma (**in** *comm-monoid*) *finprod-Un-Int*:
[| *finite A*; *finite B*; *g ∈ A −> carrier G*; *g ∈ B −> carrier G* |] ==>
finprod G g (*A Un B*) ⊗ *finprod G g* (*A Int B*) =
finprod G g A ⊗ *finprod G g B*
— The reversed orientation looks more natural, but LOOPS as a simprule!
proof (*induct set*: *finite*)
case *empty* **then show** *?case* **by** (*simp add*: *finprod-closed*)
next
case (*insert a A*)
then have *a*: *g a ∈ carrier G* **by** *fast*
from *insert* **have** *A*: *g ∈ A −> carrier G* **by** *fast*
from *insert A a* **show** *?case*
by (*simp add*: *m-ac Int-insert-left insert-absorb finprod-closed*
Int-mono2 Un-subset-iff)
qed

lemma (**in** *comm-monoid*) *finprod-Un-disjoint*:
[| *finite A*; *finite B*; *A Int B* = {};
g ∈ A −> carrier G; *g ∈ B −> carrier G* |]
==> *finprod G g* (*A Un B*) = *finprod G g A* ⊗ *finprod G g B*
apply (*subst finprod-Un-Int* [*symmetric*])
apply (*auto simp add*: *finprod-closed*)
done

lemma (**in** *comm-monoid*) *finprod-multf*:
[| *finite A*; *f ∈ A −> carrier G*; *g ∈ A −> carrier G* |] ==>
finprod G (%*x*. *f x* ⊗ *g x*) *A* = (*finprod G f A* ⊗ *finprod G g A*)
proof (*induct set*: *finite*)
case *empty* **show** *?case* **by** *simp*

```
next
  case (insert a A) then
  have fA: f ∈ A -> carrier G by fast
  from insert have fa: f a ∈ carrier G by fast
  from insert have gA: g ∈ A -> carrier G by fast
  from insert have ga: g a ∈ carrier G by fast
  from insert have fgA: (%x. f x ⊗ g x) ∈ A -> carrier G
    by (simp add: Pi-def)
  show ?case
    by (simp add: insert fA fa gA ga fgA m-ac)
qed

lemma (in comm-monoid) finprod-cong':
  [| A = B; g ∈ B -> carrier G;
      !!i. i ∈ B ==> f i = g i |] ==> finprod G f A = finprod G g B
proof -
  assume prems: A = B g ∈ B -> carrier G
    !!i. i ∈ B ==> f i = g i
  show ?thesis
  proof (cases finite B)
    case True
    then have !!A. [| A = B; g ∈ B -> carrier G;
      !!i. i ∈ B ==> f i = g i |] ==> finprod G f A = finprod G g B
    proof induct
      case empty thus ?case by simp
    next
      case (insert x B)
      then have finprod G f A = finprod G f (insert x B) by simp
      also from insert have ... = f x ⊗ finprod G f B
      proof (intro finprod-insert)
        show finite B by fact
      next
        show x ~: B by fact
      next
        assume x ~: B !!i. i ∈ insert x B ⟹ f i = g i
          g ∈ insert x B → carrier G
        thus f ∈ B -> carrier G by fastsimp
      next
        assume x ~: B !!i. i ∈ insert x B ⟹ f i = g i
          g ∈ insert x B → carrier G
        thus f x ∈ carrier G by fastsimp
      qed
      also from insert have ... = g x ⊗ finprod G g B by fastsimp
      also from insert have ... = finprod G g (insert x B)
      by (intro finprod-insert [THEN sym]) auto
      finally show ?case .
    qed
    with prems show ?thesis by simp
  next
```

```
    case False with prems show ?thesis by (simp add: finprod-def)
  qed
qed

lemma (in comm-monoid) finprod-cong:
  [| A = B; f ∈ B -> carrier G = True;
      !!i. i ∈ B ==> f i = g i |] ==> finprod G f A = finprod G g B

  by (rule finprod-cong') force+
```

Usually, if this rule causes a failed congruence proof error, the reason is that the premise $g \in B \to$ *carrier* G cannot be shown. Adding *Pi-def* to the simpset is often useful. For this reason, *comm-monoid.finprod-cong* is not added to the simpset by default.

```
declare funcsetI [rule del]
  funcset-mem [rule del]

lemma (in comm-monoid) finprod-0 [simp]:
  f ∈ {0::nat} -> carrier G ==> finprod G f {..0} = f 0
by (simp add: Pi-def)

lemma (in comm-monoid) finprod-Suc [simp]:
  f ∈ {..Suc n} -> carrier G ==>
   finprod G f {..Suc n} = (f (Suc n) ⊗ finprod G f {..n})
by (simp add: Pi-def atMost-Suc)

lemma (in comm-monoid) finprod-Suc2:
  f ∈ {..Suc n} -> carrier G ==>
   finprod G f {..Suc n} = (finprod G (%i. f (Suc i)) {..n} ⊗ f 0)
proof (induct n)
  case 0 thus ?case by (simp add: Pi-def)
next
  case Suc thus ?case by (simp add: m-assoc Pi-def)
qed

lemma (in comm-monoid) finprod-mult [simp]:
  [| f ∈ {..n} -> carrier G; g ∈ {..n} -> carrier G |] ==>
     finprod G (%i. f i ⊗ g i) {..n::nat} =
     finprod G f {..n} ⊗ finprod G g {..n}
  by (induct n) (simp-all add: m-ac Pi-def)

end
```

```
theory Exponent imports Main Primes Binomial begin
```

4 The Combinatorial Argument Underlying the First Sylow Theorem

definition *exponent* :: *nat* => *nat* => *nat* **where**
exponent p s == *if prime p then* (*GREATEST r*. *p* ^ *r dvd s*) *else 0*

4.1 Prime Theorems

lemma *prime-imp-one-less*: *prime p* ==> *Suc 0* < *p*
by (*unfold prime-def*, *force*)

lemma *prime-iff*:
(*prime p*) = (*Suc 0* < *p* & (∀ *a b*. *p dvd a*∗*b* −−> (*p dvd a*) | (*p dvd b*)))
apply (*auto simp add*: *prime-imp-one-less*)
apply (*blast dest*!: *prime-dvd-mult*)
apply (*auto simp add*: *prime-def*)
apply (*erule dvdE*)
apply (*case-tac k=0*, *simp*)
apply (*drule-tac x* = *m* **in** *spec*)
apply (*drule-tac x* = *k* **in** *spec*)
apply (*simp add*: *dvd-mult-cancel1 dvd-mult-cancel2*)
done

lemma *zero-less-prime-power*: *prime p* ==> *0* < *p* ^ *a*
by (*force simp add*: *prime-iff*)

lemma *zero-less-card-empty*: [| *finite S*; *S* ≠ {} |] ==> *0* < *card*(*S*)
by (*rule ccontr*, *simp*)

lemma *prime-dvd-cases*:
[| *p*∗*k dvd m*∗*n*; *prime p* |]
==> (∃ *x*. *k dvd x*∗*n* & *m* = *p*∗*x*) | (∃ *y*. *k dvd m*∗*y* & *n* = *p*∗*y*)
apply (*simp add*: *prime-iff*)
apply (*frule dvd-mult-left*)
apply (*subgoal-tac p dvd m* | *p dvd n*)
prefer *2* **apply** *blast*
apply (*erule disjE*)
apply (*rule disjI1*)
apply (*rule-tac* [*2*] *disjI2*)
apply (*erule-tac n* = *m* **in** *dvdE*)
apply (*erule-tac* [*2*] *n* = *n* **in** *dvdE*, *auto*)
done

lemma *prime-power-dvd-cases* [*rule-format* (*no-asm*)]: *prime p*
==> ∀ *m n*. *p* ^ *c dvd m*∗*n* −−>
(∀ *a b*. *a*+*b* = *Suc c* −−> *p* ^ *a dvd m* | *p* ^ *b dvd n*)

apply (*induct-tac c*)
 apply *clarify*
 apply (*case-tac a*)
  apply *simp*
 apply *simp*

apply *simp*
apply *clarify*
apply (*erule prime-dvd-cases* [*THEN disjE*], *assumption*, *auto*)

 apply (*case-tac a*)
  apply *simp*
 apply *clarify*
 apply (*drule spec*, *drule spec*, *erule* (*1*) *notE impE*)
 apply (*drule-tac x* = *nat* **in** *spec*)
 apply (*drule-tac x* = *b* **in** *spec*)
 apply *simp*

apply (*case-tac b*)
 apply *simp*
apply *clarify*
apply (*drule spec*, *drule spec*, *erule* (*1*) *notE impE*)
apply (*drule-tac x* = *a* **in** *spec*)
apply (*drule-tac x* = *nat* **in** *spec*, *simp*)
done

lemma *div-combine*:
  [| *prime p*; ~ (*p* ^ (*Suc r*) *dvd n*); *p*^(*a*+*r*) *dvd n*∗*k* |]
   ==> *p* ^ *a dvd k*
by (*drule-tac a* = *Suc r* **and** *b* = *a* **in** *prime-power-dvd-cases*, *assumption*, *auto*)

lemma *Suc-le-power*: *Suc 0* < *p* ==> *Suc n* <= *p*^*n*
apply (*induct-tac n*)
apply (*simp* (*no-asm-simp*))
apply *simp*
apply (*subgoal-tac 2* ∗ *n* + *2* <= *p* ∗ *p*^*n*, *simp*)
apply (*subgoal-tac 2* ∗ *p*^*n* <= *p* ∗ *p*^*n*)
apply *arith*
apply (*drule-tac k* = *2* **in** *mult-le-mono2*, *simp*)
done

lemma *power-dvd-bound*: [|*p*^*n dvd a*; *Suc 0* < *p*; *a* > *0*|] ==> *n* < *a*
apply (*drule dvd-imp-le*)
apply (*drule-tac* [*2*] *n* = *n* **in** *Suc-le-power*, *auto*)
done

4.2 Exponent Theorems

lemma *exponent-ge* [*rule-format*]:
[|*p^k dvd n*; *prime p*; *0<n*|] ==> *k* <= *exponent p n*
apply (*simp add*: *exponent-def*)
apply (*erule Greatest-le*)
apply (*blast dest*: *prime-imp-one-less power-dvd-bound*)
done

lemma *power-exponent-dvd*: *s>0* ==> (*p* ^ *exponent p s*) *dvd s*
apply (*simp add*: *exponent-def*)
apply *clarify*
apply (*rule-tac k = 0* **in** *GreatestI*)
prefer *2* **apply** (*blast dest*: *prime-imp-one-less power-dvd-bound*, *simp*)
done

lemma *power-Suc-exponent-Not-dvd*:
[|(*p* * *p* ^ *exponent p s*) *dvd s*; *prime p* |] ==> *s=0*
apply (*subgoal-tac p* ^ *Suc* (*exponent p s*) *dvd s*)
prefer *2* **apply** *simp*
apply (*rule ccontr*)
apply (*drule exponent-ge*, *auto*)
done

lemma *exponent-power-eq* [*simp*]: *prime p* ==> *exponent p* (*p^a*) = *a*
apply (*simp* (*no-asm-simp*) *add*: *exponent-def*)
apply (*rule Greatest-equality*, *simp*)
apply (*simp* (*no-asm-simp*) *add*: *prime-imp-one-less power-dvd-imp-le*)
done

lemma *exponent-equalityI*:
!*r*::*nat*. (*p^r dvd a*) = (*p^r dvd b*) ==> *exponent p a* = *exponent p b*
by (*simp* (*no-asm-simp*) *add*: *exponent-def*)

lemma *exponent-eq-0* [*simp*]: ¬ *prime p* ==> *exponent p s* = *0*
by (*simp* (*no-asm-simp*) *add*: *exponent-def*)

lemma *exponent-mult-add1*: [| *a* > *0*; *b* > *0* |]
==> (*exponent p a*) + (*exponent p b*) <= *exponent p* (*a* * *b*)
apply (*case-tac prime p*)
apply (*rule exponent-ge*)
apply (*auto simp add*: *power-add*)
apply (*blast intro*: *prime-imp-one-less power-exponent-dvd mult-dvd-mono*)
done

lemma *exponent-mult-add2*: [| *a* > *0*; *b* > *0* |]
==> *exponent p* (*a* * *b*) <= (*exponent p a*) + (*exponent p b*)

apply (*case-tac prime p*)
apply (*rule leI*, *clarify*)
apply (*cut-tac p = p* **and** *s = a*b* **in** *power-exponent-dvd*, *auto*)
apply (*subgoal-tac p ^ (Suc (exponent p a + exponent p b)) dvd a * b*)
apply (*rule-tac [2] le-imp-power-dvd [THEN dvd-trans]*)
prefer *3* **apply** *assumption*
prefer *2* **apply** *simp*
apply (*frule-tac a = Suc (exponent p a)* **and** *b = Suc (exponent p b)* **in** *prime-power-dvd-cases*)
apply (*assumption*, *force*, *simp*)
apply (*blast dest*: *power-Suc-exponent-Not-dvd*)
done

lemma *exponent-mult-add*: [| *a > 0*; *b > 0* |]
*==> exponent p (a * b) = (exponent p a) + (exponent p b)*
by (*blast intro*: *exponent-mult-add1 exponent-mult-add2 order-antisym*)

lemma *not-divides-exponent-0*: ~ (*p dvd n*) *==> exponent p n = 0*
apply (*case-tac exponent p n*, *simp*)
apply (*case-tac n*, *simp*)
apply (*cut-tac s = n* **and** *p = p* **in** *power-exponent-dvd*)
apply (*auto dest*: *dvd-mult-left*)
done

lemma *exponent-1-eq-0* [*simp*]: *exponent p (Suc 0) = 0*
apply (*case-tac prime p*)
apply (*auto simp add*: *prime-iff not-divides-exponent-0*)
done

4.3 Main Combinatorial Argument

lemma *le-extend-mult*: [| *c > 0*; *a <= b* |] *==> a <= b * (c::nat)*
apply (*rule-tac P = %x. x <= b * c* **in** *subst*)
apply (*rule mult-1-right*)
apply (*rule mult-le-mono*, *auto*)
done

lemma *p-fac-forw-lemma*:
[| *(m::nat) > 0*; *k > 0*; *k < p^a*; *(p^r) dvd (p^a)* m - k* |] *==> r <= a*
apply (*rule notnotD*)
apply (*rule notI*)
apply (*drule contrapos-nn* [*OF - leI*, *THEN notnotD*], *assumption*)
apply (*drule less-imp-le* [*of a*])
apply (*drule le-imp-power-dvd*)
apply (*drule-tac n = p ^ r* **in** *dvd-trans*, *assumption*)
apply(*metis dvd-diffD1 dvd-triv-right le-extend-mult linorder-linear linorder-not-less mult-commute nat-dvd-not-less neq0-conv*)
done

lemma *p-fac-forw*: [| (*m*::*nat*) > *0*; *k*>*0*; *k* < *p* ^ *a*; (*p* ^ *r*) *dvd* (*p* ^ *a*)∗ *m* − *k* |]
  ==> (*p* ^ *r*) *dvd* (*p* ^ *a*) − *k*
apply (*frule-tac k1* = *k* **and** *i* = *p* **in** *p-fac-forw-lemma* [*THEN le-imp-power-dvd*],
auto)
apply (*subgoal-tac p* ^ *r dvd p* ^ *a*∗*m*)
 prefer *2* **apply** (*blast intro*: *dvd-mult2*)
apply (*drule dvd-diffD1*)
  apply *assumption*
 prefer *2* **apply** (*blast intro*: *dvd-diff*)
apply (*drule gr0-implies-Suc*, *auto*)
done

lemma *r-le-a-forw*:
  [| (*k*::*nat*) > *0*; *k* < *p* ^ *a*; *p*>*0*; (*p* ^ *r*) *dvd* (*p* ^ *a*) − *k* |] ==> *r* <= *a*
by (*rule-tac m* = *Suc 0* **in** *p-fac-forw-lemma*, *auto*)

lemma *p-fac-backw*: [| *m*>*0*; *k*>*0*; (*p*::*nat*)≠*0*; *k* < *p* ^ *a*; (*p* ^ *r*) *dvd p* ^ *a* − *k* |]
  ==> (*p* ^ *r*) *dvd* (*p* ^ *a*)∗*m* − *k*
apply (*frule-tac k1* = *k* **and** *i* = *p* **in** *r-le-a-forw* [*THEN le-imp-power-dvd*], *auto*)
apply (*subgoal-tac p* ^ *r dvd p* ^ *a*∗*m*)
 prefer *2* **apply** (*blast intro*: *dvd-mult2*)
apply (*drule dvd-diffD1*)
  apply *assumption*
 prefer *2* **apply** (*blast intro*: *dvd-diff*)
apply (*drule less-imp-Suc-add*, *auto*)
done

lemma *exponent-p-a-m-k-equation*: [| *m*>*0*; *k*>*0*; (*p*::*nat*)≠*0*; *k* < *p* ^ *a* |]
  ==> *exponent p* (*p* ^ *a* ∗ *m* − *k*) = *exponent p* (*p* ^ *a* − *k*)
apply (*blast intro*: *exponent-equalityI p-fac-forw p-fac-backw*)
done

Suc rules that we have to delete from the simpset

lemmas *bad-Sucs* = *binomial-Suc-Suc mult-Suc mult-Suc-right*

lemma *p-not-div-choose-lemma* [*rule-format*]:
  [| ∀ *i*. *Suc i* < *K* −−> *exponent p* (*Suc i*) = *exponent p* (*Suc*(*j*+*i*))|]
   ==> *k*<*K* −−> *exponent p* ((*j*+*k*) *choose k*) = *0*
apply (*case-tac prime p*)
 prefer *2* **apply** *simp*
apply (*induct-tac k*)
apply (*simp* (*no-asm*))

apply (*subgoal-tac* (*Suc* (*j*+*n*) *choose Suc n*) > *0*)
 prefer *2* **apply** (*simp add*: *zero-less-binomial-iff*, *clarify*)
apply (*subgoal-tac exponent p* ((*Suc* (*j*+*n*) *choose Suc n*) ∗ *Suc n*) =

exponent p (Suc n))

First, use the assumed equation. We simplify the LHS to *exponent p (Suc (j + n) choose Suc n) + exponent p (Suc n)* the common terms cancel, proving the conclusion.

apply (*simp del*: *bad-Sucs add*: *exponent-mult-add*)

Establishing the equation requires first applying *Suc-times-binomial-eq* ...

apply (*simp del*: *bad-Sucs add*: *Suc-times-binomial-eq* [*symmetric*])

...then *exponent-mult-add* and the quantified premise.

apply (*simp del*: *bad-Sucs add*: *zero-less-binomial-iff exponent-mult-add*)
done

lemma *p-not-div-choose*:
[| *k*<*K*; *k*<=*n*;
∀*j*. *0*<*j* & *j*<*K* −−> *exponent p (n − k + (K − j)) = exponent p (K − j)*|]
==> *exponent p (n choose k) = 0*
apply (*cut-tac j = n*−*k* **and** *k = k* **and** *p = p* **in** *p-not-div-choose-lemma*)
prefer *3* **apply** *simp*
prefer *2* **apply** *assumption*
apply (*drule-tac x = K − Suc i* **in** *spec*)
apply (*simp add*: *Suc-diff-le*)
done

lemma *const-p-fac-right*:
m>*0* ==> *exponent p ((p^a * m − Suc 0) choose (p^a − Suc 0)) = 0*
apply (*case-tac prime p*)
prefer *2* **apply** *simp*
apply (*frule-tac a = a* **in** *zero-less-prime-power*)
apply (*rule-tac K = p^a* **in** *p-not-div-choose*)
apply *simp*
apply *simp*
apply (*case-tac m*)
apply (*case-tac* [*2*] *p^a*)
apply *auto*

apply (*subgoal-tac 0*<*p*)
prefer *2* **apply** (*force dest*!: *prime-imp-one-less*)
apply (*subst exponent-p-a-m-k-equation*, *auto*)
done

lemma *const-p-fac*:
m>*0* ==> *exponent p (((p^a) * m) choose p^a) = exponent p m*
apply (*case-tac prime p*)
prefer *2* **apply** *simp*

apply (*subgoal-tac* $0 < p\hat{}a * m$ & $p\hat{}a <= p\hat{}a * m$)
prefer *2* **apply** (*force simp add*: *prime-iff*)

A similar trick to the one used in *p-not-div-choose-lemma*: insert an equation; use *exponent-mult-add* on the LHS; on the RHS, first transform the binomial coefficient, then use *exponent-mult-add*.

apply (*subgoal-tac exponent p* ((($p\hat{}a$) $*$ *m*) *choose* $p\hat{}a$) $*$ $p\hat{}a$) =
a + *exponent p m*)
apply (*simp del*: *bad-Sucs add*: *zero-less-binomial-iff exponent-mult-add prime-iff*)

one subgoal left!

apply (*subst times-binomial-minus1-eq*, *simp*, *simp*)
apply (*subst exponent-mult-add*, *simp*)
apply (*simp* (*no-asm-simp*) *add*: *zero-less-binomial-iff*)
apply *arith*
apply (*simp del*: *bad-Sucs add*: *exponent-mult-add const-p-fac-right*)
done

end

theory *Coset* **imports** *Group Exponent* **begin**

5 Cosets and Quotient Groups

constdefs (**structure** *G*)
r-coset :: [-, 'a *set*, 'a] ⇒ 'a *set* (**infixl** #>$_1$ *60*)
H #> $a \equiv \bigcup h \in H.\ \{h \otimes a\}$

l-coset :: [-, 'a, 'a *set*] ⇒ 'a *set* (**infixl** <#$_1$ *60*)
a <# $H \equiv \bigcup h \in H.\ \{a \otimes h\}$

RCOSETS :: [-, 'a *set*] ⇒ ('a *set*)*set* (*rcosets*$_1$ - [*81*] *80*)
rcosets $H \equiv \bigcup a \in$ *carrier* $G.\ \{H$ #> $a\}$

set-mult :: [-, 'a *set* ,'a *set*] ⇒ 'a *set* (**infixl** <#>$_1$ *60*)
H <#> $K \equiv \bigcup h \in H.\ \bigcup k \in K.\ \{h \otimes k\}$

SET-INV :: [-,'a *set*] ⇒ 'a *set* (*set'-inv*$_1$ - [*81*] *80*)
set-inv $H \equiv \bigcup h \in H.\ \{inv\ h\}$

locale *normal* = *subgroup* + *group* +
assumes *coset-eq*: ($\forall x \in$ *carrier* $G.\ H$ #> $x = x$ <# H)

abbreviation
normal-rel :: ['a *set*, ('a, 'b) *monoid-scheme*] ⇒ *bool* (**infixl** ◁ *60*) **where**

$H \lhd G \equiv$ *normal H G*

5.1 Basic Properties of Cosets

lemma (**in** *group*) *coset-mult-assoc*:
$[|\ M \subseteq$ *carrier* $G;\ g \in$ *carrier* $G;\ h \in$ *carrier* $G\ |]$
$\Longrightarrow (M \#> g) \#> h = M \#> (g \otimes h)$
by (*force simp add*: *r-coset-def m-assoc*)

lemma (**in** *group*) *coset-mult-one* [*simp*]: $M \subseteq$ *carrier* $G \Longrightarrow M \#> \mathbf{1} = M$
by (*force simp add*: *r-coset-def*)

lemma (**in** *group*) *coset-mult-inv1*:
$[|\ M \#> (x \otimes (inv\ y)) = M;\ \ x \in$ *carrier* $G\ ;\ y \in$ *carrier* $G;$
$M \subseteq$ *carrier* $G\ |] \Longrightarrow M \#> x = M \#> y$
apply (*erule subst* [*of concl*: %*z*. $M \#> x = z \#> y$])
apply (*simp add*: *coset-mult-assoc m-assoc*)
done

lemma (**in** *group*) *coset-mult-inv2*:
$[|\ M \#> x = M \#> y;\ \ x \in$ *carrier* $G;\ \ y \in$ *carrier* $G;\ \ M \subseteq$ *carrier* $G\ |]$
$\Longrightarrow M \#> (x \otimes (inv\ y)) = M$
apply (*simp add*: *coset-mult-assoc* [*symmetric*])
apply (*simp add*: *coset-mult-assoc*)
done

lemma (**in** *group*) *coset-join1*:
$[|\ H \#> x = H;\ \ x \in$ *carrier* $G;\ \ subgroup\ H\ G\ |] \Longrightarrow x \in H$
apply (*erule subst*)
apply (*simp add*: *r-coset-def*)
apply (*blast intro*: *l-one subgroup.one-closed sym*)
done

lemma (**in** *group*) *solve-equation*:
$[\![subgroup\ H\ G;\ x \in H;\ y \in H]\!] \Longrightarrow \exists h \in H.\ y = h \otimes x$
apply (*rule bexI* [*of* - $y \otimes (inv\ x)$])
apply (*auto simp add*: *subgroup.m-closed subgroup.m-inv-closed m-assoc*
subgroup.subset [*THEN subsetD*])
done

lemma (**in** *group*) *repr-independence*:
$[\![y \in H \#> x;\ \ x \in$ *carrier* G; *subgroup* $H\ G]\!] \Longrightarrow H \#> x = H \#> y$
by (*auto simp add*: *r-coset-def m-assoc* [*symmetric*]
subgroup.subset [*THEN subsetD*]
subgroup.m-closed solve-equation)

lemma (**in** *group*) *coset-join2*:
$[\![x \in$ *carrier* $G;\ \ subgroup\ H\ G;\ \ x \in H]\!] \Longrightarrow H \#> x = H$
— Alternative proof is to put $x = \mathbf{1}$ in *repr-independence*.

by (*force simp add*: *subgroup.m-closed r-coset-def solve-equation*)

lemma (**in** *monoid*) *r-coset-subset-G*:
  [| $H \subseteq$ *carrier G*; $x \in$ *carrier G* |] ==> H #> $x \subseteq$ *carrier G*
by (*auto simp add*: *r-coset-def*)

lemma (**in** *group*) *rcosI*:
  [| $h \in H$; $H \subseteq$ *carrier G*; $x \in$ *carrier G*|] ==> $h \otimes x \in H$ #> x
by (*auto simp add*: *r-coset-def*)

lemma (**in** *group*) *rcosetsI*:
  ⟦$H \subseteq$ *carrier G*; $x \in$ *carrier G*⟧ ⟹ H #> $x \in$ *rcosets H*
by (*auto simp add*: *RCOSETS-def*)

Really needed?

lemma (**in** *group*) *transpose-inv*:
  [| $x \otimes y = z$; $x \in$ *carrier G*; $y \in$ *carrier G*; $z \in$ *carrier G* |]
   ==> (*inv x*) $\otimes z = y$
by (*force simp add*: *m-assoc* [*symmetric*])

lemma (**in** *group*) *rcos-self*: [| $x \in$ *carrier G*; *subgroup H G* |] ==> $x \in H$ #> x
apply (*simp add*: *r-coset-def*)
apply (*blast intro*: *sym l-one subgroup.subset* [*THEN subsetD*]
            subgroup.one-closed)
done

Opposite of *repr-independence*

lemma (**in** *group*) *repr-independenceD*:
 includes *subgroup H G*
 assumes *ycarr*: $y \in$ *carrier G*
   and *repr*: H #> $x = H$ #> y
 shows $y \in H$ #> x
 apply (*subst repr*)
 apply (*intro rcos-self*)
  apply (*rule ycarr*)
  apply (*rule is-subgroup*)
 done

Elements of a right coset are in the carrier

lemma (**in** *subgroup*) *elemrcos-carrier*:
 includes *group*
 assumes *acarr*: $a \in$ *carrier G*
  and a': $a' \in H$ #> a
 shows $a' \in$ *carrier G*
proof −
 from *subset* **and** *acarr*
 have H #> $a \subseteq$ *carrier G* **by** (*rule r-coset-subset-G*)
 from *this* **and** a'
 show $a' \in$ *carrier G*

```
    by fast
qed

lemma (in subgroup) rcos-const:
  includes group
  assumes hH: h ∈ H
  shows H #> h = H
  apply (unfold r-coset-def)
  apply rule
   apply rule
   apply clarsimp
   apply (intro subgroup.m-closed)
     apply (rule is-subgroup)
    apply assumption
   apply (rule hH)
  apply rule
  apply simp
proof −
  fix h'
  assume h'H: h' ∈ H
  note carr = hH[THEN mem-carrier] h'H[THEN mem-carrier]
  from carr
  have a: h' = (h' ⊗ inv h) ⊗ h by (simp add: m-assoc)
  from h'H hH
  have h' ⊗ inv h ∈ H by simp
  from this and a
  show ∃ x∈H. h' = x ⊗ h by fast
qed
```

Step one for lemma *rcos-module*

```
lemma (in subgroup) rcos-module-imp:
  includes group
  assumes xcarr: x ∈ carrier G
      and x'cos: x' ∈ H #> x
  shows (x' ⊗ inv x) ∈ H
proof −
  from xcarr x'cos
      have x'carr: x' ∈ carrier G
      by (rule elemrcos-carrier[OF is-group])
  from xcarr
      have ixcarr: inv x ∈ carrier G
      by simp
  from x'cos
      have ∃ h∈H. x' = h ⊗ x
      unfolding r-coset-def
      by fast
  from this
      obtain h
        where hH: h ∈ H
```

```
      and x': x' = h ⊗ x
    by auto
  from hH and subset
    have hcarr: h ∈ carrier G by fast
  note carr = xcarr x'carr hcarr
  from x' and carr
    have x' ⊗ (inv x) = (h ⊗ x) ⊗ (inv x) by fast
  also from carr
    have ... = h ⊗ (x ⊗ inv x) by (simp add: m-assoc)
  also from carr
    have ... = h ⊗ 1 by simp
  also from carr
    have ... = h by simp
  finally
    have x' ⊗ (inv x) = h by simp
  from hH this
    show x' ⊗ (inv x) ∈ H by simp
qed
```

Step two for lemma *rcos-module*

```
lemma (in subgroup) rcos-module-rev:
  includes group
  assumes carr: x ∈ carrier G x' ∈ carrier G
    and xixH: (x' ⊗ inv x) ∈ H
  shows x' ∈ H #> x
proof −
  from xixH
    have ∃h∈H. x' ⊗ (inv x) = h by fast
  from this
    obtain h
      where hH: h ∈ H
      and hsym: x' ⊗ (inv x) = h
    by fast
  from hH subset have hcarr: h ∈ carrier G by simp
  note carr = carr hcarr
  from hsym[symmetric] have h ⊗ x = x' ⊗ (inv x) ⊗ x by fast
  also from carr
    have ... = x' ⊗ ((inv x) ⊗ x) by (simp add: m-assoc)
  also from carr
    have ... = x' ⊗ 1 by (simp add: l-inv)
  also from carr
    have ... = x' by simp
  finally
    have h ⊗ x = x' by simp
  from this[symmetric] and hH
    show x' ∈ H #> x
    unfolding r-coset-def
    by fast
qed
```

Module property of right cosets

lemma (**in** *subgroup*) *rcos-module*:
includes *group*
assumes *carr*: $x \in$ *carrier* G $x' \in$ *carrier* G
shows $(x' \in H \#> x) = (x' \otimes inv\ x \in H)$
proof
assume $x' \in H \#> x$
from *this* **and** *carr*
show $x' \otimes inv\ x \in H$
by (*intro rcos-module-imp*[*OF is-group*])
next
assume $x' \otimes inv\ x \in H$
from *this* **and** *carr*
show $x' \in H \#> x$
by (*intro rcos-module-rev*[*OF is-group*])
qed

Right cosets are subsets of the carrier.

lemma (**in** *subgroup*) *rcosets-carrier*:
includes *group*
assumes *XH*: $X \in$ *rcosets* H
shows $X \subseteq$ *carrier* G
proof −
from *XH* **have** $\exists x \in$ *carrier* G. $X = H \#> x$
unfolding *RCOSETS-def*
by *fast*
from *this*
obtain x
where *xcarr*: $x \in$ *carrier* G
and X: $X = H \#> x$
by *fast*
from *subset* **and** *xcarr*
show $X \subseteq$ *carrier* G
unfolding X
by (*rule r-coset-subset-G*)
qed

Multiplication of general subsets

lemma (**in** *monoid*) *set-mult-closed*:
assumes *Acarr*: $A \subseteq$ *carrier* G
and *Bcarr*: $B \subseteq$ *carrier* G
shows $A <\#> B \subseteq$ *carrier* G
apply *rule* **apply** (*simp add*: *set-mult-def*, *clarsimp*)
proof −
fix a b
assume $a \in A$
from *this* **and** *Acarr*
have *acarr*: $a \in$ *carrier* G **by** *fast*

```
  assume b ∈ B
  from this and Bcarr
      have bcarr: b ∈ carrier G by fast

  from acarr bcarr
      show a ⊗ b ∈ carrier G by (rule m-closed)
qed

lemma (in comm-group) mult-subgroups:
  assumes subH: subgroup H G
      and subK: subgroup K G
  shows subgroup (H <#> K) G
apply (rule subgroup.intro)
  apply (intro set-mult-closed subgroup.subset[OF subH] subgroup.subset[OF subK])
  apply (simp add: set-mult-def) apply clarsimp defer 1
  apply (simp add: set-mult-def) defer 1
  apply (simp add: set-mult-def, clarsimp) defer 1
proof −
  fix ha hb ka kb
  assume haH: ha ∈ H and hbH: hb ∈ H and kaK: ka ∈ K and kbK: kb ∈ K
 note carr = haH[THEN subgroup.mem-carrier[OF subH]] hbH[THEN subgroup.mem-carrier[OF
subH]]
         kaK[THEN subgroup.mem-carrier[OF subK]] kbK[THEN subgroup.mem-carrier[OF
subK]]
  from carr
     have (ha ⊗ ka) ⊗ (hb ⊗ kb) = ha ⊗ (ka ⊗ hb) ⊗ kb by (simp add: m-assoc)
  also from carr
      have ... = ha ⊗ (hb ⊗ ka) ⊗ kb by (simp add: m-comm)
  also from carr
      have ... = (ha ⊗ hb) ⊗ (ka ⊗ kb) by (simp add: m-assoc)
  finally
      have eq: (ha ⊗ ka) ⊗ (hb ⊗ kb) = (ha ⊗ hb) ⊗ (ka ⊗ kb) .

  from haH hbH have hH: ha ⊗ hb ∈ H by (simp add: subgroup.m-closed[OF
subH])
  from kaK kbK have kK: ka ⊗ kb ∈ K by (simp add: subgroup.m-closed[OF
subK])

  from hH and kK and eq
      show ∃h′∈H. ∃k′∈K. (ha ⊗ ka) ⊗ (hb ⊗ kb) = h′ ⊗ k′ by fast
next
  have 1 = 1 ⊗ 1 by simp
  from subgroup.one-closed[OF subH] subgroup.one-closed[OF subK] this
      show ∃h∈H. ∃k∈K. 1 = h ⊗ k by fast
next
  fix h k
  assume hH: h ∈ H
     and kK: k ∈ K
```

from *hH*[*THEN subgroup.mem-carrier*[*OF subH*]] *kK*[*THEN subgroup.mem-carrier*[*OF subK*]]
have *inv* $(h \otimes k) = inv\ h \otimes inv\ k$ **by** (*simp add*: *inv-mult-group m-comm*)

from *subgroup.m-inv-closed*[*OF subH hH*] **and** *subgroup.m-inv-closed*[*OF subK kK*] **and** *this*
show $\exists ha{\in}H.\ \exists ka{\in}K.\ inv\ (h \otimes k) = ha \otimes ka$ **by** *fast*
qed

lemma (**in** *subgroup*) *lcos-module-rev*:
includes *group*
assumes *carr*: $x \in carrier\ G\ x' \in carrier\ G$
and *xixH*: $(inv\ x \otimes x') \in H$
shows $x' \in x <\# H$
proof −
from *xixH*
have $\exists h{\in}H.\ (inv\ x) \otimes x' = h$ **by** *fast*
from *this*
obtain *h*
where *hH*: $h \in H$
and *hsym*: $(inv\ x) \otimes x' = h$
by *fast*

from *hH subset* **have** *hcarr*: $h \in carrier\ G$ **by** *simp*
note *carr* = *carr hcarr*
from *hsym*[*symmetric*] **have** $x \otimes h = x \otimes ((inv\ x) \otimes x')$ **by** *fast*
also from *carr*
have $\ldots = (x \otimes (inv\ x)) \otimes x'$ **by** (*simp add*: *m-assoc*[*symmetric*])
also from *carr*
have $\ldots = \mathbf{1} \otimes x'$ **by** *simp*
also from *carr*
have $\ldots = x'$ **by** *simp*
finally
have $x \otimes h = x'$ **by** *simp*

from *this*[*symmetric*] **and** *hH*
show $x' \in x <\# H$
unfolding *l-coset-def*
by *fast*
qed

5.2 Normal subgroups

lemma *normal-imp-subgroup*: $H \lhd G \Longrightarrow subgroup\ H\ G$
by (*simp add*: *normal-def subgroup-def*)

lemma (**in** *group*) *normalI*:
$subgroup\ H\ G \Longrightarrow (\forall x \in carrier\ G.\ H\ \#> x = x <\# H) \Longrightarrow H \lhd G$
by (*simp add*: *normal-def normal-axioms-def prems*)

lemma (**in** *normal*) *inv-op-closed1*:
  $[\![x \in carrier\ G;\ h \in H]\!] \Longrightarrow (inv\ x) \otimes h \otimes x \in H$
apply (*insert coset-eq*)
apply (*auto simp add*: *l-coset-def r-coset-def*)
apply (*drule bspec*, *assumption*)
apply (*drule equalityD1* [*THEN subsetD*], *blast*, *clarify*)
apply (*simp add*: *m-assoc*)
apply (*simp add*: *m-assoc* [*symmetric*])
done

lemma (**in** *normal*) *inv-op-closed2*:
  $[\![x \in carrier\ G;\ h \in H]\!] \Longrightarrow x \otimes h \otimes (inv\ x) \in H$
apply (*subgoal-tac inv* $(inv\ x) \otimes h \otimes (inv\ x) \in H$)
apply (*simp add*:)
apply (*blast intro*: *inv-op-closed1*)
done

Alternative characterization of normal subgroups

lemma (**in** *group*) *normal-inv-iff*:
  $(N \lhd G) =$
   (*subgroup N G* & $(\forall x \in carrier\ G.\ \forall h \in N.\ x \otimes h \otimes (inv\ x) \in N)$)
   (**is** - = *?rhs*)
proof
 assume *N*: $N \lhd G$
 show *?rhs*
  by (*blast intro*: *N normal.inv-op-closed2 normal-imp-subgroup*)
next
 assume *?rhs*
 hence *sg*: *subgroup N G*
  and *closed*: $\bigwedge x.\ x \in carrier\ G \Longrightarrow \forall h \in N.\ x \otimes h \otimes inv\ x \in N$ **by** *auto*
 hence *sb*: $N \subseteq carrier\ G$ **by** (*simp add*: *subgroup.subset*)
 show $N \lhd G$
 proof (*intro normalI* [*OF sg*], *simp add*: *l-coset-def r-coset-def*, *clarify*)
  fix *x*
  assume *x*: $x \in carrier\ G$
  show $(\bigcup h \in N.\ \{h \otimes x\}) = (\bigcup h \in N.\ \{x \otimes h\})$
  proof
   show $(\bigcup h \in N.\ \{h \otimes x\}) \subseteq (\bigcup h \in N.\ \{x \otimes h\})$
   proof *clarify*
    fix *n*
    assume *n*: $n \in N$
    show $n \otimes x \in (\bigcup h \in N.\ \{x \otimes h\})$
    proof
     from *closed* [*of inv x*]
     show $inv\ x \otimes n \otimes x \in N$ **by** (*simp add*: *x n*)
     show $n \otimes x \in \{x \otimes (inv\ x \otimes n \otimes x)\}$
      by (*simp add*: *x n m-assoc* [*symmetric*] *sb* [*THEN subsetD*])
    qed

```
    qed
  next
    show (⋃h∈N. {x ⊗ h}) ⊆ (⋃h∈N. {h ⊗ x})
    proof clarify
      fix n
      assume n: n ∈ N
      show x ⊗ n ∈ (⋃h∈N. {h ⊗ x})
      proof
        show x ⊗ n ⊗ inv x ∈ N by (simp add: x n closed)
        show x ⊗ n ∈ {x ⊗ n ⊗ inv x ⊗ x}
          by (simp add: x n m-assoc sb [THEN subsetD])
      qed
    qed
  qed
qed
```

5.3 More Properties of Cosets

```
lemma (in group) lcos-m-assoc:
    [| M ⊆ carrier G; g ∈ carrier G; h ∈ carrier G |]
     ==> g <# (h <# M) = (g ⊗ h) <# M
by (force simp add: l-coset-def m-assoc)

lemma (in group) lcos-mult-one: M ⊆ carrier G ==> 1 <# M = M
by (force simp add: l-coset-def)

lemma (in group) l-coset-subset-G:
    [| H ⊆ carrier G; x ∈ carrier G |] ==> x <# H ⊆ carrier G
by (auto simp add: l-coset-def subsetD)

lemma (in group) l-coset-swap:
    ⟦y ∈ x <# H;  x ∈ carrier G;  subgroup H G⟧ ⟹ x ∈ y <# H
proof (simp add: l-coset-def)
  assume ∃h∈H. y = x ⊗ h
    and x: x ∈ carrier G
    and sb: subgroup H G
  then obtain h' where h': h' ∈ H & x ⊗ h' = y by blast
  show ∃h∈H. x = y ⊗ h
  proof
    show x = y ⊗ inv h' using h' x sb
      by (auto simp add: m-assoc subgroup.subset [THEN subsetD])
    show inv h' ∈ H using h' sb
      by (auto simp add: subgroup.subset [THEN subsetD] subgroup.m-inv-closed)
  qed
qed

lemma (in group) l-coset-carrier:
    [| y ∈ x <# H;  x ∈ carrier G;  subgroup H G |] ==> y ∈ carrier G
```

by (*auto simp add*: *l-coset-def m-assoc*
subgroup.subset [*THEN subsetD*] *subgroup.m-closed*)

lemma (**in** *group*) *l-repr-imp-subset*:
assumes *y*: $y \in x <\# H$ **and** *x*: $x \in carrier\ G$ **and** *sb*: *subgroup H G*
shows $y <\# H \subseteq x <\# H$
proof −
from *y*
obtain h' **where** $h' \in H\ x \otimes h' = y$ **by** (*auto simp add*: *l-coset-def*)
thus *?thesis* **using** *x sb*
by (*auto simp add*: *l-coset-def m-assoc*
subgroup.subset [*THEN subsetD*] *subgroup.m-closed*)
qed

lemma (**in** *group*) *l-repr-independence*:
assumes *y*: $y \in x <\# H$ **and** *x*: $x \in carrier\ G$ **and** *sb*: *subgroup H G*
shows $x <\# H = y <\# H$
proof
show $x <\# H \subseteq y <\# H$
by (*rule l-repr-imp-subset*,
(*blast intro*: *l-coset-swap l-coset-carrier y x sb*)+)
show $y <\# H \subseteq x <\# H$ **by** (*rule l-repr-imp-subset* [*OF y x sb*])
qed

lemma (**in** *group*) *setmult-subset-G*:
$[\![H \subseteq carrier\ G; K \subseteq carrier\ G]\!] \Longrightarrow H <\#> K \subseteq carrier\ G$
by (*auto simp add*: *set-mult-def subsetD*)

lemma (**in** *group*) *subgroup-mult-id*: *subgroup H G* $\Longrightarrow H <\#> H = H$
apply (*auto simp add*: *subgroup.m-closed set-mult-def Sigma-def image-def*)
apply (*rule-tac* $x = x$ **in** *bexI*)
apply (*rule bexI* [*of* - **1**])
apply (*auto simp add*: *subgroup.m-closed subgroup.one-closed*
r-one subgroup.subset [*THEN subsetD*])
done

5.3.1 Set of Inverses of an *r-coset*.

lemma (**in** *normal*) *rcos-inv*:
assumes *x*: $x \in carrier\ G$
shows *set-inv* $(H\ \#>\ x) = H\ \#>\ (inv\ x)$
proof (*simp add*: *r-coset-def SET-INV-def x inv-mult-group*, *safe*)
fix *h*
assume $h \in H$
show $inv\ x \otimes inv\ h \in (\bigcup j \in H.\ \{j \otimes inv\ x\})$
proof
show $inv\ x \otimes inv\ h \otimes x \in H$
by (*simp add*: *inv-op-closed1 prems*)
show $inv\ x \otimes inv\ h \in \{inv\ x \otimes inv\ h \otimes x \otimes inv\ x\}$

by (*simp add*: *prems m-assoc*)
qed
next
fix *h*
assume $h \in H$
show $h \otimes inv\ x \in (\bigcup j \in H.\ \{inv\ x \otimes inv\ j\})$
proof
show $x \otimes inv\ h \otimes inv\ x \in H$
by (*simp add*: *inv-op-closed2 prems*)
show $h \otimes inv\ x \in \{inv\ x \otimes inv\ (x \otimes inv\ h \otimes inv\ x)\}$
by (*simp add*: *prems m-assoc* [*symmetric*] *inv-mult-group*)
qed
qed

5.3.2 Theorems for <#> with #> or <#.

lemma (**in** *group*) *setmult-rcos-assoc*:
$[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G;\ x \in carrier\ G]\!]$
$\Longrightarrow H <\#> (K\ \#> x) = (H <\#> K)\ \#> x$
by (*force simp add*: *r-coset-def set-mult-def m-assoc*)

lemma (**in** *group*) *rcos-assoc-lcos*:
$[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G;\ x \in carrier\ G]\!]$
$\Longrightarrow (H\ \#> x) <\#> K = H <\#> (x <\#\ K)$
by (*force simp add*: *r-coset-def l-coset-def set-mult-def m-assoc*)

lemma (**in** *normal*) *rcos-mult-step1*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H\ \#> x) <\#> (H\ \#> y) = (H <\#> (x <\#\ H))\ \#> y$
by (*simp add*: *setmult-rcos-assoc subset*
r-coset-subset-G l-coset-subset-G rcos-assoc-lcos)

lemma (**in** *normal*) *rcos-mult-step2*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H <\#> (x <\#\ H))\ \#> y = (H <\#> (H\ \#> x))\ \#> y$
by (*insert coset-eq*, *simp add*: *normal-def*)

lemma (**in** *normal*) *rcos-mult-step3*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H <\#> (H\ \#> x))\ \#> y = H\ \#> (x \otimes y)$
by (*simp add*: *setmult-rcos-assoc coset-mult-assoc*
subgroup-mult-id normal.axioms subset prems)

lemma (**in** *normal*) *rcos-sum*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H\ \#> x) <\#> (H\ \#> y) = H\ \#> (x \otimes y)$
by (*simp add*: *rcos-mult-step1 rcos-mult-step2 rcos-mult-step3*)

lemma (**in** *normal*) *rcosets-mult-eq*: $M \in rcosets\ H \Longrightarrow H <\#> M = M$

— generalizes *subgroup-mult-id*
by (*auto simp add*: *RCOSETS-def subset*
setmult-rcos-assoc subgroup-mult-id normal.axioms prems)

5.3.3 An Equivalence Relation

constdefs (**structure** *G*)
r-congruent :: [('*a*,'*b*)*monoid-scheme*, '*a set*] ⇒ ('*a*∗'*a*)*set*
(*rcong*ı -)
rcong H ≡ {(*x*,*y*). *x* ∈ *carrier G* & *y* ∈ *carrier G* & *inv x* ⊗ *y* ∈ *H*}

lemma (**in** *subgroup*) *equiv-rcong*:
includes *group G*
shows *equiv* (*carrier G*) (*rcong H*)
proof (*intro equiv.intro*)
show *refl* (*carrier G*) (*rcong H*)
by (*auto simp add*: *r-congruent-def refl-def*)
next
show *sym* (*rcong H*)
proof (*simp add*: *r-congruent-def sym-def*, *clarify*)
fix *x y*
assume [*simp*]: *x* ∈ *carrier G y* ∈ *carrier G*
and *inv x* ⊗ *y* ∈ *H*
hence *inv* (*inv x* ⊗ *y*) ∈ *H* **by** (*simp add*: *m-inv-closed*)
thus *inv y* ⊗ *x* ∈ *H* **by** (*simp add*: *inv-mult-group*)
qed
next
show *trans* (*rcong H*)
proof (*simp add*: *r-congruent-def trans-def*, *clarify*)
fix *x y z*
assume [*simp*]: *x* ∈ *carrier G y* ∈ *carrier G z* ∈ *carrier G*
and *inv x* ⊗ *y* ∈ *H* **and** *inv y* ⊗ *z* ∈ *H*
hence (*inv x* ⊗ *y*) ⊗ (*inv y* ⊗ *z*) ∈ *H* **by** *simp*
hence *inv x* ⊗ (*y* ⊗ *inv y*) ⊗ *z* ∈ *H* **by** (*simp add*: *m-assoc del*: *r-inv*)
thus *inv x* ⊗ *z* ∈ *H* **by** *simp*
qed
qed

Equivalence classes of *rcong* correspond to left cosets. Was there a mistake in the definitions? I'd have expected them to correspond to right cosets.

lemma (**in** *subgroup*) *l-coset-eq-rcong*:
includes *group G*
assumes *a*: *a* ∈ *carrier G*
shows *a* <# *H* = *rcong H* `` {*a*}
by (*force simp add*: *r-congruent-def l-coset-def m-assoc* [*symmetric*] *a*)

5.3.4 Two Distinct Right Cosets are Disjoint

lemma (**in** *group*) *rcos-equation*:

includes *subgroup H G*
shows
$[\![ha \otimes a = h \otimes b;\ a \in carrier\ G;\ \ b \in carrier\ G;$
$h \in H;\ \ ha \in H;\ \ hb \in H]\!]$
$\Longrightarrow hb \otimes a \in (\bigcup h{\in}H.\ \{h \otimes b\})$
apply (*rule UN-I* [*of* $hb \otimes ((inv\ ha) \otimes h)$])
apply (*simp add*:)
apply (*simp add*: *m-assoc transpose-inv*)
done

lemma (**in** *group*) *rcos-disjoint*:
includes *subgroup H G*
shows $[\![a \in rcosets\ H;\ b \in rcosets\ H;\ a{\neq}b]\!] \Longrightarrow a \cap b = \{\}$
apply (*simp add*: *RCOSETS-def r-coset-def*)
apply (*blast intro*: *rcos-equation prems sym*)
done

5.4 Further lemmas for *r-congruent*

The relation is a congruence

lemma (**in** *normal*) *congruent-rcong*:
shows *congruent2* (*rcong H*) (*rcong H*) ($\lambda a\ b.\ a \otimes b <\# H$)
proof (*intro congruent2I*[*of carrier G - carrier G -*] *equiv-rcong is-group*)
fix *a b c*
assume *abrcong*: $(a, b) \in rcong\ H$
and *ccarr*: $c \in carrier\ G$

from *abrcong*
have *acarr*: $a \in carrier\ G$
and *bcarr*: $b \in carrier\ G$
and *abH*: $inv\ a \otimes b \in H$
unfolding *r-congruent-def*
by *fast+*

note $carr = acarr\ bcarr\ ccarr$

from *ccarr* **and** *abH*
have $inv\ c \otimes (inv\ a \otimes b) \otimes c \in H$ **by** (*rule inv-op-closed1*)
moreover
from *carr* **and** *inv-closed*
have $inv\ c \otimes (inv\ a \otimes b) \otimes c = (inv\ c \otimes inv\ a) \otimes (b \otimes c)$
by (*force cong*: *m-assoc*)
moreover
from *carr* **and** *inv-closed*
have $\ldots = (inv\ (a \otimes c)) \otimes (b \otimes c)$
by (*simp add*: *inv-mult-group*)
ultimately
have $(inv\ (a \otimes c)) \otimes (b \otimes c) \in H$ **by** *simp*
from *carr* **and** *this*

```
    have (b ⊗ c) ∈ (a ⊗ c) <# H
    by (simp add: lcos-module-rev[OF is-group])
  from carr and this and is-subgroup
    show (a ⊗ c) <# H = (b ⊗ c) <# H by (intro l-repr-independence, simp+)
next
  fix a b c
  assume abrcong: (a, b) ∈ rcong H
    and ccarr: c ∈ carrier G

  from ccarr have c ∈ Units G by (simp add: Units-eq)
  hence cinvc-one: inv c ⊗ c = 1 by (rule Units-l-inv)

  from abrcong
      have acarr: a ∈ carrier G
       and bcarr: b ∈ carrier G
       and abH: inv a ⊗ b ∈ H
      by (unfold r-congruent-def, fast+)

  note carr = acarr bcarr ccarr

  from carr and inv-closed
    have inv a ⊗ b = inv a ⊗ (1 ⊗ b) by simp
  also from carr and inv-closed
      have ... = inv a ⊗ (inv c ⊗ c) ⊗ b by simp
  also from carr and inv-closed
      have ... = (inv a ⊗ inv c) ⊗ (c ⊗ b) by (force cong: m-assoc)
  also from carr and inv-closed
      have ... = inv (c ⊗ a) ⊗ (c ⊗ b) by (simp add: inv-mult-group)
  finally
      have inv a ⊗ b = inv (c ⊗ a) ⊗ (c ⊗ b) .
  from abH and this
      have inv (c ⊗ a) ⊗ (c ⊗ b) ∈ H by simp

  from carr and this
    have (c ⊗ b) ∈ (c ⊗ a) <# H
    by (simp add: lcos-module-rev[OF is-group])
  from carr and this and is-subgroup
    show (c ⊗ a) <# H = (c ⊗ b) <# H by (intro l-repr-independence, simp+)
qed
```

5.5 Order of a Group and Lagrange's Theorem

```
constdefs
  order :: ('a, 'b) monoid-scheme ⇒ nat
  order S ≡ card (carrier S)

lemma (in group) rcosets-part-G:
  includes subgroup
  shows ⋃(rcosets H) = carrier G
```

apply (*rule equalityI*)
apply (*force simp add*: *RCOSETS-def r-coset-def*)
apply (*auto simp add*: *RCOSETS-def intro*: *rcos-self prems*)
done

lemma (**in** *group*) *cosets-finite*:
⟦*c* ∈ *rcosets H*; *H* ⊆ *carrier G*; *finite* (*carrier G*)⟧ ⟹ *finite c*
apply (*auto simp add*: *RCOSETS-def*)
apply (*simp add*: *r-coset-subset-G* [*THEN finite-subset*])
done

The next two lemmas support the proof of *card-cosets-equal*.

lemma (**in** *group*) *inj-on-f*:
⟦*H* ⊆ *carrier G*; *a* ∈ *carrier G*⟧ ⟹ *inj-on* (λ*y*. *y* ⊗ *inv a*) (*H* #> *a*)
apply (*rule inj-onI*)
apply (*subgoal-tac x* ∈ *carrier G* & *y* ∈ *carrier G*)
prefer *2* **apply** (*blast intro*: *r-coset-subset-G* [*THEN subsetD*])
apply (*simp add*: *subsetD*)
done

lemma (**in** *group*) *inj-on-g*:
⟦*H* ⊆ *carrier G*; *a* ∈ *carrier G*⟧ ⟹ *inj-on* (λ*y*. *y* ⊗ *a*) *H*
by (*force simp add*: *inj-on-def subsetD*)

lemma (**in** *group*) *card-cosets-equal*:
⟦*c* ∈ *rcosets H*; *H* ⊆ *carrier G*; *finite*(*carrier G*)⟧
⟹ *card c* = *card H*
apply (*auto simp add*: *RCOSETS-def*)
apply (*rule card-bij-eq*)
apply (*rule inj-on-f*, *assumption*+)
apply (*force simp add*: *m-assoc subsetD r-coset-def*)
apply (*rule inj-on-g*, *assumption*+)
apply (*force simp add*: *m-assoc subsetD r-coset-def*)

The sets *H* #> *a* and *H* are finite.

apply (*simp add*: *r-coset-subset-G* [*THEN finite-subset*])
apply (*blast intro*: *finite-subset*)
done

lemma (**in** *group*) *rcosets-subset-PowG*:
subgroup H G ⟹ *rcosets H* ⊆ *Pow*(*carrier G*)
apply (*simp add*: *RCOSETS-def*)
apply (*blast dest*: *r-coset-subset-G subgroup.subset*)
done

theorem (**in** *group*) *lagrange*:
⟦*finite*(*carrier G*); *subgroup H G*⟧
⟹ *card*(*rcosets H*) ∗ *card*(*H*) = *order*(*G*)

apply (*simp* (*no-asm-simp*) *add*: *order-def rcosets-part-G* [*symmetric*])
apply (*subst mult-commute*)
apply (*rule card-partition*)
apply (*simp add*: *rcosets-subset-PowG* [*THEN finite-subset*])
apply (*simp add*: *rcosets-part-G*)
apply (*simp add*: *card-cosets-equal subgroup.subset*)
apply (*simp add*: *rcos-disjoint*)
done

5.6 Quotient Groups: Factorization of a Group

constdefs
FactGroup :: [('*a*,'*b*) *monoid-scheme*, '*a set*] ⇒ ('*a set*) *monoid*
(**infixl** *Mod 65*)
— Actually defined for groups rather than monoids
FactGroup G H ≡
(|*carrier* = *rcosets*$_G$ *H*, *mult* = *set-mult G*, *one* = *H*|)

lemma (**in** *normal*) *setmult-closed*:
⟦*K1* ∈ *rcosets H*; *K2* ∈ *rcosets H*⟧ ⟹ *K1* <#> *K2* ∈ *rcosets H*
by (*auto simp add*: *rcos-sum RCOSETS-def*)

lemma (**in** *normal*) *setinv-closed*:
K ∈ *rcosets H* ⟹ *set-inv K* ∈ *rcosets H*
by (*auto simp add*: *rcos-inv RCOSETS-def*)

lemma (**in** *normal*) *rcosets-assoc*:
⟦*M1* ∈ *rcosets H*; *M2* ∈ *rcosets H*; *M3* ∈ *rcosets H*⟧
⟹ *M1* <#> *M2* <#> *M3* = *M1* <#> (*M2* <#> *M3*)
by (*auto simp add*: *RCOSETS-def rcos-sum m-assoc*)

lemma (**in** *subgroup*) *subgroup-in-rcosets*:
includes *group G*
shows *H* ∈ *rcosets H*
proof −
from - *subgroup-axioms* **have** *H* #> **1** = *H*
by (*rule coset-join2*) *auto*
then show *?thesis*
by (*auto simp add*: *RCOSETS-def*)
qed

lemma (**in** *normal*) *rcosets-inv-mult-group-eq*:
M ∈ *rcosets H* ⟹ *set-inv M* <#> *M* = *H*
by (*auto simp add*: *RCOSETS-def rcos-inv rcos-sum subgroup.subset normal.axioms prems*)

theorem (**in** *normal*) *factorgroup-is-group*:
group (*G Mod H*)
apply (*simp add*: *FactGroup-def*)

apply (*rule groupI*)
apply (*simp add*: *setmult-closed*)
apply (*simp add*: *normal-imp-subgroup subgroup-in-rcosets* [*OF is-group*])
apply (*simp add*: *restrictI setmult-closed rcosets-assoc*)
apply (*simp add*: *normal-imp-subgroup*
subgroup-in-rcosets rcosets-mult-eq)
apply (*auto dest*: *rcosets-inv-mult-group-eq simp add*: *setinv-closed*)
done

lemma *mult-FactGroup* [*simp*]: $X \otimes_{(G\ Mod\ H)} X' = X <\#>_G X'$
by (*simp add*: *FactGroup-def*)

lemma (**in** *normal*) *inv-FactGroup*:
$X \in carrier\ (G\ Mod\ H) \Longrightarrow inv_{G\ Mod\ H}\ X = set\text{-}inv\ X$
apply (*rule group.inv-equality* [*OF factorgroup-is-group*])
apply (*simp-all add*: *FactGroup-def setinv-closed rcosets-inv-mult-group-eq*)
done

The coset map is a homomorphism from G to the quotient group $G\ Mod\ H$

lemma (**in** *normal*) *r-coset-hom-Mod*:
$(\lambda a.\ H\ \#>\ a) \in hom\ G\ (G\ Mod\ H)$
by (*auto simp add*: *FactGroup-def RCOSETS-def Pi-def hom-def rcos-sum*)

5.7 The First Isomorphism Theorem

The quotient by the kernel of a homomorphism is isomorphic to the range of that homomorphism.

constdefs
kernel :: ('*a*, '*m*) *monoid-scheme* ⇒ ('*b*, '*n*) *monoid-scheme* ⇒
('*a* ⇒ '*b*) ⇒ '*a set*
— the kernel of a homomorphism
$kernel\ G\ H\ h \equiv \{x.\ x \in carrier\ G\ \&\ h\ x = \mathbf{1}_H\}$

lemma (**in** *group-hom*) *subgroup-kernel*: *subgroup* (*kernel G H h*) *G*
apply (*rule subgroup.intro*)
apply (*auto simp add*: *kernel-def group.intro prems*)
done

The kernel of a homomorphism is a normal subgroup

lemma (**in** *group-hom*) *normal-kernel*: (*kernel G H h*) ◁ *G*
apply (*simp add*: *G.normal-inv-iff subgroup-kernel*)
apply (*simp add*: *kernel-def*)
done

lemma (**in** *group-hom*) *FactGroup-nonempty*:
assumes *X*: *X* ∈ *carrier* (*G Mod kernel G H h*)
shows *X* ≠ {}
proof −

from X
obtain g **where** $g \in carrier\ G$
and $X = kernel\ G\ H\ h\ \#>\ g$
by (*auto simp add*: *FactGroup-def RCOSETS-def*)
thus *?thesis*
by (*auto simp add*: *kernel-def r-coset-def image-def intro*: *hom-one*)
qed

lemma (**in** *group-hom*) *FactGroup-contents-mem*:
assumes X: $X \in carrier\ (G\ Mod\ (kernel\ G\ H\ h))$
shows $contents\ (h`X) \in carrier\ H$
proof −
from X
obtain g **where** g: $g \in carrier\ G$
and $X = kernel\ G\ H\ h\ \#>\ g$
by (*auto simp add*: *FactGroup-def RCOSETS-def*)
hence $h\ `\ X = \{h\ g\}$ **by** (*auto simp add*: *kernel-def r-coset-def image-def g*)
thus *?thesis* **by** (*auto simp add*: *g*)
qed

lemma (**in** *group-hom*) *FactGroup-hom*:
$(\lambda X.\ contents\ (h`X)) \in hom\ (G\ Mod\ (kernel\ G\ H\ h))\ H$
apply (*simp add*: *hom-def FactGroup-contents-mem normal.factorgroup-is-group*
[*OF normal-kernel*] *group.axioms monoid.m-closed*)
proof (*simp add*: *hom-def funcsetI FactGroup-contents-mem*, *intro ballI*)
fix X **and** X'
assume X: $X \in carrier\ (G\ Mod\ kernel\ G\ H\ h)$
and X': $X' \in carrier\ (G\ Mod\ kernel\ G\ H\ h)$
then
obtain g **and** g'
where $g \in carrier\ G$ **and** $g' \in carrier\ G$
and $X = kernel\ G\ H\ h\ \#>\ g$ **and** $X' = kernel\ G\ H\ h\ \#>\ g'$
by (*auto simp add*: *FactGroup-def RCOSETS-def*)
hence *all*: $\forall x{\in}X.\ h\ x = h\ g\ \forall x{\in}X'.\ h\ x = h\ g'$
and *Xsub*: $X \subseteq carrier\ G$ **and** *X'sub*: $X' \subseteq carrier\ G$
by (*force simp add*: *kernel-def r-coset-def image-def*)+
hence $h\ `\ (X <\#> X') = \{h\ g \otimes_H h\ g'\}$ **using** $X\ X'$
by (*auto dest*!: *FactGroup-nonempty*
simp add: *set-mult-def image-eq-UN*
subsetD [*OF Xsub*] *subsetD* [*OF X'sub*])
thus $contents\ (h\ `\ (X <\#> X')) = contents\ (h\ `\ X) \otimes_H contents\ (h\ `\ X')$
by (*simp add*: *all image-eq-UN FactGroup-nonempty X X'*)
qed

Lemma for the following injectivity result

lemma (**in** *group-hom*) *FactGroup-subset*:
$[\![g \in carrier\ G;\ g' \in carrier\ G;\ h\ g = h\ g']\!]$
$\Longrightarrow\ kernel\ G\ H\ h\ \#>\ g \subseteq kernel\ G\ H\ h\ \#>\ g'$

apply (*clarsimp simp add*: *kernel-def r-coset-def image-def*)
apply (*rename-tac y*)
apply (*rule-tac x*=$y \otimes g \otimes inv\ g'$ **in** *exI*)
apply (*simp add*: *G.m-assoc*)
done

lemma (**in** *group-hom*) *FactGroup-inj-on*:
  inj-on ($\lambda X.$ *contents* (*h* ‘ *X*)) (*carrier* (*G Mod kernel G H h*))
proof (*simp add*: *inj-on-def*, *clarify*)
  fix *X* **and** *X'*
  assume *X*: *X* ∈ *carrier* (*G Mod kernel G H h*)
    and *X'*: *X'* ∈ *carrier* (*G Mod kernel G H h*)
  then
  obtain *g* **and** *g'*
      where *gX*: *g* ∈ *carrier G* *g'* ∈ *carrier G*
        X = *kernel G H h* #> *g* *X'* = *kernel G H h* #> *g'*
    by (*auto simp add*: *FactGroup-def RCOSETS-def*)
  hence *all*: ∀ *x*∈*X*. *h x* = *h g* ∀ *x*∈*X'*. *h x* = *h g'*
    by (*force simp add*: *kernel-def r-coset-def image-def*)+
  assume *contents* (*h* ‘ *X*) = *contents* (*h* ‘ *X'*)
  hence *h*: *h g* = *h g'*
    by (*simp add*: *image-eq-UN all FactGroup-nonempty X X'*)
  show *X*=*X'* **by** (*rule equalityI*) (*simp-all add*: *FactGroup-subset h gX*)
qed

If the homomorphism h is onto H, then so is the homomorphism from the quotient group

lemma (**in** *group-hom*) *FactGroup-onto*:
  assumes *h*: *h* ‘ *carrier G* = *carrier H*
  shows ($\lambda X.$ *contents* (*h* ‘ *X*)) ‘ *carrier* (*G Mod kernel G H h*) = *carrier H*
proof
  show ($\lambda X.$ *contents* (*h* ‘ *X*)) ‘ *carrier* (*G Mod kernel G H h*) ⊆ *carrier H*
    by (*auto simp add*: *FactGroup-contents-mem*)
  show *carrier H* ⊆ ($\lambda X.$ *contents* (*h* ‘ *X*)) ‘ *carrier* (*G Mod kernel G H h*)
  proof
    fix *y*
    assume *y*: *y* ∈ *carrier H*
    with *h* **obtain** *g* **where** *g*: *g* ∈ *carrier G h g* = *y*
      by (*blast elim*: *equalityE*)
    hence ($\bigcup x$∈*kernel G H h* #> *g*. {*h x*}) = {*y*}
      by (*auto simp add*: *y kernel-def r-coset-def*)
    with *g* **show** *y* ∈ ($\lambda X.$ *contents* (*h* ‘ *X*)) ‘ *carrier* (*G Mod kernel G H h*)
      by (*auto intro*!: *bexI simp add*: *FactGroup-def RCOSETS-def image-eq-UN*)
  qed
qed

If h is a homomorphism from G onto H, then the quotient group $G\ Mod\ kernel\ G\ H\ h$ is isomorphic to H.

theorem (**in** *group-hom*) *FactGroup-iso*:

h ‘ carrier G = carrier H
⟹ (λX. contents (h‘X)) ∈ (G Mod (kernel G H h)) ≅ H
by (*simp add*: *iso-def FactGroup-hom FactGroup-inj-on bij-betw-def*
FactGroup-onto)

end

theory *Sylow* **imports** *Coset* **begin**

6 Sylow's Theorem

See also [3].

The combinatorial argument is in theory Exponent

locale *sylow = group +*
fixes *p* **and** *a* **and** *m* **and** *calM* **and** *RelM*
assumes *prime-p*: *prime p*
and *order-G*: *order(G) = (p ^ a) * m*
and *finite-G* [*iff*]: *finite (carrier G)*
defines *calM == {s. s ⊆ carrier(G) & card(s) = p ^ a}*
and *RelM == {(N1,N2). N1 ∈ calM & N2 ∈ calM &*
(∃ g ∈ carrier(G). N1 = (N2 #> g))}

lemma (**in** *sylow*) *RelM-refl*: *refl calM RelM*
apply (*auto simp add*: *refl-def RelM-def calM-def*)
apply (*blast intro*!: *coset-mult-one* [*symmetric*])
done

lemma (**in** *sylow*) *RelM-sym*: *sym RelM*
proof (*unfold sym-def RelM-def*, *clarify*)
fix *y g*
assume *y ∈ calM*
and *g*: *g ∈ carrier G*
hence *y = y #> g #> (inv g)* **by** (*simp add*: *coset-mult-assoc calM-def*)
thus *∃ g'∈carrier G. y = y #> g #> g'*
by (*blast intro*: *g inv-closed*)
qed

lemma (**in** *sylow*) *RelM-trans*: *trans RelM*
by (*auto simp add*: *trans-def RelM-def calM-def coset-mult-assoc*)

lemma (**in** *sylow*) *RelM-equiv*: *equiv calM RelM*
apply (*unfold equiv-def*)
apply (*blast intro*: *RelM-refl RelM-sym RelM-trans*)
done

lemma (**in** *sylow*) *M-subset-calM-prep*: $M' \in calM\ //\ RelM \Longrightarrow M' \subseteq calM$
apply (*unfold RelM-def*)
apply (*blast elim*!: *quotientE*)
done

6.1 Main Part of the Proof

locale *sylow-central* = *sylow* +
  fixes *H* **and** *M1* **and** *M*
  assumes *M-in-quot*: $M \in calM\ //\ RelM$
    and *not-dvd-M*: $\sim(p\ \hat{}\ Suc(exponent\ p\ m)\ dvd\ card(M))$
    and *M1-in-M*: $M1 \in M$
  defines $H == \{g.\ g \in carrier\ G\ \&\ M1\ \#>\ g = M1\}$

lemma (**in** *sylow-central*) *M-subset-calM*: $M \subseteq calM$
by (*rule M-in-quot* [*THEN M-subset-calM-prep*])

lemma (**in** *sylow-central*) *card-M1*: $card(M1) = p\hat{}a$
apply (*cut-tac M-subset-calM M1-in-M*)
apply (*simp add*: *calM-def*, *blast*)
done

lemma *card-nonempty*: $0 < card(S) \Longrightarrow S \neq \{\}$
by *force*

lemma (**in** *sylow-central*) *exists-x-in-M1*: $\exists x.\ x \in M1$
apply (*subgoal-tac* $0 < card\ M1$)
 apply (*blast dest*: *card-nonempty*)
apply (*cut-tac prime-p* [*THEN prime-imp-one-less*])
apply (*simp* (*no-asm-simp*) *add*: *card-M1*)
done

lemma (**in** *sylow-central*) *M1-subset-G* [*simp*]: $M1 \subseteq carrier\ G$
apply (*rule subsetD* [*THEN PowD*])
apply (*rule-tac* [*2*] *M1-in-M*)
apply (*rule M-subset-calM* [*THEN subset-trans*])
apply (*auto simp add*: *calM-def*)
done

lemma (**in** *sylow-central*) *M1-inj-H*: $\exists f \in H \rightarrow M1.\ inj\text{-}on\ f\ H$
  proof −
    from *exists-x-in-M1* **obtain** *m1* **where** *m1M*: $m1 \in M1$..
    have *m1G*: $m1 \in carrier\ G$ **by** (*simp add*: *m1M M1-subset-G* [*THEN subsetD*])
    show *?thesis*
    proof
      show $inj\text{-}on\ (\lambda z \in H.\ m1 \otimes z)\ H$
        by (*simp add*: *inj-on-def l-cancel* [*of m1 x y*, *THEN iffD1*] *H-def m1G*)
      show $restrict\ (op \otimes\ m1)\ H \in H \rightarrow M1$

```
    proof (rule restrictI)
      fix z assume zH: z ∈ H
      show m1 ⊗ z ∈ M1
      proof −
        from zH
        have zG: z ∈ carrier G and M1zeq: M1 #> z = M1
          by (auto simp add: H-def)
        show ?thesis
          by (rule subst [OF M1zeq], simp add: m1M zG rcosI)
      qed
    qed
  qed
qed
```

6.2 Discharging the Assumptions of *sylow-central*

```
lemma (in sylow) EmptyNotInEquivSet: {} ∉ calM // RelM
by (blast elim!: quotientE dest: RelM-equiv [THEN equiv-class-self])

lemma (in sylow) existsM1inM: M ∈ calM // RelM ==> ∃M1. M1 ∈ M
apply (subgoal-tac M ≠ {})
 apply blast
apply (cut-tac EmptyNotInEquivSet, blast)
done

lemma (in sylow) zero-less-o-G: 0 < order(G)
apply (unfold order-def)
apply (blast intro: one-closed zero-less-card-empty)
done

lemma (in sylow) zero-less-m: m > 0
apply (cut-tac zero-less-o-G)
apply (simp add: order-G)
done

lemma (in sylow) card-calM: card(calM) = (p^a) * m choose p^a
by (simp add: calM-def n-subsets order-G [symmetric] order-def)

lemma (in sylow) zero-less-card-calM: card calM > 0
by (simp add: card-calM zero-less-binomial le-extend-mult zero-less-m)

lemma (in sylow) max-p-div-calM:
    ~ (p ^ Suc(exponent p m) dvd card(calM))
apply (subgoal-tac exponent p m = exponent p (card calM) )
 apply (cut-tac zero-less-card-calM prime-p)
 apply (force dest: power-Suc-exponent-Not-dvd)
apply (simp add: card-calM zero-less-m [THEN const-p-fac])
done
```

lemma (**in** *sylow*) *finite-calM*: *finite calM*
apply (*unfold calM-def*)
apply (*rule-tac B = Pow (carrier G)* **in** *finite-subset*)
apply *auto*
done

lemma (**in** *sylow*) *lemma-A1*:
  $\exists M \in calM\ //\ RelM.\ \sim (p\ \hat{}\ Suc(exponent\ p\ m)\ dvd\ card(M))$
apply (*rule max-p-div-calM* [*THEN contrapos-np*])
apply (*simp add*: *finite-calM equiv-imp-dvd-card* [*OF - RelM-equiv*])
done

6.2.1 Introduction and Destruct Rules for H

lemma (**in** *sylow-central*) *H-I*: $[|g \in carrier\ G;\ M1\ \#>\ g = M1|] ==> g \in H$
by (*simp add*: *H-def*)

lemma (**in** *sylow-central*) *H-into-carrier-G*: $x \in H ==> x \in carrier\ G$
by (*simp add*: *H-def*)

lemma (**in** *sylow-central*) *in-H-imp-eq*: $g : H ==> M1\ \#>\ g = M1$
by (*simp add*: *H-def*)

lemma (**in** *sylow-central*) *H-m-closed*: $[|\ x{\in}H;\ y{\in}H|] ==> x \otimes y \in H$
apply (*unfold H-def*)
apply (*simp add*: *coset-mult-assoc* [*symmetric*] *m-closed*)
done

lemma (**in** *sylow-central*) *H-not-empty*: $H \neq \{\}$
apply (*simp add*: *H-def*)
apply (*rule exI* [*of* - **1**], *simp*)
done

lemma (**in** *sylow-central*) *H-is-subgroup*: *subgroup H G*
apply (*rule subgroupI*)
apply (*rule subsetI*)
apply (*erule H-into-carrier-G*)
apply (*rule H-not-empty*)
apply (*simp add*: *H-def*, *clarify*)
apply (*erule-tac P = %z. ?lhs(z) = M1* **in** *subst*)
apply (*simp add*: *coset-mult-assoc*)
apply (*blast intro*: *H-m-closed*)
done

lemma (**in** *sylow-central*) *rcosetGM1g-subset-G*:
  $[|\ g \in carrier\ G;\ x \in M1\ \#>\ g\ |] ==> x \in carrier\ G$
by (*blast intro*: *M1-subset-G* [*THEN r-coset-subset-G*, *THEN subsetD*])

lemma (**in** *sylow-central*) *finite-M1*: *finite M1*
by (*rule finite-subset* [*OF M1-subset-G finite-G*])

lemma (**in** *sylow-central*) *finite-rcosetGM1g*: *g*∈*carrier G* ==> *finite* (*M1 #> g*)
apply (*rule finite-subset*)
apply (*rule subsetI*)
apply (*erule rcosetGM1g-subset-G*, *assumption*)
apply (*rule finite-G*)
done

lemma (**in** *sylow-central*) *M1-cardeq-rcosetGM1g*:
g ∈ *carrier G* ==> *card*(*M1 #> g*) = *card*(*M1*)
by (*simp* (*no-asm-simp*) *add*: *M1-subset-G card-cosets-equal rcosetsI*)

lemma (**in** *sylow-central*) *M1-RelM-rcosetGM1g*:
g ∈ *carrier G* ==> (*M1*, *M1 #> g*) ∈ *RelM*
apply (*simp* (*no-asm*) *add*: *RelM-def calM-def card-M1 M1-subset-G*)
apply (*rule conjI*)
apply (*blast intro*: *rcosetGM1g-subset-G*)
apply (*simp* (*no-asm-simp*) *add*: *card-M1 M1-cardeq-rcosetGM1g*)
apply (*rule bexI* [*of - inv g*])
apply (*simp-all add*: *coset-mult-assoc M1-subset-G*)
done

6.3 Equal Cardinalities of M and the Set of Cosets

Injections between M and $rcosets_G\ H$ show that their cardinalities are equal.

lemma *ElemClassEquiv*:
[| *equiv A r*; *C* ∈ *A* // *r* |] ==> ∀ *x* ∈ *C*. ∀ *y* ∈ *C*. (*x*,*y*)∈*r*
by (*unfold equiv-def quotient-def sym-def trans-def*, *blast*)

lemma (**in** *sylow-central*) *M-elem-map*:
M2 ∈ *M* ==> ∃ *g*. *g* ∈ *carrier G* & *M1 #> g* = *M2*
apply (*cut-tac M1-in-M M-in-quot* [*THEN RelM-equiv* [*THEN ElemClassEquiv*]])
apply (*simp add*: *RelM-def*)
apply (*blast dest*!: *bspec*)
done

lemmas (**in** *sylow-central*) *M-elem-map-carrier* =
M-elem-map [*THEN someI-ex*, *THEN conjunct1*]

lemmas (**in** *sylow-central*) *M-elem-map-eq* =
M-elem-map [*THEN someI-ex*, *THEN conjunct2*]

lemma (**in** *sylow-central*) *M-funcset-rcosets-H*:
(%*x*:*M*. *H #>* (*SOME g*. *g* ∈ *carrier G* & *M1 #> g* = *x*)) ∈ *M* → *rcosets H*
apply (*rule rcosetsI* [*THEN restrictI*])
apply (*rule H-is-subgroup* [*THEN subgroup.subset*])
apply (*erule M-elem-map-carrier*)

done

lemma (**in** *sylow-central*) *inj-M-GmodH*: $\exists f \in M \to rcosets\ H.\ inj\text{-}on\ f\ M$
apply (*rule bexI*)
apply (*rule-tac* [*2*] *M-funcset-rcosets-H*)
apply (*rule inj-onI*, *simp*)
apply (*rule trans* [*OF - M-elem-map-eq*])
prefer *2* **apply** *assumption*
apply (*rule M-elem-map-eq* [*symmetric*, *THEN trans*], *assumption*)
apply (*rule coset-mult-inv1*)
apply (*erule-tac* [*2*] *M-elem-map-carrier*)+
apply (*rule-tac* [*2*] *M1-subset-G*)
apply (*rule coset-join1* [*THEN in-H-imp-eq*])
apply (*rule-tac* [*3*] *H-is-subgroup*)
prefer *2* **apply** (*blast intro*: *m-closed M-elem-map-carrier inv-closed*)
apply (*simp add*: *coset-mult-inv2 H-def M-elem-map-carrier subset-eq*)
done

6.3.1 The Opposite Injection

lemma (**in** *sylow-central*) *H-elem-map*:
$H1 \in rcosets\ H \Longrightarrow \exists g.\ g \in carrier\ G\ \&\ H\ \#>\ g = H1$
by (*auto simp add*: *RCOSETS-def*)

lemmas (**in** *sylow-central*) *H-elem-map-carrier* =
H-elem-map [*THEN someI-ex*, *THEN conjunct1*]

lemmas (**in** *sylow-central*) *H-elem-map-eq* =
H-elem-map [*THEN someI-ex*, *THEN conjunct2*]

lemma *EquivElemClass*:
$[|equiv\ A\ r;\ M \in A//r;\ M1 \in M;\ (M1,M2) \in r\ |] \Longrightarrow M2 \in M$
by (*unfold equiv-def quotient-def sym-def trans-def*, *blast*)

lemma (**in** *sylow-central*) *rcosets-H-funcset-M*:
$(\lambda C \in rcosets\ H.\ M1\ \#>\ (@g.\ g \in carrier\ G \wedge H\ \#>\ g = C)) \in rcosets\ H \to M$
apply (*simp add*: *RCOSETS-def*)
apply (*fast intro*: *someI2*
intro!: *restrictI M1-in-M*
EquivElemClass [*OF RelM-equiv M-in-quot - M1-RelM-rcosetGM1g*])
done

close to a duplicate of *inj-M-GmodH*

lemma (**in** *sylow-central*) *inj-GmodH-M*:
$\exists g \in rcosets\ H \to M.\ inj\text{-}on\ g\ (rcosets\ H)$
apply (*rule bexI*)

apply (*rule-tac* [*2*] *rcosets-H-funcset-M*)
apply (*rule inj-onI*)
apply (*simp*)
apply (*rule trans* [*OF - H-elem-map-eq*])
prefer *2* **apply** *assumption*
apply (*rule H-elem-map-eq* [*symmetric*, *THEN trans*], *assumption*)
apply (*rule coset-mult-inv1*)
apply (*erule-tac* [*2*] *H-elem-map-carrier*)+
apply (*rule-tac* [*2*] *H-is-subgroup* [*THEN subgroup.subset*])
apply (*rule coset-join2*)
apply (*blast intro*: *m-closed inv-closed H-elem-map-carrier*)
apply (*rule H-is-subgroup*)
apply (*simp add*: *H-I coset-mult-inv2 M1-subset-G H-elem-map-carrier*)
done

lemma (**in** *sylow-central*) *calM-subset-PowG*: *calM* ⊆ *Pow*(*carrier G*)
by (*auto simp add*: *calM-def*)

lemma (**in** *sylow-central*) *finite-M*: *finite M*
apply (*rule finite-subset*)
apply (*rule M-subset-calM* [*THEN subset-trans*])
apply (*rule calM-subset-PowG*, *blast*)
done

lemma (**in** *sylow-central*) *cardMeqIndexH*: *card*(*M*) = *card*(*rcosets H*)
apply (*insert inj-M-GmodH inj-GmodH-M*)
apply (*blast intro*: *card-bij finite-M H-is-subgroup*
rcosets-subset-PowG [*THEN finite-subset*]
finite-Pow-iff [*THEN iffD2*])
done

lemma (**in** *sylow-central*) *index-lem*: *card*(*M*) ∗ *card*(*H*) = *order*(*G*)
by (*simp add*: *cardMeqIndexH lagrange H-is-subgroup*)

lemma (**in** *sylow-central*) *lemma-leq1*: *p* ^ *a* ≤ *card*(*H*)
apply (*rule dvd-imp-le*)
apply (*rule div-combine* [*OF prime-p not-dvd-M*])
prefer *2* **apply** (*blast intro*: *subgroup.finite-imp-card-positive H-is-subgroup*)
apply (*simp add*: *index-lem order-G power-add mult-dvd-mono power-exponent-dvd*
zero-less-m)
done

lemma (**in** *sylow-central*) *lemma-leq2*: *card*(*H*) ≤ *p* ^ *a*
apply (*subst card-M1* [*symmetric*])
apply (*cut-tac M1-inj-H*)
apply (*blast intro*!: *M1-subset-G intro*:
card-inj H-into-carrier-G finite-subset [*OF - finite-G*])
done

lemma (**in** *sylow-central*) *card-H-eq*: *card*(*H*) = *p* ^ *a*
by (*blast intro*: *le-anti-sym lemma-leq1 lemma-leq2*)

lemma (**in** *sylow*) *sylow-thm*: ∃ *H*. *subgroup H G* & *card*(*H*) = *p* ^ *a*
apply (*cut-tac lemma-A1*, *clarify*)
apply (*frule existsM1inM*, *clarify*)
apply (*subgoal-tac sylow-central G p a m M1 M*)
apply (*blast dest*: *sylow-central.H-is-subgroup sylow-central.card-H-eq*)
apply (*simp add*: *sylow-central-def sylow-central-axioms-def prems*)
done

Needed because the locale's automatic definition refers to *semigroup G* and *group-axioms G* rather than simply to *group G*.

lemma *sylow-eq*: *sylow G p a m* = (*group G* & *sylow-axioms G p a m*)
by (*simp add*: *sylow-def group-def*)

6.4 Sylow's Theorem

theorem *sylow-thm*:
[| *prime p*; *group*(*G*); *order*(*G*) = (*p* ^ *a*) ∗ *m*; *finite* (*carrier G*)|]
==> ∃ *H*. *subgroup H G* & *card*(*H*) = *p* ^ *a*
apply (*rule sylow.sylow-thm* [*of G p a m*])
apply (*simp add*: *sylow-eq sylow-axioms-def*)
done

end

theory *Bij* **imports** *Group* **begin**

7 Bijections of a Set, Permutation Groups and Automorphism Groups

constdefs
Bij :: *'a set* ⇒ (*'a* ⇒ *'a*) *set*
— Only extensional functions, since otherwise we get too many.
Bij S ≡ *extensional S* ∩ {*f*. *bij-betw f S S*}

BijGroup :: *'a set* ⇒ (*'a* ⇒ *'a*) *monoid*
BijGroup S ≡
(|*carrier* = *Bij S*,
mult = λ*g* ∈ *Bij S*. λ*f* ∈ *Bij S*. *compose S g f*,
one = λ*x* ∈ *S*. *x*|)

declare *Id-compose* [*simp*] *compose-Id* [*simp*]

lemma *Bij-imp-extensional*: $f \in Bij\ S \Longrightarrow f \in extensional\ S$
by (*simp add*: *Bij-def*)

lemma *Bij-imp-funcset*: $f \in Bij\ S \Longrightarrow f \in S \to S$
by (*auto simp add*: *Bij-def bij-betw-imp-funcset*)

7.1 Bijections Form a Group

lemma *restrict-Inv-Bij*: $f \in Bij\ S \Longrightarrow (\lambda x \in S.\ (Inv\ S\ f)\ x) \in Bij\ S$
by (*simp add*: *Bij-def bij-betw-Inv*)

lemma *id-Bij*: $(\lambda x{\in}S.\ x) \in Bij\ S$
by (*auto simp add*: *Bij-def bij-betw-def inj-on-def*)

lemma *compose-Bij*: $[\![x \in Bij\ S;\ y \in Bij\ S]\!] \Longrightarrow compose\ S\ x\ y \in Bij\ S$
by (*auto simp add*: *Bij-def bij-betw-compose*)

lemma *Bij-compose-restrict-eq*:
$f \in Bij\ S \Longrightarrow compose\ S\ (restrict\ (Inv\ S\ f)\ S)\ f = (\lambda x{\in}S.\ x)$
by (*simp add*: *Bij-def compose-Inv-id*)

theorem *group-BijGroup*: *group* (*BijGroup S*)
apply (*simp add*: *BijGroup-def*)
apply (*rule groupI*)
apply (*simp add*: *compose-Bij*)
apply (*simp add*: *id-Bij*)
apply (*simp add*: *compose-Bij*)
apply (*blast intro*: *compose-assoc* [*symmetric*] *Bij-imp-funcset*)
apply (*simp add*: *id-Bij Bij-imp-funcset Bij-imp-extensional*, *simp*)
apply (*blast intro*: *Bij-compose-restrict-eq restrict-Inv-Bij*)
done

7.2 Automorphisms Form a Group

lemma *Bij-Inv-mem*: $[\![\ f \in Bij\ S;\ \ x \in S]\!] \Longrightarrow Inv\ S\ f\ x \in S$
by (*simp add*: *Bij-def bij-betw-def Inv-mem*)

lemma *Bij-Inv-lemma*:
assumes *eq*: $\bigwedge x\ y.\ [\![x \in S;\ y \in S]\!] \Longrightarrow h(g\ x\ y) = g\ (h\ x)\ (h\ y)$
shows $[\![h \in Bij\ S;\ \ g \in S \to S \to S;\ \ x \in S;\ \ y \in S]\!]$
$\Longrightarrow Inv\ S\ h\ (g\ x\ y) = g\ (Inv\ S\ h\ x)\ (Inv\ S\ h\ y)$
apply (*simp add*: *Bij-def bij-betw-def*)
apply (*subgoal-tac* $\exists x'{\in}S.\ \exists y'{\in}S.\ x = h\ x'\ \&\ y = h\ y'$, *clarify*)
apply (*simp add*: *eq* [*symmetric*] *Inv-f-f funcset-mem* [*THEN funcset-mem*], *blast*)
done

constdefs

```
  auto :: ('a, 'b) monoid-scheme ⇒ ('a ⇒ 'a) set
  auto G ≡ hom G G ∩ Bij (carrier G)

  AutoGroup :: ('a, 'c) monoid-scheme ⇒ ('a ⇒ 'a) monoid
  AutoGroup G ≡ BijGroup (carrier G) (|carrier := auto G|)

lemma (in group) id-in-auto: (λx ∈ carrier G. x) ∈ auto G
  by (simp add: auto-def hom-def restrictI group.axioms id-Bij)

lemma (in group) mult-funcset: mult G ∈ carrier G → carrier G → carrier G
  by (simp add:  Pi-I group.axioms)

lemma (in group) restrict-Inv-hom:
     ⟦h ∈ hom G G; h ∈ Bij (carrier G)⟧
      ⟹ restrict (Inv (carrier G) h) (carrier G) ∈ hom G G
  by (simp add: hom-def Bij-Inv-mem restrictI mult-funcset
              group.axioms Bij-Inv-lemma)

lemma inv-BijGroup:
    f ∈ Bij S ⟹ m-inv (BijGroup S) f = (λx ∈ S. (Inv S f) x)
apply (rule group.inv-equality)
apply (rule group-BijGroup)
apply (simp-all add: BijGroup-def restrict-Inv-Bij Bij-compose-restrict-eq)
done

lemma (in group) subgroup-auto:
      subgroup (auto G) (BijGroup (carrier G))
proof (rule subgroup.intro)
  show auto G ⊆ carrier (BijGroup (carrier G))
    by (force simp add: auto-def BijGroup-def)
next
  fix x y
  assume x ∈ auto G y ∈ auto G
  thus x ⊗_BijGroup (carrier G) y ∈ auto G
    by (force simp add: BijGroup-def is-group auto-def Bij-imp-funcset
                        group.hom-compose compose-Bij)
next
  show 1_BijGroup (carrier G) ∈ auto G by (simp add:  BijGroup-def id-in-auto)
next
  fix x
  assume x ∈ auto G
  thus inv_BijGroup (carrier G) x ∈ auto G
    by (simp del: restrict-apply
            add: inv-BijGroup auto-def restrict-Inv-Bij restrict-Inv-hom)
qed

theorem (in group) AutoGroup: group (AutoGroup G)
by (simp add: AutoGroup-def subgroup.subgroup-is-group subgroup-auto
              group-BijGroup)
```

end

theory *Ring* **imports** *FiniteProduct*
uses (*ringsimp.ML*) **begin**

8 Abelian Groups

record *'a ring = 'a monoid +*
  zero :: 'a (**0**$_1$)
  add :: ['a, 'a] => 'a (**infixl** $\oplus_1$ *65*)

Derived operations.

constdefs (**structure** *R*)
  a-inv :: [('a, 'm) ring-scheme, 'a] => 'a ($\ominus_1$ - [*81*] *80*)
  a-inv R == m-inv (| carrier = carrier R, mult = add R, one = zero R |)

  a-minus :: [('a, 'm) ring-scheme, 'a, 'a] => 'a (**infixl** $\ominus_1$ *65*)
  [| x ∈ carrier R; y ∈ carrier R |] ==> x ⊖ y == x ⊕ (⊖ y)

locale *abelian-monoid =*
  fixes *G* (**structure**)
  assumes *a-comm-monoid*:
    comm-monoid (| carrier = carrier G, mult = add G, one = zero G |)

The following definition is redundant but simple to use.

locale *abelian-group = abelian-monoid +*
  assumes *a-comm-group*:
    comm-group (| carrier = carrier G, mult = add G, one = zero G |)

8.1 Basic Properties

lemma *abelian-monoidI*:
  fixes *R* (**structure**)
  assumes *a-closed*:
    !!x y. [| x ∈ carrier R; y ∈ carrier R |] ==> x ⊕ y ∈ carrier R
   and *zero-closed*: **0** *∈ carrier R*
   and *a-assoc*:
    !!x y z. [| x ∈ carrier R; y ∈ carrier R; z ∈ carrier R |] ==>
    (x ⊕ y) ⊕ z = x ⊕ (y ⊕ z)
   and *l-zero*: *!!x. x ∈ carrier R ==>* **0** *⊕ x = x*
   and *a-comm*:
    !!x y. [| x ∈ carrier R; y ∈ carrier R |] ==> x ⊕ y = y ⊕ x
  shows *abelian-monoid R*
  by (*auto intro*!: *abelian-monoid.intro comm-monoidI intro*: *prems*)

lemma *abelian-groupI*:
fixes *R* (**structure**)
assumes *a-closed*:
!!*x y*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> *x* ⊕ *y* ∈ *carrier R*
and *zero-closed*: *zero R* ∈ *carrier R*
and *a-assoc*:
!!*x y z*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R*; *z* ∈ *carrier R* |] ==>
(*x* ⊕ *y*) ⊕ *z* = *x* ⊕ (*y* ⊕ *z*)
and *a-comm*:
!!*x y*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> *x* ⊕ *y* = *y* ⊕ *x*
and *l-zero*: !!*x*. *x* ∈ *carrier R* ==> **0** ⊕ *x* = *x*
and *l-inv-ex*: !!*x*. *x* ∈ *carrier R* ==> *EX y* : *carrier R*. *y* ⊕ *x* = **0**
shows *abelian-group R*
by (*auto intro*!: *abelian-group.intro abelian-monoidI*
abelian-group-axioms.intro comm-monoidI comm-groupI
intro: *prems*)

lemma (**in** *abelian-monoid*) *a-monoid*:
monoid (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)
by (*rule comm-monoid.axioms*, *rule a-comm-monoid*)

lemma (**in** *abelian-group*) *a-group*:
group (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)
by (*simp add*: *group-def a-monoid*)
(*simp add*: *comm-group.axioms group.axioms a-comm-group*)

lemmas *monoid-record-simps* = *partial-object.simps monoid.simps*

lemma (**in** *abelian-monoid*) *a-closed* [*intro*, *simp*]:
⟦ *x* ∈ *carrier G*; *y* ∈ *carrier G* ⟧ ⟹ *x* ⊕ *y* ∈ *carrier G*
by (*rule monoid.m-closed* [*OF a-monoid*, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *zero-closed* [*intro*, *simp*]:
0 ∈ *carrier G*
by (*rule monoid.one-closed* [*OF a-monoid*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-inv-closed* [*intro*, *simp*]:
x ∈ *carrier G* ==> ⊖ *x* ∈ *carrier G*
by (*simp add*: *a-inv-def*
group.inv-closed [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *minus-closed* [*intro*, *simp*]:
[| *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==> *x* ⊖ *y* ∈ *carrier G*
by (*simp add*: *a-minus-def*)

lemma (**in** *abelian-group*) *a-l-cancel* [*simp*]:
[| *x* ∈ *carrier G*; *y* ∈ *carrier G*; *z* ∈ *carrier G* |] ==>
(*x* ⊕ *y* = *x* ⊕ *z*) = (*y* = *z*)
by (*rule group.l-cancel* [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-r-cancel* [*simp*]:
[| $x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* G |] ==>
$(y \oplus x = z \oplus x) = (y = z)$
by (*rule group.r-cancel* [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *a-assoc*:
$[\![x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* $G]\!] \Longrightarrow$
$(x \oplus y) \oplus z = x \oplus (y \oplus z)$
by (*rule monoid.m-assoc* [*OF a-monoid*, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *l-zero* [*simp*]:
$x \in$ *carrier* G ==> $\mathbf{0} \oplus x = x$
by (*rule monoid.l-one* [*OF a-monoid*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *l-neg*:
$x \in$ *carrier* G ==> $\ominus x \oplus x = \mathbf{0}$
by (*simp add*: *a-inv-def*
group.l-inv [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *a-comm*:
$[\![x \in$ *carrier* G; $y \in$ *carrier* $G]\!] \Longrightarrow x \oplus y = y \oplus x$
by (*rule comm-monoid.m-comm* [*OF a-comm-monoid*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *a-lcomm*:
$[\![x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* $G]\!] \Longrightarrow$
$x \oplus (y \oplus z) = y \oplus (x \oplus z)$
by (*rule comm-monoid.m-lcomm* [*OF a-comm-monoid*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *r-zero* [*simp*]:
$x \in$ *carrier* G ==> $x \oplus \mathbf{0} = x$
using *monoid.r-one* [*OF a-monoid*]
by *simp*

lemma (**in** *abelian-group*) *r-neg*:
$x \in$ *carrier* G ==> $x \oplus (\ominus x) = \mathbf{0}$
using *group.r-inv* [*OF a-group*]
by (*simp add*: *a-inv-def*)

lemma (**in** *abelian-group*) *minus-zero* [*simp*]:
$\ominus \mathbf{0} = \mathbf{0}$
by (*simp add*: *a-inv-def*
group.inv-one [*OF a-group*, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *minus-minus* [*simp*]:
$x \in$ *carrier* G ==> $\ominus (\ominus x) = x$
using *group.inv-inv* [*OF a-group*, *simplified monoid-record-simps*]

by (*simp add*: *a-inv-def*)

lemma (**in** *abelian-group*) *a-inv-inj*:
inj-on (*a-inv G*) (*carrier G*)
using *group.inv-inj* [*OF a-group*, *simplified monoid-record-simps*]
by (*simp add*: *a-inv-def*)

lemma (**in** *abelian-group*) *minus-add*:
[| $x \in$ *carrier G*; $y \in$ *carrier G* |] ==> $\ominus (x \oplus y) = \ominus x \oplus \ominus y$
using *comm-group.inv-mult* [*OF a-comm-group*]
by (*simp add*: *a-inv-def*)

lemma (**in** *abelian-group*) *minus-equality*:
[| $x \in$ *carrier G*; $y \in$ *carrier G*; $y \oplus x = \mathbf{0}$ |] ==> $\ominus x = y$
using *group.inv-equality* [*OF a-group*]
by (*auto simp add*: *a-inv-def*)

lemma (**in** *abelian-monoid*) *minus-unique*:
[| $x \in$ *carrier G*; $y \in$ *carrier G*; $y' \in$ *carrier G*;
$y \oplus x = \mathbf{0}$; $x \oplus y' = \mathbf{0}$ |] ==> $y = y'$
using *monoid.inv-unique* [*OF a-monoid*]
by (*simp add*: *a-inv-def*)

lemmas (**in** *abelian-monoid*) *a-ac* = *a-assoc a-comm a-lcomm*

Derive an *abelian-group* from a *comm-group*

lemma *comm-group-abelian-groupI*:
fixes *G* (**structure**)
assumes *cg*: *comm-group* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)
shows *abelian-group G*
proof −
interpret *comm-group* [(|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)]
by (*rule cg*)
show *abelian-group G* **by** (*unfold-locales*)
qed

8.2 Sums over Finite Sets

This definition makes it easy to lift lemmas from *finprod*.

constdefs
finsum :: [('*b*, '*m*) *ring-scheme*, '*a* => '*b*, '*a set*] => '*b*
finsum G f A == *finprod* (| *carrier* = *carrier G*,
mult = *add G*, *one* = *zero G* |) *f A*

syntax
-*finsum* :: *index* => *idt* => '*a set* => '*b* => '*b*
((3$\bigoplus$ --:-. -) [*1000*, *0*, *51*, *10*] *10*)
syntax (*xsymbols*)
-*finsum* :: *index* => *idt* => '*a set* => '*b* => '*b*

((3⨁ --∈-. -) [1000, 0, 51, 10] 10)
syntax (*HTML* **output**)
-*finsum* :: *index* => *idt* => *'a set* => *'b* => *'b*
((3⨁ --∈-. -) [1000, 0, 51, 10] 10)
translations
⨁1*i*:*A*. *b* == *finsum* ◇1 (%*i*. *b*) *A*
— Beware of argument permutation!

lemma (**in** *abelian-monoid*) *finsum-empty* [*simp*]:
finsum G f {} = **0**
by (*rule comm-monoid.finprod-empty* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-insert* [*simp*]:
[| *finite F*; *a* ∉ *F*; *f* ∈ *F* -> *carrier G*; *f a* ∈ *carrier G* |]
==> *finsum G f* (*insert a F*) = *f a* ⊕ *finsum G f F*
by (*rule comm-monoid.finprod-insert* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-zero* [*simp*]:
finite A ==> (⨁ *i*∈*A*. **0**) = **0**
by (*rule comm-monoid.finprod-one* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-closed* [*simp*]:
fixes *A*
assumes *fin*: *finite A* **and** *f*: *f* ∈ *A* -> *carrier G*
shows *finsum G f A* ∈ *carrier G*
apply (*rule comm-monoid.finprod-closed* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])
apply (*rule fin*)
apply (*rule f*)
done

lemma (**in** *abelian-monoid*) *finsum-Un-Int*:
[| *finite A*; *finite B*; *g* ∈ *A* -> *carrier G*; *g* ∈ *B* -> *carrier G* |] ==>
finsum G g (*A Un B*) ⊕ *finsum G g* (*A Int B*) =
finsum G g A ⊕ *finsum G g B*
by (*rule comm-monoid.finprod-Un-Int* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-Un-disjoint*:
[| *finite A*; *finite B*; *A Int B* = {};
g ∈ *A* -> *carrier G*; *g* ∈ *B* -> *carrier G* |]
==> *finsum G g* (*A Un B*) = *finsum G g A* ⊕ *finsum G g B*
by (*rule comm-monoid.finprod-Un-disjoint* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-addf*:
[| *finite A*; *f* ∈ *A* −> *carrier G*; *g* ∈ *A* −> *carrier G* |] ==>
finsum G (%*x*. *f x* ⊕ *g x*) *A* = (*finsum G f A* ⊕ *finsum G g A*)
by (*rule comm-monoid.finprod-multf* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *finsum-cong'*:
[| *A* = *B*; *g* : *B* −> *carrier G*;
!!*i*. *i* : *B* ==> *f i* = *g i* |] ==> *finsum G f A* = *finsum G g B*
by (*rule comm-monoid.finprod-cong'* [*OF a-comm-monoid*,
folded finsum-def, *simplified monoid-record-simps*]) *auto*

lemma (**in** *abelian-monoid*) *finsum-0* [*simp*]:
f : {*0*::*nat*} −> *carrier G* ==> *finsum G f* {..*0*} = *f 0*
by (*rule comm-monoid.finprod-0* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-Suc* [*simp*]:
f : {..*Suc n*} −> *carrier G* ==>
finsum G f {..*Suc n*} = (*f* (*Suc n*) ⊕ *finsum G f* {..*n*})
by (*rule comm-monoid.finprod-Suc* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-Suc2*:
f : {..*Suc n*} −> *carrier G* ==>
finsum G f {..*Suc n*} = (*finsum G* (%*i*. *f* (*Suc i*)) {..*n*} ⊕ *f 0*)
by (*rule comm-monoid.finprod-Suc2* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-add* [*simp*]:
[| *f* : {..*n*} −> *carrier G*; *g* : {..*n*} −> *carrier G* |] ==>
finsum G (%*i*. *f i* ⊕ *g i*) {..*n*::*nat*} =
finsum G f {..*n*} ⊕ *finsum G g* {..*n*}
by (*rule comm-monoid.finprod-mult* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps])

lemma (**in** *abelian-monoid*) *finsum-cong*:
[| *A* = *B*; *f* : *B* −> *carrier G*;
!!*i*. *i* : *B* =*simp*=> *f i* = *g i* |] ==> *finsum G f A* = *finsum G g B*
by (*rule comm-monoid.finprod-cong* [*OF a-comm-monoid*, *folded finsum-def*,
simplified monoid-record-simps]) (*auto simp add*: *simp-implies-def*)

Usually, if this rule causes a failed congruence proof error, the reason is that the premise *g* ∈ *B* −> *carrier G* cannot be shown. Adding *Pi-def* to the simpset is often useful.

9 The Algebraic Hierarchy of Rings

9.1 Basic Definitions

locale *ring* = *abelian-group R* + *monoid R* +
  assumes *l-distr*: [| $x \in$ *carrier R*; $y \in$ *carrier R*; $z \in$ *carrier R* |]
    ==> $(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z$
  and *r-distr*: [| $x \in$ *carrier R*; $y \in$ *carrier R*; $z \in$ *carrier R* |]
    ==> $z \otimes (x \oplus y) = z \otimes x \oplus z \otimes y$

locale *cring* = *ring* + *comm-monoid R*

locale *domain* = *cring* +
  assumes *one-not-zero* [*simp*]: $\mathbf{1} \sim= \mathbf{0}$
  and *integral*: [| $a \otimes b = \mathbf{0}$; $a \in$ *carrier R*; $b \in$ *carrier R* |] ==>
      $a = \mathbf{0} \mid b = \mathbf{0}$

locale *field* = *domain* +
  assumes *field-Units*: *Units R* = *carrier R* − {**0**}

9.2 Rings

lemma *ringI*:
  fixes *R* (**structure**)
  assumes *abelian-group*: *abelian-group R*
  and *monoid*: *monoid R*
  and *l-distr*: !!*x y z*. [| $x \in$ *carrier R*; $y \in$ *carrier R*; $z \in$ *carrier R* |]
    ==> $(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z$
  and *r-distr*: !!*x y z*. [| $x \in$ *carrier R*; $y \in$ *carrier R*; $z \in$ *carrier R* |]
    ==> $z \otimes (x \oplus y) = z \otimes x \oplus z \otimes y$
  shows *ring R*
  by (*auto intro*: *ring.intro*
    abelian-group.axioms ring-axioms.intro prems)

lemma (**in** *ring*) *is-abelian-group*:
  abelian-group R
  by (*auto intro*!: *abelian-groupI a-assoc a-comm l-neg*)

lemma (**in** *ring*) *is-monoid*:
  monoid R
  by (*auto intro*!: *monoidI m-assoc*)

lemma (**in** *ring*) *is-ring*:
  ring R
  by (*rule ring-axioms*)

lemmas *ring-record-simps* = *monoid-record-simps ring.simps*

lemma *cringI*:
  fixes *R* (**structure**)

```
  assumes abelian-group: abelian-group R
    and comm-monoid: comm-monoid R
    and l-distr: !!x y z. [| x ∈ carrier R; y ∈ carrier R; z ∈ carrier R |]
      ==> (x ⊕ y) ⊗ z = x ⊗ z ⊕ y ⊗ z
  shows cring R
proof (intro cring.intro ring.intro)
  show ring-axioms R
    — Right-distributivity follows from left-distributivity and commutativity.
  proof (rule ring-axioms.intro)
    fix x y z
    assume R: x ∈ carrier R y ∈ carrier R z ∈ carrier R
    note [simp] = comm-monoid.axioms [OF comm-monoid]
      abelian-group.axioms [OF abelian-group]
      abelian-monoid.a-closed

    from R have z ⊗ (x ⊕ y) = (x ⊕ y) ⊗ z
      by (simp add: comm-monoid.m-comm [OF comm-monoid.intro])
    also from R have ... = x ⊗ z ⊕ y ⊗ z by (simp add: l-distr)
    also from R have ... = z ⊗ x ⊕ z ⊗ y
      by (simp add: comm-monoid.m-comm [OF comm-monoid.intro])
    finally show z ⊗ (x ⊕ y) = z ⊗ x ⊕ z ⊗ y .
  qed (rule l-distr)
qed (auto intro: cring.intro
  abelian-group.axioms comm-monoid.axioms ring-axioms.intro prems)

lemma (in cring) is-comm-monoid:
  comm-monoid R
  by (auto intro!: comm-monoidI m-assoc m-comm)

lemma (in cring) is-cring:
  cring R by (rule cring-axioms)
```

9.2.1 Normaliser for Rings

```
lemma (in abelian-group) r-neg2:
  [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊕ (⊖ x ⊕ y) = y
proof −
  assume G: x ∈ carrier G y ∈ carrier G
  then have (x ⊕ ⊖ x) ⊕ y = y
    by (simp only: r-neg l-zero)
  with G show ?thesis
    by (simp add: a-ac)
qed

lemma (in abelian-group) r-neg1:
  [| x ∈ carrier G; y ∈ carrier G |] ==> ⊖ x ⊕ (x ⊕ y) = y
proof −
  assume G: x ∈ carrier G y ∈ carrier G
  then have (⊖ x ⊕ x) ⊕ y = y
```

by (*simp only*: *l-neg l-zero*)
with *G* **show** *?thesis* **by** (*simp add*: *a-ac*)
qed

The following proofs are from Jacobson, Basic Algebra I, pp. 88–89

lemma (**in** *ring*) *l-null* [*simp*]:
$x \in$ *carrier* $R \Longrightarrow \mathbf{0} \otimes x = \mathbf{0}$
proof −
assume *R*: $x \in$ *carrier* R
then have $\mathbf{0} \otimes x \oplus \mathbf{0} \otimes x = (\mathbf{0} \oplus \mathbf{0}) \otimes x$
by (*simp add*: *l-distr del*: *l-zero r-zero*)
also from *R* **have** $\ldots = \mathbf{0} \otimes x \oplus \mathbf{0}$ **by** *simp*
finally have $\mathbf{0} \otimes x \oplus \mathbf{0} \otimes x = \mathbf{0} \otimes x \oplus \mathbf{0}$ **.**
with *R* **show** *?thesis* **by** (*simp del*: *r-zero*)
qed

lemma (**in** *ring*) *r-null* [*simp*]:
$x \in$ *carrier* $R \Longrightarrow x \otimes \mathbf{0} = \mathbf{0}$
proof −
assume *R*: $x \in$ *carrier* R
then have $x \otimes \mathbf{0} \oplus x \otimes \mathbf{0} = x \otimes (\mathbf{0} \oplus \mathbf{0})$
by (*simp add*: *r-distr del*: *l-zero r-zero*)
also from *R* **have** $\ldots = x \otimes \mathbf{0} \oplus \mathbf{0}$ **by** *simp*
finally have $x \otimes \mathbf{0} \oplus x \otimes \mathbf{0} = x \otimes \mathbf{0} \oplus \mathbf{0}$ **.**
with *R* **show** *?thesis* **by** (*simp del*: *r-zero*)
qed

lemma (**in** *ring*) *l-minus*:
$[\![x \in$ *carrier* $R; y \in$ *carrier* $R]\!] \Longrightarrow \ominus x \otimes y = \ominus (x \otimes y)$
proof −
assume *R*: $x \in$ *carrier* R $y \in$ *carrier* R
then have $(\ominus x) \otimes y \oplus x \otimes y = (\ominus x \oplus x) \otimes y$ **by** (*simp add*: *l-distr*)
also from *R* **have** $\ldots = \mathbf{0}$ **by** (*simp add*: *l-neg l-null*)
finally have $(\ominus x) \otimes y \oplus x \otimes y = \mathbf{0}$ **.**
with *R* **have** $(\ominus x) \otimes y \oplus x \otimes y \oplus \ominus (x \otimes y) = \mathbf{0} \oplus \ominus (x \otimes y)$ **by** *simp*
with *R* **show** *?thesis* **by** (*simp add*: *a-assoc r-neg*)
qed

lemma (**in** *ring*) *r-minus*:
$[\![x \in$ *carrier* $R; y \in$ *carrier* $R]\!] \Longrightarrow x \otimes \ominus y = \ominus (x \otimes y)$
proof −
assume *R*: $x \in$ *carrier* R $y \in$ *carrier* R
then have $x \otimes (\ominus y) \oplus x \otimes y = x \otimes (\ominus y \oplus y)$ **by** (*simp add*: *r-distr*)
also from *R* **have** $\ldots = \mathbf{0}$ **by** (*simp add*: *l-neg r-null*)
finally have $x \otimes (\ominus y) \oplus x \otimes y = \mathbf{0}$ **.**
with *R* **have** $x \otimes (\ominus y) \oplus x \otimes y \oplus \ominus (x \otimes y) = \mathbf{0} \oplus \ominus (x \otimes y)$ **by** *simp*
with *R* **show** *?thesis* **by** (*simp add*: *a-assoc r-neg*)
qed

lemma (**in** *abelian-group*) *minus-eq*:
[| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==> $x \ominus y = x \oplus \ominus y$
by (*simp only*: *a-minus-def*)

Setup algebra method: compute distributive normal form in locale contexts

use *ringsimp.ML*

setup *Algebra.setup*

lemmas (**in** *ring*) *ring-simprules*
[*algebra ring zero R add R a-inv R a-minus R one R mult R*] =
a-closed zero-closed a-inv-closed minus-closed m-closed one-closed
a-assoc l-zero l-neg a-comm m-assoc l-one l-distr minus-eq
r-zero r-neg r-neg2 r-neg1 minus-add minus-minus minus-zero
a-lcomm r-distr l-null r-null l-minus r-minus

lemmas (**in** *cring*)
[*algebra del*: *ring zero R add R a-inv R a-minus R one R mult R*] =
-

lemmas (**in** *cring*) *cring-simprules*
[*algebra add*: *cring zero R add R a-inv R a-minus R one R mult R*] =
a-closed zero-closed a-inv-closed minus-closed m-closed one-closed
a-assoc l-zero l-neg a-comm m-assoc l-one l-distr m-comm minus-eq
r-zero r-neg r-neg2 r-neg1 minus-add minus-minus minus-zero
a-lcomm m-lcomm r-distr l-null r-null l-minus r-minus

lemma (**in** *cring*) *nat-pow-zero*:
(n::*nat*) ^~= *0* ==> **0** (^) n = **0**
by (*induct n*) *simp-all*

lemma (**in** *ring*) *one-zeroD*:
assumes *onezero*: **1** = **0**
shows *carrier R* = {**0**}
proof (*rule*, *rule*)
fix *x*
assume *xcarr*: $x \in$ *carrier R*
from *xcarr*
have $x = x \otimes$ **1** **by** *simp*
from *this* **and** *onezero*
have $x = x \otimes$ **0** **by** *simp*
from *this* **and** *xcarr*
have $x =$ **0** **by** *simp*
thus $x \in$ {**0**} **by** *fast*
qed *fast*

lemma (**in** *ring*) *one-zeroI*:
assumes *carrzero*: *carrier R* = {**0**}

shows $\mathbf{1} = \mathbf{0}$
proof −
from *one-closed* **and** *carrzero*
show $\mathbf{1} = \mathbf{0}$ **by** *simp*
qed

lemma (**in** *ring*) *one-zero*:
shows (*carrier* $R = \{\mathbf{0}\}$) = ($\mathbf{1} = \mathbf{0}$)
by (*rule*, *erule one-zeroI*, *erule one-zeroD*)

lemma (**in** *ring*) *one-not-zero*:
shows (*carrier* $R \neq \{\mathbf{0}\}$) = ($\mathbf{1} \neq \mathbf{0}$)
by (*simp add*: *one-zero*)

Two examples for use of method algebra

lemma
includes *ring* R + *cring* S
shows [| $a \in$ *carrier* R; $b \in$ *carrier* R; $c \in$ *carrier* S; $d \in$ *carrier* S |] ==>
$a \oplus \ominus (a \oplus \ominus b) = b$ & $c \otimes_S d = d \otimes_S c$
by *algebra*

lemma
includes *cring*
shows [| $a \in$ *carrier* R; $b \in$ *carrier* R |] ==> $a \ominus (a \ominus b) = b$
by *algebra*

9.2.2 Sums over Finite Sets

lemma (**in** *cring*) *finsum-ldistr*:
[| *finite* A; $a \in$ *carrier* R; $f \in A$ −> *carrier* R |] ==>
finsum R f $A \otimes a$ = *finsum* R (%i. $f\ i \otimes a$) A
proof (*induct set*: *finite*)
case *empty* **then show** *?case* **by** *simp*
next
case (*insert* x F) **then show** *?case* **by** (*simp add*: *Pi-def l-distr*)
qed

lemma (**in** *cring*) *finsum-rdistr*:
[| *finite* A; $a \in$ *carrier* R; $f \in A$ −> *carrier* R |] ==>
$a \otimes$ *finsum* R f A = *finsum* R (%i. $a \otimes f\ i$) A
proof (*induct set*: *finite*)
case *empty* **then show** *?case* **by** *simp*
next
case (*insert* x F) **then show** *?case* **by** (*simp add*: *Pi-def r-distr*)
qed

9.3 Integral Domains

lemma (**in** *domain*) *zero-not-one* [*simp*]:

0 $\tilde{=}$ **1**
by (*rule not-sym*) *simp*

lemma (**in** *domain*) *integral-iff*:
[| $a \in$ *carrier* R; $b \in$ *carrier* R |] ==> $(a \otimes b = \mathbf{0}) = (a = \mathbf{0} \mid b = \mathbf{0})$
proof
assume $a \in$ *carrier* R $b \in$ *carrier* R $a \otimes b = \mathbf{0}$
then show $a = \mathbf{0} \mid b = \mathbf{0}$ **by** (*simp add*: *integral*)
next
assume $a \in$ *carrier* R $b \in$ *carrier* R $a = \mathbf{0} \mid b = \mathbf{0}$
then show $a \otimes b = \mathbf{0}$ **by** *auto*
qed

lemma (**in** *domain*) *m-lcancel*:
assumes *prem*: $a \tilde{=} \mathbf{0}$
and R: $a \in$ *carrier* R $b \in$ *carrier* R $c \in$ *carrier* R
shows $(a \otimes b = a \otimes c) = (b = c)$
proof
assume *eq*: $a \otimes b = a \otimes c$
with R **have** $a \otimes (b \ominus c) = \mathbf{0}$ **by** *algebra*
with R **have** $a = \mathbf{0} \mid (b \ominus c) = \mathbf{0}$ **by** (*simp add*: *integral-iff*)
with *prem* **and** R **have** $b \ominus c = \mathbf{0}$ **by** *auto*
with R **have** $b = b \ominus (b \ominus c)$ **by** *algebra*
also from R **have** $b \ominus (b \ominus c) = c$ **by** *algebra*
finally show $b = c$ **.**
next
assume $b = c$ **then show** $a \otimes b = a \otimes c$ **by** *simp*
qed

lemma (**in** *domain*) *m-rcancel*:
assumes *prem*: $a \tilde{=} \mathbf{0}$
and R: $a \in$ *carrier* R $b \in$ *carrier* R $c \in$ *carrier* R
shows *conc*: $(b \otimes a = c \otimes a) = (b = c)$
proof −
from *prem* **and** R **have** $(a \otimes b = a \otimes c) = (b = c)$ **by** (*rule m-lcancel*)
with R **show** *?thesis* **by** *algebra*
qed

9.4 Fields

Field would not need to be derived from domain, the properties for domain follow from the assumptions of field

lemma (**in** *cring*) *cring-fieldI*:
assumes *field-Units*: *Units* R = *carrier* $R - \{\mathbf{0}\}$
shows *field* R
proof *unfold-locales*
from *field-Units*
have a: $\mathbf{0} \notin$ *Units* R **by** *fast*
have $\mathbf{1} \in$ *Units* R **by** *fast*

```
  from this and a
  show 1 ≠ 0 by force
next
  fix a b
  assume acarr: a ∈ carrier R
    and bcarr: b ∈ carrier R
    and ab: a ⊗ b = 0
  show a = 0 ∨ b = 0
  proof (cases a = 0, simp)
    assume a ≠ 0
    from this and field-Units and acarr
    have aUnit: a ∈ Units R by fast
    from bcarr
    have b = 1 ⊗ b by algebra
    also from aUnit acarr
    have ... = (inv a ⊗ a) ⊗ b by (simp add: Units-l-inv)
    also from acarr bcarr aUnit[THEN Units-inv-closed]
    have ... = (inv a) ⊗ (a ⊗ b) by algebra
    also from ab and acarr bcarr aUnit
    have ... = (inv a) ⊗ 0 by simp
    also from aUnit[THEN Units-inv-closed]
    have ... = 0 by algebra
    finally
    have b = 0 .
    thus a = 0 ∨ b = 0 by simp
  qed
qed (rule field-Units)
```

Another variant to show that something is a field

```
lemma (in cring) cring-fieldI2:
  assumes notzero: 0 ≠ 1
  and invex: ⋀a. [[a ∈ carrier R; a ≠ 0]] ⟹ ∃ b∈carrier R. a ⊗ b = 1
  shows field R
  apply (rule cring-fieldI, simp add: Units-def)
  apply (rule, clarsimp)
  apply (simp add: notzero)
proof (clarsimp)
  fix x
  assume xcarr: x ∈ carrier R
    and x ≠ 0
  from this
  have ∃ y∈carrier R. x ⊗ y = 1 by (rule invex)
  from this
  obtain y
    where ycarr: y ∈ carrier R
    and xy: x ⊗ y = 1
    by fast
  from xy xcarr ycarr have y ⊗ x = 1 by (simp add: m-comm)
  from ycarr and this and xy
```

show $\exists\, y{\in}carrier\ R.\ y \otimes x = \mathbf{1} \wedge x \otimes y = \mathbf{1}$ **by** *fast*
qed

9.5 Morphisms

constdefs (**structure** R S)
ring-hom :: $[('a, 'm)\ ring\text{-}scheme, ('b, 'n)\ ring\text{-}scheme] \Rightarrow ('a \Rightarrow 'b)\ set$
ring-hom R S == $\{h.\ h \in carrier\ R \rightarrow carrier\ S\ \&$
$(ALL\ x\ y.\ x \in carrier\ R\ \&\ y \in carrier\ R \longrightarrow$
$h\ (x \otimes y) = h\ x \otimes_S h\ y\ \&\ h\ (x \oplus y) = h\ x \oplus_S h\ y)\ \&$
$h\ \mathbf{1} = \mathbf{1}_S\}$

lemma *ring-hom-memI*:
fixes R (**structure**) **and** S (**structure**)
assumes *hom-closed*: $!!x.\ x \in carrier\ R \Longrightarrow h\ x \in carrier\ S$
and *hom-mult*: $!!x\ y.\ [|\ x \in carrier\ R;\ y \in carrier\ R\ |] \Longrightarrow$
$h\ (x \otimes y) = h\ x \otimes_S h\ y$
and *hom-add*: $!!x\ y.\ [|\ x \in carrier\ R;\ y \in carrier\ R\ |] \Longrightarrow$
$h\ (x \oplus y) = h\ x \oplus_S h\ y$
and *hom-one*: $h\ \mathbf{1} = \mathbf{1}_S$
shows $h \in ring\text{-}hom\ R\ S$
by (*auto simp add*: *ring-hom-def prems Pi-def*)

lemma *ring-hom-closed*:
$[|\ h \in ring\text{-}hom\ R\ S;\ x \in carrier\ R\ |] \Longrightarrow h\ x \in carrier\ S$
by (*auto simp add*: *ring-hom-def funcset-mem*)

lemma *ring-hom-mult*:
fixes R (**structure**) **and** S (**structure**)
shows
$[|\ h \in ring\text{-}hom\ R\ S;\ x \in carrier\ R;\ y \in carrier\ R\ |] \Longrightarrow$
$h\ (x \otimes y) = h\ x \otimes_S h\ y$
by (*simp add*: *ring-hom-def*)

lemma *ring-hom-add*:
fixes R (**structure**) **and** S (**structure**)
shows
$[|\ h \in ring\text{-}hom\ R\ S;\ x \in carrier\ R;\ y \in carrier\ R\ |] \Longrightarrow$
$h\ (x \oplus y) = h\ x \oplus_S h\ y$
by (*simp add*: *ring-hom-def*)

lemma *ring-hom-one*:
fixes R (**structure**) **and** S (**structure**)
shows $h \in ring\text{-}hom\ R\ S \Longrightarrow h\ \mathbf{1} = \mathbf{1}_S$
by (*simp add*: *ring-hom-def*)

locale *ring-hom-cring* = *cring* R + *cring* S +
fixes h
assumes *homh* [*simp*, *intro*]: $h \in ring\text{-}hom\ R\ S$

```
  notes hom-closed [simp, intro] = ring-hom-closed [OF homh]
    and hom-mult [simp] = ring-hom-mult [OF homh]
    and hom-add [simp] = ring-hom-add [OF homh]
    and hom-one [simp] = ring-hom-one [OF homh]

lemma (in ring-hom-cring) hom-zero [simp]:
  h 𝟬 = 𝟬_S
proof −
  have h 𝟬 ⊕_S h 𝟬 = h 𝟬 ⊕_S 𝟬_S
    by (simp add: hom-add [symmetric] del: hom-add)
  then show ?thesis by (simp del: S.r-zero)
qed

lemma (in ring-hom-cring) hom-a-inv [simp]:
  x ∈ carrier R ==> h (⊖ x) = ⊖_S h x
proof −
  assume R: x ∈ carrier R
  then have h x ⊕_S h (⊖ x) = h x ⊕_S (⊖_S h x)
    by (simp add: hom-add [symmetric] R.r-neg S.r-neg del: hom-add)
  with R show ?thesis by simp
qed

lemma (in ring-hom-cring) hom-finsum [simp]:
  [| finite A; f ∈ A -> carrier R |] ==>
  h (finsum R f A) = finsum S (h o f) A
proof (induct set: finite)
  case empty then show ?case by simp
next
  case insert then show ?case by (simp add: Pi-def)
qed

lemma (in ring-hom-cring) hom-finprod:
  [| finite A; f ∈ A -> carrier R |] ==>
  h (finprod R f A) = finprod S (h o f) A
proof (induct set: finite)
  case empty then show ?case by simp
next
  case insert then show ?case by (simp add: Pi-def)
qed

declare ring-hom-cring.hom-finprod [simp]

lemma id-ring-hom [simp]:
  id ∈ ring-hom R R
  by (auto intro!: ring-hom-memI)

end
```

theory *Module* **imports** *Ring* **begin**

10 Modules over an Abelian Group

10.1 Definitions

record $('a, 'b)$ *module* $=$ $'b$ *ring* $+$
  smult $::$ $['a, 'b] \Rightarrow 'b$ (**infixl** $\odot_1$ *70*)

locale *module* $=$ *cring* R $+$ *abelian-group* M $+$
  assumes *smult-closed* [*simp*, *intro*]:
    $[\![\ a \in carrier\ R;\ x \in carrier\ M\ ]\!] \Longrightarrow a \odot_M x \in carrier\ M$
   and *smult-l-distr*:
    $[\![\ a \in carrier\ R;\ b \in carrier\ R;\ x \in carrier\ M\ ]\!] \Longrightarrow$
    $(a \oplus b) \odot_M x = a \odot_M x \oplus_M b \odot_M x$
   and *smult-r-distr*:
    $[\![\ a \in carrier\ R;\ x \in carrier\ M;\ y \in carrier\ M\ ]\!] \Longrightarrow$
    $a \odot_M (x \oplus_M y) = a \odot_M x \oplus_M a \odot_M y$
   and *smult-assoc1*:
    $[\![\ a \in carrier\ R;\ b \in carrier\ R;\ x \in carrier\ M\ ]\!] \Longrightarrow$
    $(a \otimes b) \odot_M x = a \odot_M (b \odot_M x)$
   and *smult-one* [*simp*]:
    $x \in carrier\ M \Longrightarrow \mathbf{1} \odot_M x = x$

locale *algebra* $=$ *module* R M $+$ *cring* M $+$
  assumes *smult-assoc2*:
    $[\![\ a \in carrier\ R;\ x \in carrier\ M;\ y \in carrier\ M\ ]\!] \Longrightarrow$
    $(a \odot_M x) \otimes_M y = a \odot_M (x \otimes_M y)$

lemma *moduleI*:
  fixes R (**structure**) **and** M (**structure**)
  assumes *cring*: *cring* R
   and *abelian-group*: *abelian-group* M
   and *smult-closed*:
    $!!a\ x.\ [\![\ a \in carrier\ R;\ x \in carrier\ M\ ]\!] \Longrightarrow a \odot_M x \in carrier\ M$
   and *smult-l-distr*:
    $!!a\ b\ x.\ [\![\ a \in carrier\ R;\ b \in carrier\ R;\ x \in carrier\ M\ ]\!] \Longrightarrow$
    $(a \oplus b) \odot_M x = (a \odot_M x) \oplus_M (b \odot_M x)$
   and *smult-r-distr*:
    $!!a\ x\ y.\ [\![\ a \in carrier\ R;\ x \in carrier\ M;\ y \in carrier\ M\ ]\!] \Longrightarrow$
    $a \odot_M (x \oplus_M y) = (a \odot_M x) \oplus_M (a \odot_M y)$
   and *smult-assoc1*:
    $!!a\ b\ x.\ [\![\ a \in carrier\ R;\ b \in carrier\ R;\ x \in carrier\ M\ ]\!] \Longrightarrow$
    $(a \otimes b) \odot_M x = a \odot_M (b \odot_M x)$
   and *smult-one*:
    $!!x.\ x \in carrier\ M \Longrightarrow \mathbf{1} \odot_M x = x$
  shows *module* R M
  by (*auto intro*: *module.intro cring.axioms abelian-group.axioms*

module-axioms.intro prems)

lemma *algebraI*:
fixes *R* (**structure**) **and** *M* (**structure**)
assumes *R-cring*: *cring R*
and *M-cring*: *cring M*
and *smult-closed*:
!!*a x*. [| *a* ∈ *carrier R*; *x* ∈ *carrier M* |] ==> *a* ⊙$_M$ *x* ∈ *carrier M*
and *smult-l-distr*:
!!*a b x*. [| *a* ∈ *carrier R*; *b* ∈ *carrier R*; *x* ∈ *carrier M* |] ==>
(*a* ⊕ *b*) ⊙$_M$ *x* = (*a* ⊙$_M$ *x*) ⊕$_M$ (*b* ⊙$_M$ *x*)
and *smult-r-distr*:
!!*a x y*. [| *a* ∈ *carrier R*; *x* ∈ *carrier M*; *y* ∈ *carrier M* |] ==>
a ⊙$_M$ (*x* ⊕$_M$ *y*) = (*a* ⊙$_M$ *x*) ⊕$_M$ (*a* ⊙$_M$ *y*)
and *smult-assoc1*:
!!*a b x*. [| *a* ∈ *carrier R*; *b* ∈ *carrier R*; *x* ∈ *carrier M* |] ==>
(*a* ⊗ *b*) ⊙$_M$ *x* = *a* ⊙$_M$ (*b* ⊙$_M$ *x*)
and *smult-one*:
!!*x*. *x* ∈ *carrier M* ==> (*one R*) ⊙$_M$ *x* = *x*
and *smult-assoc2*:
!!*a x y*. [| *a* ∈ *carrier R*; *x* ∈ *carrier M*; *y* ∈ *carrier M* |] ==>
(*a* ⊙$_M$ *x*) ⊗$_M$ *y* = *a* ⊙$_M$ (*x* ⊗$_M$ *y*)
shows *algebra R M*
apply *intro-locales*
apply (*rule cring.axioms ring.axioms abelian-group.axioms comm-monoid.axioms prems*)+
apply (*rule module-axioms.intro*)
apply (*simp add*: *smult-closed*)
apply (*simp add*: *smult-l-distr*)
apply (*simp add*: *smult-r-distr*)
apply (*simp add*: *smult-assoc1*)
apply (*simp add*: *smult-one*)
apply (*rule cring.axioms ring.axioms abelian-group.axioms comm-monoid.axioms prems*)+
apply (*rule algebra-axioms.intro*)
apply (*simp add*: *smult-assoc2*)
done

lemma (**in** *algebra*) *R-cring*:
cring R
by *unfold-locales*

lemma (**in** *algebra*) *M-cring*:
cring M
by *unfold-locales*

lemma (**in** *algebra*) *module*:
module R M
by (*auto intro*: *moduleI R-cring is-abelian-group*

smult-l-distr smult-r-distr smult-assoc1)

10.2 Basic Properties of Algebras

lemma (**in** *algebra*) *smult-l-null* [*simp*]:
$x \in carrier\ M \Longrightarrow \mathbf{0} \odot_M x = \mathbf{0}_M$
proof −
assume *M*: $x \in carrier\ M$
note *facts* = *M smult-closed* [*OF R.zero-closed*]
from *facts* **have** $\mathbf{0} \odot_M x = (\mathbf{0} \odot_M x \oplus_M \mathbf{0} \odot_M x) \oplus_M \ominus_M (\mathbf{0} \odot_M x)$ **by** *algebra*
also from *M* **have** $... = (\mathbf{0} \oplus \mathbf{0}) \odot_M x \oplus_M \ominus_M (\mathbf{0} \odot_M x)$
by (*simp add*: *smult-l-distr del*: *R.l-zero R.r-zero*)
also from *facts* **have** $... = \mathbf{0}_M$ **apply** *algebra* **apply** *algebra* **done**
finally show *?thesis* **.**
qed

lemma (**in** *algebra*) *smult-r-null* [*simp*]:
$a \in carrier\ R \Longrightarrow a \odot_M \mathbf{0}_M = \mathbf{0}_M$
proof −
assume *R*: $a \in carrier\ R$
note *facts* = *R smult-closed*
from *facts* **have** $a \odot_M \mathbf{0}_M = (a \odot_M \mathbf{0}_M \oplus_M a \odot_M \mathbf{0}_M) \oplus_M \ominus_M (a \odot_M \mathbf{0}_M)$
by *algebra*
also from *R* **have** $... = a \odot_M (\mathbf{0}_M \oplus_M \mathbf{0}_M) \oplus_M \ominus_M (a \odot_M \mathbf{0}_M)$
by (*simp add*: *smult-r-distr del*: *M.l-zero M.r-zero*)
also from *facts* **have** $... = \mathbf{0}_M$ **by** *algebra*
finally show *?thesis* **.**
qed

lemma (**in** *algebra*) *smult-l-minus*:
$[|\ a \in carrier\ R; x \in carrier\ M\ |] \Longrightarrow (\ominus a) \odot_M x = \ominus_M (a \odot_M x)$
proof −
assume *RM*: $a \in carrier\ R\ x \in carrier\ M$
from *RM* **have** *a-smult*: $a \odot_M x \in carrier\ M$ **by** *simp*
from *RM* **have** *ma-smult*: $\ominus a \odot_M x \in carrier\ M$ **by** *simp*
note *facts* = *RM a-smult ma-smult*
from *facts* **have** $(\ominus a) \odot_M x = (\ominus a \odot_M x \oplus_M a \odot_M x) \oplus_M \ominus_M (a \odot_M x)$
by *algebra*
also from *RM* **have** $... = (\ominus a \oplus a) \odot_M x \oplus_M \ominus_M (a \odot_M x)$
by (*simp add*: *smult-l-distr*)
also from *facts smult-l-null* **have** $... = \ominus_M (a \odot_M x)$
apply *algebra* **apply** *algebra* **done**
finally show *?thesis* **.**
qed

lemma (**in** *algebra*) *smult-r-minus*:
$[|\ a \in carrier\ R; x \in carrier\ M\ |] \Longrightarrow a \odot_M (\ominus_M x) = \ominus_M (a \odot_M x)$
proof −
assume *RM*: $a \in carrier\ R\ x \in carrier\ M$

note *facts* = *RM smult-closed*
from *facts* **have** $a \odot_M (\ominus_M x) = (a \odot_M \ominus_M x \oplus_M a \odot_M x) \oplus_M \ominus_M (a \odot_M x)$
by *algebra*
also from *RM* **have** ... = $a \odot_M (\ominus_M x \oplus_M x) \oplus_M \ominus_M (a \odot_M x)$
by (*simp add*: *smult-r-distr*)
also from *facts smult-r-null* **have** ... = $\ominus_M (a \odot_M x)$ **by** *algebra*
finally show *?thesis* **.**
qed

end

theory *UnivPoly* **imports** *Module* **begin**

11 Univariate Polynomials

Polynomials are formalised as modules with additional operations for extracting coefficients from polynomials and for obtaining monomials from coefficients and exponents (record *up-ring*). The carrier set is a set of bounded functions from Nat to the coefficient domain. Bounded means that these functions return zero above a certain bound (the degree). There is a chapter on the formalisation of polynomials in the PhD thesis [1], which was implemented with axiomatic type classes. This was later ported to Locales.

11.1 The Constructor for Univariate Polynomials

Functions with finite support.

locale *bound* =
fixes $z :: 'a$
and $n ::$ *nat*
and $f ::$ *nat* => $'a$
assumes *bound*: $!!m.\ n < m \Longrightarrow f\ m = z$

declare *bound.intro* [*intro*!]
and *bound.bound* [*dest*]

lemma *bound-below*:
assumes *bound*: *bound z m f* **and** *nonzero*: $f\ n \neq z$ **shows** $n \leq m$
proof (*rule classical*)
assume ~ *?thesis*
then have $m < n$ **by** *arith*
with *bound* **have** $f\ n = z$ **..**
with *nonzero* **show** *?thesis* **by** *contradiction*
qed

record ($'a$, $'p$) *up-ring* = ($'a$, $'p$) *module* +

```
  monom :: ['a, nat] => 'p
  coeff :: ['p, nat] => 'a

constdefs (structure R)
  up :: ('a, 'm) ring-scheme => (nat => 'a) set
  up R == {f. f ∈ UNIV -> carrier R & (EX n. bound 0 n f)}
  UP :: ('a, 'm) ring-scheme => ('a, nat => 'a) up-ring
  UP R == (|
    carrier = up R,
    mult = (%p:up R. %q:up R. %n. ⨁ i ∈ {..n}. p i ⊗ q (n-i)),
    one = (%i. if i=0 then 1 else 0),
    zero = (%i. 0),
    add = (%p:up R. %q:up R. %i. p i ⊕ q i),
    smult = (%a:carrier R. %p:up R. %i. a ⊗ p i),
    monom = (%a:carrier R. %n i. if i=n then a else 0),
    coeff = (%p:up R. %n. p n) |)
```

Properties of the set of polynomials *up*.

```
lemma mem-upI [intro]:
  [| !!n. f n ∈ carrier R; EX n. bound (zero R) n f |] ==> f ∈ up R
  by (simp add: up-def Pi-def)

lemma mem-upD [dest]:
  f ∈ up R ==> f n ∈ carrier R
  by (simp add: up-def Pi-def)

lemma (in cring) bound-upD [dest]:
  f ∈ up R ==> EX n. bound 0 n f
  by (simp add: up-def)

lemma (in cring) up-one-closed:
    (%n. if n = 0 then 1 else 0) ∈ up R
  using up-def by force

lemma (in cring) up-smult-closed:
  [| a ∈ carrier R; p ∈ up R |] ==> (%i. a ⊗ p i) ∈ up R
  by force

lemma (in cring) up-add-closed:
  [| p ∈ up R; q ∈ up R |] ==> (%i. p i ⊕ q i) ∈ up R
proof
  fix n
  assume p ∈ up R and q ∈ up R
  then show p n ⊕ q n ∈ carrier R
    by auto
next
  assume UP: p ∈ up R q ∈ up R
  show EX n. bound 0 n (%i. p i ⊕ q i)
  proof -
```

```
    from UP obtain n where boundn: bound 0 n p by fast
    from UP obtain m where boundm: bound 0 m q by fast
    have bound 0 (max n m) (%i. p i ⊕ q i)
    proof
      fix i
      assume max n m < i
      with boundn and boundm and UP show p i ⊕ q i = 0 by fastsimp
    qed
    then show ?thesis ..
  qed
qed

lemma (in cring) up-a-inv-closed:
  p ∈ up R ==> (%i. ⊖ (p i)) ∈ up R
proof
  assume R: p ∈ up R
  then obtain n where bound 0 n p by auto
  then have bound 0 n (%i. ⊖ p i) by auto
  then show EX n. bound 0 n (%i. ⊖ p i) by auto
qed auto

lemma (in cring) up-mult-closed:
  [| p ∈ up R; q ∈ up R |] ==>
  (%n. ⨁ i ∈ {..n}. p i ⊗ q (n−i)) ∈ up R
proof
  fix n
  assume p ∈ up R q ∈ up R
  then show (⨁ i ∈ {..n}. p i ⊗ q (n−i)) ∈ carrier R
    by (simp add: mem-upD funcsetI)
next
  assume UP: p ∈ up R q ∈ up R
  show EX n. bound 0 n (%n. ⨁ i ∈ {..n}. p i ⊗ q (n−i))
  proof −
    from UP obtain n where boundn: bound 0 n p by fast
    from UP obtain m where boundm: bound 0 m q by fast
    have bound 0 (n + m) (%n. ⨁ i ∈ {..n}. p i ⊗ q (n − i))
    proof
      fix k assume bound: n + m < k
      {
        fix i
        have p i ⊗ q (k−i) = 0
        proof (cases n < i)
          case True
          with boundn have p i = 0 by auto
          moreover from UP have q (k−i) ∈ carrier R by auto
          ultimately show ?thesis by simp
        next
          case False
          with bound have m < k−i by arith
```

```
        with boundm have q (k−i) = 0 by auto
        moreover from UP have p i ∈ carrier R by auto
        ultimately show ?thesis by simp
      qed
    }
    then show (⨁ i ∈ {..k}. p i ⊗ q (k−i)) = 0
      by (simp add: Pi-def)
  qed
  then show ?thesis by fast
 qed
qed
```

11.2 Effect of Operations on Coefficients

```
locale UP =
 fixes R (structure) and P (structure)
 defines P-def: P == UP R

locale UP-cring = UP + cring R

locale UP-domain = UP-cring + domain R
```

Temporarily declare $P \equiv UP\ R$ as simp rule.

```
declare (in UP) P-def [simp]

lemma (in UP-cring) coeff-monom [simp]:
 a ∈ carrier R ==>
 coeff P (monom P a m) n = (if m=n then a else 0)
proof −
 assume R: a ∈ carrier R
 then have (%n. if n = m then a else 0) ∈ up R
   using up-def by force
 with R show ?thesis by (simp add: UP-def)
qed

lemma (in UP-cring) coeff-zero [simp]:
 coeff P 0_P n = 0
 by (auto simp add: UP-def)

lemma (in UP-cring) coeff-one [simp]:
 coeff P 1_P n = (if n=0 then 1 else 0)
 using up-one-closed by (simp add: UP-def)

lemma (in UP-cring) coeff-smult [simp]:
 [| a ∈ carrier R; p ∈ carrier P |] ==>
 coeff P (a ⊙_P p) n = a ⊗ coeff P p n
 by (simp add: UP-def up-smult-closed)

lemma (in UP-cring) coeff-add [simp]:
```

[| $p \in$ *carrier* P; $q \in$ *carrier* P |] ==>
coeff P ($p \oplus_P q$) n = *coeff* P p $n \oplus$ *coeff* P q n
by (*simp add*: *UP-def up-add-closed*)

lemma (**in** *UP-cring*) *coeff-mult* [*simp*]:
[| $p \in$ *carrier* P; $q \in$ *carrier* P |] ==>
coeff P ($p \otimes_P q$) n = ($\bigoplus i \in \{..n\}$. *coeff* P p $i \otimes$ *coeff* P q $(n-i)$)
by (*simp add*: *UP-def up-mult-closed*)

lemma (**in** *UP*) *up-eqI*:
assumes *prem*: !!n. *coeff* P p n = *coeff* P q n
and R: $p \in$ *carrier* P $q \in$ *carrier* P
shows $p = q$
proof
fix x
from *prem* **and** R **show** $p\ x = q\ x$ **by** (*simp add*: *UP-def*)
qed

11.3 Polynomials Form a Commutative Ring.

Operations are closed over P.

lemma (**in** *UP-cring*) *UP-mult-closed* [*simp*]:
[| $p \in$ *carrier* P; $q \in$ *carrier* P |] ==> $p \otimes_P q \in$ *carrier* P
by (*simp add*: *UP-def up-mult-closed*)

lemma (**in** *UP-cring*) *UP-one-closed* [*simp*]:
$\mathbf{1}_P \in$ *carrier* P
by (*simp add*: *UP-def up-one-closed*)

lemma (**in** *UP-cring*) *UP-zero-closed* [*intro*, *simp*]:
$\mathbf{0}_P \in$ *carrier* P
by (*auto simp add*: *UP-def*)

lemma (**in** *UP-cring*) *UP-a-closed* [*intro*, *simp*]:
[| $p \in$ *carrier* P; $q \in$ *carrier* P |] ==> $p \oplus_P q \in$ *carrier* P
by (*simp add*: *UP-def up-add-closed*)

lemma (**in** *UP-cring*) *monom-closed* [*simp*]:
$a \in$ *carrier* R ==> *monom* P a $n \in$ *carrier* P
by (*auto simp add*: *UP-def up-def Pi-def*)

lemma (**in** *UP-cring*) *UP-smult-closed* [*simp*]:
[| $a \in$ *carrier* R; $p \in$ *carrier* P |] ==> $a \odot_P p \in$ *carrier* P
by (*simp add*: *UP-def up-smult-closed*)

lemma (**in** *UP*) *coeff-closed* [*simp*]:
$p \in$ *carrier* P ==> *coeff* P p $n \in$ *carrier* R
by (*auto simp add*: *UP-def*)

declare (**in** *UP*) *P-def* [*simp del*]

Algebraic ring properties

lemma (**in** *UP-cring*) *UP-a-assoc*:
assumes *R*: $p \in carrier\ P\ q \in carrier\ P\ r \in carrier\ P$
shows $(p \oplus_P q) \oplus_P r = p \oplus_P (q \oplus_P r)$
by (*rule up-eqI*, *simp add*: *a-assoc R*, *simp-all add*: *R*)

lemma (**in** *UP-cring*) *UP-l-zero* [*simp*]:
assumes *R*: $p \in carrier\ P$
shows $\mathbf{0}_P \oplus_P p = p$
by (*rule up-eqI*, *simp-all add*: *R*)

lemma (**in** *UP-cring*) *UP-l-neg-ex*:
assumes *R*: $p \in carrier\ P$
shows $EX\ q : carrier\ P.\ q \oplus_P p = \mathbf{0}_P$
proof −
let *?q* = $\%i.\ \ominus\ (p\ i)$
from *R* **have** *closed*: *?q* $\in$ *carrier P*
by (*simp add*: *UP-def P-def up-a-inv-closed*)
from *R* **have** *coeff*: $!!n.\ coeff\ P\ ?q\ n = \ominus\ (coeff\ P\ p\ n)$
by (*simp add*: *UP-def P-def up-a-inv-closed*)
show *?thesis*
proof
show $?q \oplus_P p = \mathbf{0}_P$
by (*auto intro*!: *up-eqI simp add*: *R closed coeff R.l-neg*)
qed (*rule closed*)
qed

lemma (**in** *UP-cring*) *UP-a-comm*:
assumes *R*: $p \in carrier\ P\ q \in carrier\ P$
shows $p \oplus_P q = q \oplus_P p$
by (*rule up-eqI*, *simp add*: *a-comm R*, *simp-all add*: *R*)

lemma (**in** *UP-cring*) *UP-m-assoc*:
assumes *R*: $p \in carrier\ P\ q \in carrier\ P\ r \in carrier\ P$
shows $(p \otimes_P q) \otimes_P r = p \otimes_P (q \otimes_P r)$
proof (*rule up-eqI*)
fix *n*
{
fix *k* **and** *a b c* :: *nat*=>'*a*
assume *R*: *a* $\in$ *UNIV* −> *carrier R b* $\in$ *UNIV* −> *carrier R*
c $\in$ *UNIV* −> *carrier R*
then have *k* <= *n* ==>
$(\bigoplus j \in \{..k\}.\ (\bigoplus i \in \{..j\}.\ a\ i \otimes b\ (j-i)) \otimes c\ (n-j)) =$
$(\bigoplus j \in \{..k\}.\ a\ j \otimes (\bigoplus i \in \{..k-j\}.\ b\ i \otimes c\ (n-j-i)))$
(**is** - $\Longrightarrow$ *?eq k*)
proof (*induct k*)
case *0* **then show** *?case* **by** (*simp add*: *Pi-def m-assoc*)

```
    next
      case (Suc k)
      then have k <= n by arith
      from this R have ?eq k by (rule Suc)
      with R show ?case
        by (simp cong: finsum-cong
            add: Suc-diff-le Pi-def l-distr r-distr m-assoc)
          (simp cong: finsum-cong add: Pi-def a-ac finsum-ldistr m-assoc)
    qed
  }
  with R show coeff P ((p ⊗_P q) ⊗_P r) n = coeff P (p ⊗_P (q ⊗_P r)) n
    by (simp add: Pi-def)
qed (simp-all add: R)

lemma (in UP-cring) UP-l-one [simp]:
  assumes R: p ∈ carrier P
  shows 1_P ⊗_P p = p
proof (rule up-eqI)
  fix n
  show coeff P (1_P ⊗_P p) n = coeff P p n
  proof (cases n)
    case 0 with R show ?thesis by simp
  next
    case Suc with R show ?thesis
      by (simp del: finsum-Suc add: finsum-Suc2 Pi-def)
  qed
qed (simp-all add: R)

lemma (in UP-cring) UP-l-distr:
  assumes R: p ∈ carrier P q ∈ carrier P r ∈ carrier P
  shows (p ⊕_P q) ⊗_P r = (p ⊗_P r) ⊕_P (q ⊗_P r)
  by (rule up-eqI) (simp add: l-distr R Pi-def, simp-all add: R)

lemma (in UP-cring) UP-m-comm:
  assumes R: p ∈ carrier P q ∈ carrier P
  shows p ⊗_P q = q ⊗_P p
proof (rule up-eqI)
  fix n
  {
    fix k and a b :: nat=>'a
    assume R: a ∈ UNIV -> carrier R b ∈ UNIV -> carrier R
    then have k <= n ==>
      (⨁ i ∈ {..k}. a i ⊗ b (n-i)) =
      (⨁ i ∈ {..k}. a (k-i) ⊗ b (i+n-k))
      (is - ⟹ ?eq k)
    proof (induct k)
      case 0 then show ?case by (simp add: Pi-def)
    next
      case (Suc k) then show ?case
```

```
      by (subst (2) finsum-Suc2) (simp add: Pi-def a-comm)+
    qed
  }
  note l = this
  from R show coeff P (p ⊗_P q) n =  coeff P (q ⊗_P p) n
    apply (simp add: Pi-def)
    apply (subst l)
    apply (auto simp add: Pi-def)
    apply (simp add: m-comm)
    done
qed (simp-all add: R)

theorem (in UP-cring) UP-cring:
  cring P
  by (auto intro!: cringI abelian-groupI comm-monoidI UP-a-assoc UP-l-zero
    UP-l-neg-ex UP-a-comm UP-m-assoc UP-l-one UP-m-comm UP-l-distr)

lemma (in UP-cring) UP-ring:

  ring P
  by (auto intro: ring.intro cring.axioms UP-cring)

lemma (in UP-cring) UP-a-inv-closed [intro, simp]:
  p ∈ carrier P ==> ⊖_P p ∈ carrier P
  by (rule abelian-group.a-inv-closed
    [OF ring.is-abelian-group [OF UP-ring]])

lemma (in UP-cring) coeff-a-inv [simp]:
  assumes R: p ∈ carrier P
  shows coeff P (⊖_P p) n = ⊖ (coeff P p n)
proof −
  from R coeff-closed UP-a-inv-closed have
    coeff P (⊖_P p) n = ⊖ coeff P p n ⊕ (coeff P p n ⊕ coeff P (⊖_P p) n)
    by algebra
  also from R have ... =  ⊖ (coeff P p n)
    by (simp del: coeff-add add: coeff-add [THEN sym]
      abelian-group.r-neg [OF ring.is-abelian-group [OF UP-ring]])
  finally show ?thesis .
qed
```

Interpretation of lemmas from *cring*. Saves lifting 43 lemmas manually.

```
interpretation UP-cring < cring P
  by intro-locales
   (rule cring.axioms ring.axioms abelian-group.axioms comm-monoid.axioms UP-cring)+
```

11.4 Polynomials Form an Algebra

```
lemma (in UP-cring) UP-smult-l-distr:
  [| a ∈ carrier R; b ∈ carrier R; p ∈ carrier P |] ==>
```

$(a \oplus b) \odot_P p = a \odot_P p \oplus_P b \odot_P p$
by (*rule up-eqI*) (*simp-all add*: *R.l-distr*)

lemma (**in** *UP-cring*) *UP-smult-r-distr*:
$[\!|\ a \in$ *carrier R*; $p \in$ *carrier P*; $q \in$ *carrier P* $|\!] ==>$
$a \odot_P (p \oplus_P q) = a \odot_P p \oplus_P a \odot_P q$
by (*rule up-eqI*) (*simp-all add*: *R.r-distr*)

lemma (**in** *UP-cring*) *UP-smult-assoc1*:
$[\!|\ a \in$ *carrier R*; $b \in$ *carrier R*; $p \in$ *carrier P* $|\!] ==>$
$(a \otimes b) \odot_P p = a \odot_P (b \odot_P p)$
by (*rule up-eqI*) (*simp-all add*: *R.m-assoc*)

lemma (**in** *UP-cring*) *UP-smult-one* [*simp*]:
$p \in$ *carrier P* $==> \mathbf{1} \odot_P p = p$
by (*rule up-eqI*) *simp-all*

lemma (**in** *UP-cring*) *UP-smult-assoc2*:
$[\!|\ a \in$ *carrier R*; $p \in$ *carrier P*; $q \in$ *carrier P* $|\!] ==>$
$(a \odot_P p) \otimes_P q = a \odot_P (p \otimes_P q)$
by (*rule up-eqI*) (*simp-all add*: *R.finsum-rdistr R.m-assoc Pi-def*)

Interpretation of lemmas from *algebra*.

lemma (**in** *cring*) *cring*:
cring R
by (*fast intro*: *cring.intro prems*)

lemma (**in** *UP-cring*) *UP-algebra*:
algebra R P
by (*auto intro*: *algebraI R.cring UP-cring UP-smult-l-distr UP-smult-r-distr UP-smult-assoc1 UP-smult-assoc2*)

interpretation *UP-cring* < *algebra R P*
by *intro-locales*
(*rule module.axioms algebra.axioms UP-algebra*)+

11.5 Further Lemmas Involving Monomials

lemma (**in** *UP-cring*) *monom-zero* [*simp*]:
monom P $\mathbf{0}$ *n* $= \mathbf{0}_P$
by (*simp add*: *UP-def P-def*)

lemma (**in** *UP-cring*) *monom-mult-is-smult*:
assumes *R*: $a \in$ *carrier R* $p \in$ *carrier P*
shows *monom P a 0* $\otimes_P p = a \odot_P p$
proof (*rule up-eqI*)
fix *n*
have *coeff P* $(p \otimes_P$ *monom P a 0*$)$ *n* $=$ *coeff P* $(a \odot_P p)$ *n*
proof (*cases n*)

```
    case 0 with R show ?thesis by (simp add: R.m-comm)
  next
    case Suc with R show ?thesis
      by (simp cong: R.finsum-cong add: R.r-null Pi-def)
        (simp add: R.m-comm)
  qed
  with R show coeff P (monom P a 0 ⊗_P p) n = coeff P (a ⊙_P p) n
    by (simp add: UP-m-comm)
qed (simp-all add: R)

lemma (in UP-cring) monom-add [simp]:
  [| a ∈ carrier R; b ∈ carrier R |] ==>
  monom P (a ⊕ b) n = monom P a n ⊕_P monom P b n
  by (rule up-eqI) simp-all

lemma (in UP-cring) monom-one-Suc:
  monom P 1 (Suc n) = monom P 1 n ⊗_P monom P 1 1
proof (rule up-eqI)
  fix k
  show coeff P (monom P 1 (Suc n)) k = coeff P (monom P 1 n ⊗_P monom P
1 1) k
  proof (cases k = Suc n)
    case True show ?thesis
    proof -
      fix m
      from True have less-add-diff:
        !!i. [| n < i; i <= n + m |] ==> n + m - i < m by arith
      from True have coeff P (monom P 1 (Suc n)) k = 1 by simp
      also from True
      have ... = (⊕ i ∈ {..<n} ∪ {n}. coeff P (monom P 1 n) i ⊗
        coeff P (monom P 1 1) (k - i))
        by (simp cong: R.finsum-cong add: Pi-def)
      also have ... = (⊕ i ∈ {..n}. coeff P (monom P 1 n) i ⊗
        coeff P (monom P 1 1) (k - i))
        by (simp only: ivl-disj-un-singleton)
      also from True
      have ... = (⊕ i ∈ {..n} ∪ {n<..k}. coeff P (monom P 1 n) i ⊗
        coeff P (monom P 1 1) (k - i))
        by (simp cong: R.finsum-cong add: R.finsum-Un-disjoint ivl-disj-int-one
          order-less-imp-not-eq Pi-def)
      also from True have ... = coeff P (monom P 1 n ⊗_P monom P 1 1) k
        by (simp add: ivl-disj-un-one)
      finally show ?thesis .
    qed
  next
    case False
    note neq = False
    let ?s =
      λi. (if n = i then 1 else 0) ⊗ (if Suc 0 = k - i then 1 else 0)
```

```
    from neq have coeff P (monom P 1 (Suc n)) k = 0 by simp
    also have ... = (⨁ i ∈ {..k}. ?s i)
    proof −
      have f1: (⨁ i ∈ {..<n}. ?s i) = 0
        by (simp cong: R.finsum-cong add: Pi-def)
      from neq have f2: (⨁ i ∈ {n}. ?s i) = 0
        by (simp cong: R.finsum-cong add: Pi-def) arith
      have f3: n < k ==> (⨁ i ∈ {n<..k}. ?s i) = 0
        by (simp cong: R.finsum-cong add: order-less-imp-not-eq Pi-def)
      show ?thesis
      proof (cases k < n)
        case True then show ?thesis by (simp cong: R.finsum-cong add: Pi-def)
      next
        case False then have n-le-k: n <= k by arith
        show ?thesis
        proof (cases n = k)
          case True
          then have 0 = (⨁ i ∈ {..<n} ∪ {n}. ?s i)
            by (simp cong: R.finsum-cong add: ivl-disj-int-singleton Pi-def)
          also from True have ... = (⨁ i ∈ {..k}. ?s i)
            by (simp only: ivl-disj-un-singleton)
          finally show ?thesis .
        next
          case False with n-le-k have n-less-k: n < k by arith
          with neq have 0 = (⨁ i ∈ {..<n} ∪ {n}. ?s i)
            by (simp add: R.finsum-Un-disjoint f1 f2
              ivl-disj-int-singleton Pi-def del: Un-insert-right)
          also have ... = (⨁ i ∈ {..n}. ?s i)
            by (simp only: ivl-disj-un-singleton)
          also from n-less-k neq have ... = (⨁ i ∈ {..n} ∪ {n<..k}. ?s i)
            by (simp add: R.finsum-Un-disjoint f3 ivl-disj-int-one Pi-def)
          also from n-less-k have ... = (⨁ i ∈ {..k}. ?s i)
            by (simp only: ivl-disj-un-one)
          finally show ?thesis .
        qed
      qed
    qed
    also have ... = coeff P (monom P 1 n ⊗P monom P 1 1) k by simp
    finally show ?thesis .
  qed
qed (simp-all)

lemma (in UP-cring) monom-mult-smult:
  [| a ∈ carrier R; b ∈ carrier R |] ==> monom P (a ⊗ b) n = a ⊙P monom P
b n
  by (rule up-eqI) simp-all

lemma (in UP-cring) monom-one [simp]:
  monom P 1 0 = 1P
```

by (*rule up-eqI*) *simp-all*

lemma (**in** *UP-cring*) *monom-one-mult*:
monom P **1** (*n* + *m*) = *monom P* **1** *n* ⊗$_P$ *monom P* **1** *m*
proof (*induct n*)
case *0* **show** *?case* **by** *simp*
next
case *Suc* **then show** *?case*
by (*simp only*: *add-Suc monom-one-Suc*) (*simp add*: *P.m-ac*)
qed

lemma (**in** *UP-cring*) *monom-mult* [*simp*]:
assumes *R*: *a* ∈ *carrier R b* ∈ *carrier R*
shows *monom P* (*a* ⊗ *b*) (*n* + *m*) = *monom P a n* ⊗$_P$ *monom P b m*
proof −
from *R* **have** *monom P* (*a* ⊗ *b*) (*n* + *m*) = *monom P* (*a* ⊗ *b* ⊗ **1**) (*n* + *m*)
by *simp*
also from *R* **have** ... = *a* ⊗ *b* ⊙$_P$ *monom P* **1** (*n* + *m*)
by (*simp add*: *monom-mult-smult del*: *R.r-one*)
also have ... = *a* ⊗ *b* ⊙$_P$ (*monom P* **1** *n* ⊗$_P$ *monom P* **1** *m*)
by (*simp only*: *monom-one-mult*)
also from *R* **have** ... = *a* ⊙$_P$ (*b* ⊙$_P$ (*monom P* **1** *n* ⊗$_P$ *monom P* **1** *m*))
by (*simp add*: *UP-smult-assoc1*)
also from *R* **have** ... = *a* ⊙$_P$ (*b* ⊙$_P$ (*monom P* **1** *m* ⊗$_P$ *monom P* **1** *n*))
by (*simp add*: *P.m-comm*)
also from *R* **have** ... = *a* ⊙$_P$ ((*b* ⊙$_P$ *monom P* **1** *m*) ⊗$_P$ *monom P* **1** *n*)
by (*simp add*: *UP-smult-assoc2*)
also from *R* **have** ... = *a* ⊙$_P$ (*monom P* **1** *n* ⊗$_P$ (*b* ⊙$_P$ *monom P* **1** *m*))
by (*simp add*: *P.m-comm*)
also from *R* **have** ... = (*a* ⊙$_P$ *monom P* **1** *n*) ⊗$_P$ (*b* ⊙$_P$ *monom P* **1** *m*)
by (*simp add*: *UP-smult-assoc2*)
also from *R* **have** ... = *monom P* (*a* ⊗ **1**) *n* ⊗$_P$ *monom P* (*b* ⊗ **1**) *m*
by (*simp add*: *monom-mult-smult del*: *R.r-one*)
also from *R* **have** ... = *monom P a n* ⊗$_P$ *monom P b m* **by** *simp*
finally show *?thesis* **.**
qed

lemma (**in** *UP-cring*) *monom-a-inv* [*simp*]:
a ∈ *carrier R* ==> *monom P* (⊖ *a*) *n* = ⊖$_P$ *monom P a n*
by (*rule up-eqI*) *simp-all*

lemma (**in** *UP-cring*) *monom-inj*:
inj-on (%*a*. *monom P a n*) (*carrier R*)
proof (*rule inj-onI*)
fix *x y*
assume *R*: *x* ∈ *carrier R y* ∈ *carrier R* **and** *eq*: *monom P x n* = *monom P y n*
then have *coeff P* (*monom P x n*) *n* = *coeff P* (*monom P y n*) *n* **by** *simp*
with *R* **show** *x* = *y* **by** *simp*
qed

11.6 The Degree Function

constdefs (**structure** *R*)
deg :: [('*a*, '*m*) *ring-scheme*, *nat* => '*a*] => *nat*
deg R p == *LEAST n*. *bound* **0** *n* (*coeff* (*UP R*) *p*)

lemma (**in** *UP-cring*) *deg-aboveI*:
[| (!!*m*. *n* < *m* ==> *coeff P p m* = **0**); *p* ∈ *carrier P* |] ==> *deg R p* <= *n*
by (*unfold deg-def P-def*) (*fast intro*: *Least-le*)

lemma (**in** *UP-cring*) *deg-aboveD*:
assumes *deg R p* < *m* **and** *p* ∈ *carrier P*
shows *coeff P p m* = **0**
proof −
from ‹*p* ∈ *carrier P*› **obtain** *n* **where** *bound* **0** *n* (*coeff P p*)
by (*auto simp add*: *UP-def P-def*)
then have *bound* **0** (*deg R p*) (*coeff P p*)
by (*auto simp*: *deg-def P-def dest*: *LeastI*)
from *this* **and** ‹*deg R p* < *m*› **show** *?thesis* **..**
qed

lemma (**in** *UP-cring*) *deg-belowI*:
assumes *non-zero*: *n* ~= *0* ==> *coeff P p n* ~= **0**
and *R*: *p* ∈ *carrier P*
shows *n* <= *deg R p*
— Logically, this is a slightly stronger version of *deg-aboveD*
proof (*cases n=0*)
case *True* **then show** *?thesis* **by** *simp*
next
case *False* **then have** *coeff P p n* ~= **0** **by** (*rule non-zero*)
then have ~ *deg R p* < *n* **by** (*fast dest*: *deg-aboveD intro*: *R*)
then show *?thesis* **by** *arith*
qed

lemma (**in** *UP-cring*) *lcoeff-nonzero-deg*:
assumes *deg*: *deg R p* ~= *0* **and** *R*: *p* ∈ *carrier P*
shows *coeff P p* (*deg R p*) ~= **0**
proof −
from *R* **obtain** *m* **where** *deg R p* <= *m* **and** *m-coeff*: *coeff P p m* ~= **0**
proof −
have *minus*: !!(*n*::*nat*) *m*. *n* ~= *0* ==> (*n* − *Suc 0* < *m*) = (*n* <= *m*)
by *arith*

from *deg* **have** *deg R p* − *1* < (*LEAST n*. *bound* **0** *n* (*coeff P p*))
by (*unfold deg-def P-def*) *arith*
then have ~ *bound* **0** (*deg R p* − *1*) (*coeff P p*) **by** (*rule not-less-Least*)
then have *EX m*. *deg R p* − *1* < *m* & *coeff P p m* ~= **0**
by (*unfold bound-def*) *fast*

then have *EX m. deg R p <= m & coeff P p m ~= 0* **by** (*simp add*: *deg minus*)
then show *?thesis* **by** (*auto intro*: *that*)
qed
with *deg-belowI R* **have** *deg R p = m* **by** *fastsimp*
with *m-coeff* **show** *?thesis* **by** *simp*
qed

lemma (**in** *UP-cring*) *lcoeff-nonzero-nonzero*:
assumes *deg*: *deg R p = 0* **and** *nonzero*: *p ~= 0$_P$* **and** *R*: *p ∈ carrier P*
shows *coeff P p 0 ~= 0*
proof −
have *EX m. coeff P p m ~= 0*
proof (*rule classical*)
assume *~ ?thesis*
with *R* **have** *p = 0$_P$* **by** (*auto intro*: *up-eqI*)
with *nonzero* **show** *?thesis* **by** *contradiction*
qed
then obtain *m* **where** *coeff*: *coeff P p m ~= 0* **..**
from *this* **and** *R* **have** *m <= deg R p* **by** (*rule deg-belowI*)
then have *m = 0* **by** (*simp add*: *deg*)
with *coeff* **show** *?thesis* **by** *simp*
qed

lemma (**in** *UP-cring*) *lcoeff-nonzero*:
assumes *neq*: *p ~= 0$_P$* **and** *R*: *p ∈ carrier P*
shows *coeff P p* (*deg R p*) *~= 0*
proof (*cases deg R p = 0*)
case *True* **with** *neq R* **show** *?thesis* **by** (*simp add*: *lcoeff-nonzero-nonzero*)
next
case *False* **with** *neq R* **show** *?thesis* **by** (*simp add*: *lcoeff-nonzero-deg*)
qed

lemma (**in** *UP-cring*) *deg-eqI*:
[| !!*m*. *n < m ==> coeff P p m = 0*;
!!*n*. *n ~= 0 ==> coeff P p n ~= 0*; *p ∈ carrier P* |] *==> deg R p = n*
by (*fast intro*: *le-anti-sym deg-aboveI deg-belowI*)

Degree and polynomial operations

lemma (**in** *UP-cring*) *deg-add* [*simp*]:
assumes *R*: *p ∈ carrier P q ∈ carrier P*
shows *deg R* (*p ⊕$_P$ q*) *<= max* (*deg R p*) (*deg R q*)
proof (*cases deg R p <= deg R q*)
case *True* **show** *?thesis*
by (*rule deg-aboveI*) (*simp-all add*: *True R deg-aboveD*)
next
case *False* **show** *?thesis*
by (*rule deg-aboveI*) (*simp-all add*: *False R deg-aboveD*)
qed

lemma (**in** *UP-cring*) *deg-monom-le*:
a ∈ *carrier R* ==> *deg R* (*monom P a n*) <= *n*
by (*intro deg-aboveI*) *simp-all*

lemma (**in** *UP-cring*) *deg-monom* [*simp*]:
[| *a* ~= **0**; *a* ∈ *carrier R* |] ==> *deg R* (*monom P a n*) = *n*
by (*fastsimp intro*: *le-anti-sym deg-aboveI deg-belowI*)

lemma (**in** *UP-cring*) *deg-const* [*simp*]:
assumes *R*: *a* ∈ *carrier R* **shows** *deg R* (*monom P a 0*) = *0*
proof (*rule le-anti-sym*)
show *deg R* (*monom P a 0*) <= *0* **by** (*rule deg-aboveI*) (*simp-all add*: *R*)
next
show *0* <= *deg R* (*monom P a 0*) **by** (*rule deg-belowI*) (*simp-all add*: *R*)
qed

lemma (**in** *UP-cring*) *deg-zero* [*simp*]:
deg R **0**$_P$ = *0*
proof (*rule le-anti-sym*)
show *deg R* **0**$_P$ <= *0* **by** (*rule deg-aboveI*) *simp-all*
next
show *0* <= *deg R* **0**$_P$ **by** (*rule deg-belowI*) *simp-all*
qed

lemma (**in** *UP-cring*) *deg-one* [*simp*]:
deg R **1**$_P$ = *0*
proof (*rule le-anti-sym*)
show *deg R* **1**$_P$ <= *0* **by** (*rule deg-aboveI*) *simp-all*
next
show *0* <= *deg R* **1**$_P$ **by** (*rule deg-belowI*) *simp-all*
qed

lemma (**in** *UP-cring*) *deg-uminus* [*simp*]:
assumes *R*: *p* ∈ *carrier P* **shows** *deg R* (⊖$_P$ *p*) = *deg R p*
proof (*rule le-anti-sym*)
show *deg R* (⊖$_P$ *p*) <= *deg R p* **by** (*simp add*: *deg-aboveI deg-aboveD R*)
next
show *deg R p* <= *deg R* (⊖$_P$ *p*)
by (*simp add*: *deg-belowI lcoeff-nonzero-deg*
inj-on-iff [*OF R.a-inv-inj*, *of* - **0**, *simplified*] *R*)
qed

lemma (**in** *UP-domain*) *deg-smult-ring*:
[| *a* ∈ *carrier R*; *p* ∈ *carrier P* |] ==>
deg R (*a* ⊙$_P$ *p*) <= (*if a* = **0** *then 0 else deg R p*)
by (*cases a* = **0**) (*simp add*: *deg-aboveI deg-aboveD*)+

lemma (**in** *UP-domain*) *deg-smult* [*simp*]:

```
  assumes R: a ∈ carrier R p ∈ carrier P
  shows deg R (a ⊙_P p) = (if a = 𝟬 then 0 else deg R p)
proof (rule le-anti-sym)
  show deg R (a ⊙_P p) <= (if a = 𝟬 then 0 else deg R p)
    using R by (rule deg-smult-ring)
next
  show (if a = 𝟬 then 0 else deg R p) <= deg R (a ⊙_P p)
  proof (cases a = 𝟬)
  qed (simp, simp add: deg-belowI lcoeff-nonzero-deg integral-iff R)
qed

lemma (in UP-cring) deg-mult-cring:
  assumes R: p ∈ carrier P q ∈ carrier P
  shows deg R (p ⊗_P q) <= deg R p + deg R q
proof (rule deg-aboveI)
  fix m
  assume boundm: deg R p + deg R q < m
  {
    fix k i
    assume boundk: deg R p + deg R q < k
    then have coeff P p i ⊗ coeff P q (k − i) = 𝟬
    proof (cases deg R p < i)
      case True then show ?thesis by (simp add: deg-aboveD R)
    next
      case False with boundk have deg R q < k − i by arith
      then show ?thesis by (simp add: deg-aboveD R)
    qed
  }
  with boundm R show coeff P (p ⊗_P q) m = 𝟬 by simp
qed (simp add: R)

lemma (in UP-domain) deg-mult [simp]:
  [| p ~= 𝟬_P; q ~= 𝟬_P; p ∈ carrier P; q ∈ carrier P |] ==>
  deg R (p ⊗_P q) = deg R p + deg R q
proof (rule le-anti-sym)
  assume p ∈ carrier P  q ∈ carrier P
  then show deg R (p ⊗_P q) <= deg R p + deg R q by (rule deg-mult-cring)
next
  let ?s = (%i. coeff P p i ⊗ coeff P q (deg R p + deg R q − i))
  assume R: p ∈ carrier P q ∈ carrier P and nz: p ~= 𝟬_P q ~= 𝟬_P
  have less-add-diff: !!(k::nat) n m. k < n ==> m < n + m − k by arith
  show deg R p + deg R q <= deg R (p ⊗_P q)
  proof (rule deg-belowI, simp add: R)
    have (⨁ i ∈ {.. deg R p + deg R q}. ?s i)
      = (⨁ i ∈ {..< deg R p} ∪ {deg R p .. deg R p + deg R q}. ?s i)
      by (simp only: ivl-disj-un-one)
    also have ... = (⨁ i ∈ {deg R p .. deg R p + deg R q}. ?s i)
      by (simp cong: R.finsum-cong add: R.finsum-Un-disjoint ivl-disj-int-one
        deg-aboveD less-add-diff R Pi-def)
```

```
    also have ...= (⨁ i ∈ {deg R p} ∪ {deg R p <.. deg R p + deg R q}. ?s i)
      by (simp only: ivl-disj-un-singleton)
    also have ... = coeff P p (deg R p) ⊗ coeff P q (deg R q)
      by (simp cong: R.finsum-cong
        add: ivl-disj-int-singleton deg-aboveD R Pi-def)
    finally have (⨁ i ∈ {.. deg R p + deg R q}. ?s i)
      = coeff P p (deg R p) ⊗ coeff P q (deg R q) .
    with nz show (⨁ i ∈ {.. deg R p + deg R q}. ?s i) ~= 0
      by (simp add: integral-iff lcoeff-nonzero R)
    qed (simp add: R)
  qed

lemma (in UP-cring) coeff-finsum:
  assumes fin: finite A
  shows p ∈ A -> carrier P ==>
    coeff P (finsum P p A) k = (⨁ i ∈ A. coeff P (p i) k)
  using fin by induct (auto simp: Pi-def)

lemma (in UP-cring) up-repr:
  assumes R: p ∈ carrier P
  shows (⨁ P i ∈ {..deg R p}. monom P (coeff P p i) i) = p
proof (rule up-eqI)
  let ?s = (%i. monom P (coeff P p i) i)
  fix k
  from R have RR: !!i. (if i = k then coeff P p i else 0) ∈ carrier R
    by simp
  show coeff P (⨁ P i ∈ {..deg R p}. ?s i) k = coeff P p k
  proof (cases k <= deg R p)
    case True
    hence coeff P (⨁ P i ∈ {..deg R p}. ?s i) k =
        coeff P (⨁ P i ∈ {..k} ∪ {k<..deg R p}. ?s i) k
      by (simp only: ivl-disj-un-one)
    also from True
    have ... = coeff P (⨁ P i ∈ {..k}. ?s i) k
      by (simp cong: R.finsum-cong add: R.finsum-Un-disjoint
        ivl-disj-int-one order-less-imp-not-eq2 coeff-finsum R RR Pi-def)
    also
    have ... = coeff P (⨁ P i ∈ {..<k} ∪ {k}. ?s i) k
      by (simp only: ivl-disj-un-singleton)
    also have ... = coeff P p k
      by (simp cong: R.finsum-cong
        add: ivl-disj-int-singleton coeff-finsum deg-aboveD R RR Pi-def)
    finally show ?thesis .
  next
    case False
    hence coeff P (⨁ P i ∈ {..deg R p}. ?s i) k =
        coeff P (⨁ P i ∈ {..<deg R p} ∪ {deg R p}. ?s i) k
      by (simp only: ivl-disj-un-singleton)
    also from False have ... = coeff P p k
```

```
    by (simp cong: R.finsum-cong
      add: ivl-disj-int-singleton coeff-finsum deg-aboveD R Pi-def)
   finally show ?thesis .
 qed
qed (simp-all add: R Pi-def)

lemma (in UP-cring) up-repr-le:
 [| deg R p <= n; p ∈ carrier P |] ==>
 (⨁_P i ∈ {..n}. monom P (coeff P p i) i) = p
proof −
 let ?s = (%i. monom P (coeff P p i) i)
 assume R: p ∈ carrier P and deg R p <= n
 then have finsum P ?s {..n} = finsum P ?s ({..deg R p} ∪ {deg R p<..n})
   by (simp only: ivl-disj-un-one)
 also have ... = finsum P ?s {..deg R p}
   by (simp cong: P.finsum-cong add: P.finsum-Un-disjoint ivl-disj-int-one
     deg-aboveD R Pi-def)
 also have ... = p using R by (rule up-repr)
 finally show ?thesis .
qed
```

11.7 Polynomials over Integral Domains

```
lemma domainI:
 assumes cring: cring R
   and one-not-zero: one R ~= zero R
   and integral: !!a b. [| mult R a b = zero R; a ∈ carrier R;
     b ∈ carrier R |] ==> a = zero R | b = zero R
 shows domain R
 by (auto intro!: domain.intro domain-axioms.intro cring.axioms prems
   del: disjCI)

lemma (in UP-domain) UP-one-not-zero:
 1_P ~= 0_P
proof
 assume 1_P = 0_P
 hence coeff P 1_P 0 = (coeff P 0_P 0) by simp
 hence 1 = 0 by simp
 with one-not-zero show False by contradiction
qed

lemma (in UP-domain) UP-integral:
 [| p ⊗_P q = 0_P; p ∈ carrier P; q ∈ carrier P |] ==> p = 0_P | q = 0_P
proof −
 fix p q
 assume pq: p ⊗_P q = 0_P and R: p ∈ carrier P q ∈ carrier P
 show p = 0_P | q = 0_P
 proof (rule classical)
   assume c: ~ (p = 0_P | q = 0_P)
```

```
    with R have deg R p + deg R q = deg R (p ⊗_P q) by simp
    also from pq have ... = 0 by simp
    finally have deg R p + deg R q = 0 .
    then have f1: deg R p = 0 & deg R q = 0 by simp
    from f1 R have p = (⨁_P i ∈ {..0}. monom P (coeff P p i) i)
      by (simp only: up-repr-le)
    also from R have ... = monom P (coeff P p 0) 0 by simp
    finally have p: p = monom P (coeff P p 0) 0 .
    from f1 R have q = (⨁_P i ∈ {..0}. monom P (coeff P q i) i)
      by (simp only: up-repr-le)
    also from R have ... = monom P (coeff P q 0) 0 by simp
    finally have q: q = monom P (coeff P q 0) 0 .
    from R have coeff P p 0 ⊗ coeff P q 0 = coeff P (p ⊗_P q) 0 by simp
    also from pq have ... = 0 by simp
    finally have coeff P p 0 ⊗ coeff P q 0 = 0 .
    with R have coeff P p 0 = 0 | coeff P q 0 = 0
      by (simp add: R.integral-iff)
    with p q show p = 0_P | q = 0_P by fastsimp
  qed
qed

theorem (in UP-domain) UP-domain:
  domain P
  by (auto intro!: domainI UP-cring UP-one-not-zero UP-integral del: disjCI)
```

Interpretation of theorems from *domain*.

```
interpretation UP-domain < domain P
  by intro-locales (rule domain.axioms UP-domain)+
```

11.8 The Evaluation Homomorphism and Universal Property

```
theorem (in cring) diagonal-sum:
  [| f ∈ {..n + m::nat} -> carrier R; g ∈ {..n + m} -> carrier R |] ==>
  (⨁k ∈ {..n + m}. ⨁i ∈ {..k}. f i ⊗ g (k - i)) =
  (⨁k ∈ {..n + m}. ⨁i ∈ {..n + m - k}. f k ⊗ g i)
proof -
  assume Rf: f ∈ {..n + m} -> carrier R and Rg: g ∈ {..n + m} -> carrier R
  {
    fix j
    have j <= n + m ==>
      (⨁k ∈ {..j}. ⨁i ∈ {..k}. f i ⊗ g (k - i)) =
      (⨁k ∈ {..j}. ⨁i ∈ {..j - k}. f k ⊗ g i)
    proof (induct j)
      case 0 from Rf Rg show ?case by (simp add: Pi-def)
    next
      case (Suc j)
      have R6: !!i k. [| k <= j; i <= Suc j - k |] ==> g i ∈ carrier R
        using Suc by (auto intro!: funcset-mem [OF Rg])
```

```
    have R8: !!i k. [| k <= Suc j; i <= k |] ==> g (k - i) ∈ carrier R
      using Suc by (auto intro!: funcset-mem [OF Rg])
    have R9: !!i k. [| k <= Suc j |] ==> f k ∈ carrier R
      using Suc by (auto intro!: funcset-mem [OF Rf])
    have R10: !!i k. [| k <= Suc j; i <= Suc j - k |] ==> g i ∈ carrier R
      using Suc by (auto intro!: funcset-mem [OF Rg])
    have R11: g 0 ∈ carrier R
      using Suc by (auto intro!: funcset-mem [OF Rg])
    from Suc show ?case
      by (simp cong: finsum-cong add: Suc-diff-le a-ac
        Pi-def R6 R8 R9 R10 R11)
  qed
 }
 then show ?thesis by fast
qed

lemma (in abelian-monoid) boundD-carrier:
 [| bound 0 n f; n < m |] ==> f m ∈ carrier G
 by auto

theorem (in cring) cauchy-product:
 assumes bf: bound 0 n f and bg: bound 0 m g
   and Rf: f ∈ {..n} -> carrier R and Rg: g ∈ {..m} -> carrier R
 shows (⨁ k ∈ {..n + m}. ⨁ i ∈ {..k}. f i ⊗ g (k - i)) =
   (⨁ i ∈ {..n}. f i) ⊗ (⨁ i ∈ {..m}. g i)
proof -
 have f: !!x. f x ∈ carrier R
 proof -
   fix x
   show f x ∈ carrier R
     using Rf bf boundD-carrier by (cases x <= n) (auto simp: Pi-def)
 qed
 have g: !!x. g x ∈ carrier R
 proof -
   fix x
   show g x ∈ carrier R
     using Rg bg boundD-carrier by (cases x <= m) (auto simp: Pi-def)
 qed
 from f g have (⨁ k ∈ {..n + m}. ⨁ i ∈ {..k}. f i ⊗ g (k - i)) =
     (⨁ k ∈ {..n + m}. ⨁ i ∈ {..n + m - k}. f k ⊗ g i)
   by (simp add: diagonal-sum Pi-def)
 also have ... = (⨁ k ∈ {..n} ∪ {n<..n + m}. ⨁ i ∈ {..n + m - k}. f k ⊗ g i)
   by (simp only: ivl-disj-un-one)
 also from f g have ... = (⨁ k ∈ {..n}. ⨁ i ∈ {..n + m - k}. f k ⊗ g i)
   by (simp cong: finsum-cong
     add: bound.bound [OF bf] finsum-Un-disjoint ivl-disj-int-one Pi-def)
 also from f g
 have ... = (⨁ k ∈ {..n}. ⨁ i ∈ {..m} ∪ {m<..n + m - k}. f k ⊗ g i)
   by (simp cong: finsum-cong add: ivl-disj-un-one le-add-diff Pi-def)
```

also from *f g* **have** ... = $(\bigoplus k \in \{..n\}.\ \bigoplus i \in \{..m\}.\ f\ k \otimes g\ i)$
by (*simp cong*: *finsum-cong*
add: *bound.bound* [*OF bg*] *finsum-Un-disjoint ivl-disj-int-one Pi-def*)
also from *f g* **have** ... = $(\bigoplus i \in \{..n\}.\ f\ i) \otimes (\bigoplus i \in \{..m\}.\ g\ i)$
by (*simp add*: *finsum-ldistr diagonal-sum Pi-def*,
simp cong: *finsum-cong add*: *finsum-rdistr Pi-def*)
finally show *?thesis* **.**
qed

lemma (**in** *UP-cring*) *const-ring-hom*:
(%*a*. *monom P a 0*) ∈ *ring-hom R P*
by (*auto intro*!: *ring-hom-memI intro*: *up-eqI simp*: *monom-mult-is-smult*)

constdefs (**structure** *S*)
eval :: [('*a*, '*m*) *ring-scheme*, ('*b*, '*n*) *ring-scheme*,
'*a* => '*b*, '*b*, *nat* => '*a*] => '*b*
eval R S phi s == λ*p* ∈ *carrier* (*UP R*).
$\bigoplus i \in \{..deg\ R\ p\}.\ phi\ (coeff\ (UP\ R)\ p\ i) \otimes s\ (\hat{})\ i$

lemma (**in** *UP*) *eval-on-carrier*:
fixes *S* (**structure**)
shows *p* ∈ *carrier P* ==>
eval R S phi s p = $(\bigoplus_S i \in \{..deg\ R\ p\}.\ phi\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{})_S\ i)$
by (*unfold eval-def*, *fold P-def*) *simp*

lemma (**in** *UP*) *eval-extensional*:
eval R S phi p ∈ *extensional* (*carrier P*)
by (*unfold eval-def*, *fold P-def*) *simp*

The universal property of the polynomial ring

locale *UP-pre-univ-prop* = *ring-hom-cring R S h* + *UP-cring R P*

locale *UP-univ-prop* = *UP-pre-univ-prop* +
fixes *s* **and** *Eval*
assumes *indet-img-carrier* [*simp*, *intro*]: *s* ∈ *carrier S*
defines *Eval-def*: *Eval* == *eval R S h s*

theorem (**in** *UP-pre-univ-prop*) *eval-ring-hom*:
assumes *S*: *s* ∈ *carrier S*
shows *eval R S h s* ∈ *ring-hom P S*
proof (*rule ring-hom-memI*)
fix *p*
assume *R*: *p* ∈ *carrier P*
then show *eval R S h s p* ∈ *carrier S*
by (*simp only*: *eval-on-carrier*) (*simp add*: *S Pi-def*)
next
fix *p q*
assume *R*: *p* ∈ *carrier P q* ∈ *carrier P*

```
  then show eval R S h s (p ⊗_P q) = eval R S h s p ⊗_S eval R S h s q
  proof (simp only: eval-on-carrier UP-mult-closed)
    from R S have
      (⨁_S i ∈ {..deg R (p ⊗_P q)}. h (coeff P (p ⊗_P q) i) ⊗_S s (^)_S i) =
      (⨁_S i ∈ {..deg R (p ⊗_P q)} ∪ {deg R (p ⊗_P q)<..deg R p + deg R q}.
        h (coeff P (p ⊗_P q) i) ⊗_S s (^)_S i)
      by (simp cong: S.finsum-cong
        add: deg-aboveD S.finsum-Un-disjoint ivl-disj-int-one Pi-def
        del: coeff-mult)
    also from R have ... =
      (⨁_S i ∈ {..deg R p + deg R q}. h (coeff P (p ⊗_P q) i) ⊗_S s (^)_S i)
      by (simp only: ivl-disj-un-one deg-mult-cring)
    also from R S have ... =
      (⨁_S i ∈ {..deg R p + deg R q}.
        ⨁_S k ∈ {..i}.
          h (coeff P p k) ⊗_S h (coeff P q (i − k)) ⊗_S
          (s (^)_S k ⊗_S s (^)_S (i − k)))
      by (simp cong: S.finsum-cong add: S.nat-pow-mult Pi-def
        S.m-ac S.finsum-rdistr)
    also from R S have ... =
      (⨁_S i∈{..deg R p}. h (coeff P p i) ⊗_S s (^)_S i) ⊗_S
      (⨁_S i∈{..deg R q}. h (coeff P q i) ⊗_S s (^)_S i)
      by (simp add: S.cauchy-product [THEN sym] bound.intro deg-aboveD S.m-ac
        Pi-def)
    finally show
      (⨁_S i ∈ {..deg R (p ⊗_P q)}. h (coeff P (p ⊗_P q) i) ⊗_S s (^)_S i) =
      (⨁_S i ∈ {..deg R p}. h (coeff P p i) ⊗_S s (^)_S i) ⊗_S
      (⨁_S i ∈ {..deg R q}. h (coeff P q i) ⊗_S s (^)_S i) .
  qed
next
  fix p q
  assume R: p ∈ carrier P q ∈ carrier P
  then show eval R S h s (p ⊕_P q) = eval R S h s p ⊕_S eval R S h s q
  proof (simp only: eval-on-carrier P.a-closed)
    from S R have
      (⨁_S i∈{..deg R (p ⊕_P q)}. h (coeff P (p ⊕_P q) i) ⊗_S s (^)_S i) =
      (⨁_S i∈{..deg R (p ⊕_P q)} ∪ {deg R (p ⊕_P q)<..max (deg R p) (deg R q)}.
        h (coeff P (p ⊕_P q) i) ⊗_S s (^)_S i)
      by (simp cong: S.finsum-cong
        add: deg-aboveD S.finsum-Un-disjoint ivl-disj-int-one Pi-def
        del: coeff-add)
    also from R have ... =
        (⨁_S i ∈ {..max (deg R p) (deg R q)}.
          h (coeff P (p ⊕_P q) i) ⊗_S s (^)_S i)
      by (simp add: ivl-disj-un-one)
    also from R S have ... =
      (⨁_S i∈{..max (deg R p) (deg R q)}. h (coeff P p i) ⊗_S s (^)_S i) ⊕_S
      (⨁_S i∈{..max (deg R p) (deg R q)}. h (coeff P q i) ⊗_S s (^)_S i)
      by (simp cong: S.finsum-cong
```

```
      add: S.l-distr deg-aboveD ivl-disj-int-one Pi-def)
    also have ... =
      (⨁_S i ∈ {..deg R p} ∪ {deg R p<..max (deg R p) (deg R q)}.
        h (coeff P p i) ⊗_S s (^)_S i) ⊕_S
      (⨁_S i ∈ {..deg R q} ∪ {deg R q<..max (deg R p) (deg R q)}.
        h (coeff P q i) ⊗_S s (^)_S i)
      by (simp only: ivl-disj-un-one le-maxI1 le-maxI2)
    also from R S have ... =
      (⨁_S i ∈ {..deg R p}. h (coeff P p i) ⊗_S s (^)_S i) ⊕_S
      (⨁_S i ∈ {..deg R q}. h (coeff P q i) ⊗_S s (^)_S i)
      by (simp cong: S.finsum-cong
        add: deg-aboveD S.finsum-Un-disjoint ivl-disj-int-one Pi-def)
    finally show
      (⨁_S i ∈ {..deg R (p ⊕_P q)}. h (coeff P (p ⊕_P q) i) ⊗_S s (^)_S i) =
      (⨁_S i ∈ {..deg R p}. h (coeff P p i) ⊗_S s (^)_S i) ⊕_S
      (⨁_S i ∈ {..deg R q}. h (coeff P q i) ⊗_S s (^)_S i) .
  qed
next
  show eval R S h s 1_P = 1_S
    by (simp only: eval-on-carrier UP-one-closed) simp
qed
```

Interpretation of ring homomorphism lemmas.

```
interpretation UP-univ-prop < ring-hom-cring P S Eval
  apply (unfold Eval-def)
  apply intro-locales
  apply (rule ring-hom-cring.axioms)
  apply (rule ring-hom-cring.intro)
  apply unfold-locales
  apply (rule eval-ring-hom)
  apply rule
  done
```

Further properties of the evaluation homomorphism.

The following lemma could be proved in *UP-cring* with the additional assumption that h is closed.

```
lemma (in UP-pre-univ-prop) eval-const:
  [| s ∈ carrier S; r ∈ carrier R |] ==> eval R S h s (monom P r 0) = h r
  by (simp only: eval-on-carrier monom-closed) simp
```

The following proof is complicated by the fact that in arbitrary rings one might have $\mathbf{1}_R = \mathbf{0}_R$.

```
lemma (in UP-pre-univ-prop) eval-monom1:
  assumes S: s ∈ carrier S
  shows eval R S h s (monom P 1 1) = s
proof (simp only: eval-on-carrier monom-closed R.one-closed)
  from S have
```

```
    (⨁_S i∈{..deg R (monom P 1 1)}. h (coeff P (monom P 1 1) i) ⊗_S s (^)_S i)
=
    (⨁_S i∈{..deg R (monom P 1 1)} ∪ {deg R (monom P 1 1)<..1}.
      h (coeff P (monom P 1 1) i) ⊗_S s (^)_S i)
    by (simp cong: S.finsum-cong del: coeff-monom
      add: deg-aboveD S.finsum-Un-disjoint ivl-disj-int-one Pi-def)
  also have ... =
    (⨁_S i ∈ {..1}. h (coeff P (monom P 1 1) i) ⊗_S s (^)_S i)
    by (simp only: ivl-disj-un-one deg-monom-le R.one-closed)
  also have ... = s
  proof (cases s = 0_S)
    case True then show ?thesis by (simp add: Pi-def)
  next
    case False then show ?thesis by (simp add: S Pi-def)
  qed
  finally show (⨁_S i ∈ {..deg R (monom P 1 1)}.
    h (coeff P (monom P 1 1) i) ⊗_S s (^)_S i) = s .
qed

lemma (in UP-cring) monom-pow:
  assumes R: a ∈ carrier R
  shows (monom P a n) (^)_P m = monom P (a (^) m) (n * m)
proof (induct m)
  case 0 from R show ?case by simp
next
  case Suc with R show ?case
    by (simp del: monom-mult add: monom-mult [THEN sym] add-commute)
qed

lemma (in ring-hom-cring) hom-pow [simp]:
  x ∈ carrier R ==> h (x (^) n) = h x (^)_S (n::nat)
  by (induct n) simp-all

lemma (in UP-univ-prop) Eval-monom:
  r ∈ carrier R ==> Eval (monom P r n) = h r ⊗_S s (^)_S n
proof −
  assume R: r ∈ carrier R
  from R have Eval (monom P r n) = Eval (monom P r 0 ⊗_P (monom P 1 1)
(^)_P n)
    by (simp del: monom-mult add: monom-mult [THEN sym] monom-pow)
  also
  from R eval-monom1 [where s = s, folded Eval-def]
  have ... = h r ⊗_S s (^)_S n
    by (simp add: eval-const [where s = s, folded Eval-def])
  finally show ?thesis .
qed

lemma (in UP-pre-univ-prop) eval-monom:
  assumes R: r ∈ carrier R and S: s ∈ carrier S
```

```
  shows eval R S h s (monom P r n) = h r ⊗_S s ( ^)_S n
proof −
  interpret UP-univ-prop [R S h P s -]
    using UP-pre-univ-prop-axioms P-def R S
    by (auto intro: UP-univ-prop.intro UP-univ-prop-axioms.intro)
  from R
  show ?thesis by (rule Eval-monom)
qed

lemma (in UP-univ-prop) Eval-smult:
  [| r ∈ carrier R; p ∈ carrier P |] ==> Eval (r ⊙_P p) = h r ⊗_S Eval p
proof −
  assume R: r ∈ carrier R and P: p ∈ carrier P
  then show ?thesis
    by (simp add: monom-mult-is-smult [THEN sym]
      eval-const [where s = s, folded Eval-def])
qed

lemma ring-hom-cringI:
  assumes cring R
    and cring S
    and h ∈ ring-hom R S
  shows ring-hom-cring R S h
  by (fast intro: ring-hom-cring.intro ring-hom-cring-axioms.intro
    cring.axioms prems)

lemma (in UP-pre-univ-prop) UP-hom-unique:
  includes ring-hom-cring P S Phi
  assumes Phi: Phi (monom P 1 (Suc 0)) = s
      !!r. r ∈ carrier R ==> Phi (monom P r 0) = h r
  includes ring-hom-cring P S Psi
  assumes Psi: Psi (monom P 1 (Suc 0)) = s
      !!r. r ∈ carrier R ==> Psi (monom P r 0) = h r
    and P: p ∈ carrier P and S: s ∈ carrier S
  shows Phi p = Psi p
proof −
  have Phi p =
      Phi (⨁_P i ∈ {..deg R p}. monom P (coeff P p i) 0 ⊗_P monom P 1 1 ( ^)_P
i)
    by (simp add: up-repr P monom-mult [THEN sym] monom-pow del: monom-mult)
  also
  have ... =
      Psi (⨁_P i∈{..deg R p}. monom P (coeff P p i) 0 ⊗_P monom P 1 1 ( ^)_P i)
    by (simp add: Phi Psi P Pi-def comp-def)
  also have ... = Psi p
    by (simp add: up-repr P monom-mult [THEN sym] monom-pow del: monom-mult)
  finally show ?thesis .
qed
```

lemma (**in** *UP-pre-univ-prop*) *ring-homD*:
assumes *Phi*: *Phi* ∈ *ring-hom P S*
shows *ring-hom-cring P S Phi*
proof (*rule ring-hom-cring.intro*)
show *ring-hom-cring-axioms P S Phi*
by (*rule ring-hom-cring-axioms.intro*) (*rule Phi*)
qed *unfold-locales*

theorem (**in** *UP-pre-univ-prop*) *UP-universal-property*:
assumes *S*: *s* ∈ *carrier S*
shows *EX*! *Phi*. *Phi* ∈ *ring-hom P S* ∩ *extensional* (*carrier P*) &
Phi (*monom P* **1** *1*) = *s* &
(*ALL r* : *carrier R*. *Phi* (*monom P r 0*) = *h r*)
using *S eval-monom1*
apply (*auto intro*: *eval-ring-hom eval-const eval-extensional*)
apply (*rule extensionalityI*)
apply (*auto intro*: *UP-hom-unique ring-homD*)
done

11.9 Sample Application of Evaluation Homomorphism

lemma *UP-pre-univ-propI*:
assumes *cring R*
and *cring S*
and *h* ∈ *ring-hom R S*
shows *UP-pre-univ-prop R S h*
using *assms*
by (*auto intro*!: *UP-pre-univ-prop.intro ring-hom-cring.intro*
ring-hom-cring-axioms.intro UP-cring.intro)

constdefs
INTEG :: *int ring*
INTEG == (| *carrier* = *UNIV*, *mult* = *op* ∗, *one* = *1*, *zero* = *0*, *add* = *op* +
|)

lemma *INTEG-cring*:
cring INTEG
by (*unfold INTEG-def*) (*auto intro*!: *cringI abelian-groupI comm-monoidI*
zadd-zminus-inverse2 zadd-zmult-distrib)

lemma *INTEG-id-eval*:
UP-pre-univ-prop INTEG INTEG id
by (*fast intro*: *UP-pre-univ-propI INTEG-cring id-ring-hom*)

Interpretation now enables to import all theorems and lemmas valid in the context of homomorphisms between *INTEG* and *UP INTEG* globally.

interpretation *INTEG*: *UP-pre-univ-prop* [*INTEG INTEG id*]
apply *simp*
using *INTEG-id-eval*

apply *simp*
done

lemma *INTEG-closed* [*intro*, *simp*]:
z ∈ *carrier INTEG*
by (*unfold INTEG-def*) *simp*

lemma *INTEG-mult* [*simp*]:
mult INTEG z w = *z* ∗ *w*
by (*unfold INTEG-def*) *simp*

lemma *INTEG-pow* [*simp*]:
pow INTEG z n = *z* ^ *n*
by (*induct n*) (*simp-all add*: *INTEG-def nat-pow-def*)

lemma *eval INTEG INTEG id 10* (*monom* (*UP INTEG*) *5 2*) = *500*
by (*simp add*: *INTEG.eval-monom*)

end

theory *AbelCoset*
imports *Coset Ring*
begin

12 More Lifting from Groups to Abelian Groups

12.1 Definitions

Hiding <+> from *Sum-Type* until I come up with better syntax here

hide *const Plus*

constdefs (**structure** *G*)
a-r-coset :: [-, *'a set*, *'a*] ⇒ *'a set* (**infixl** +>$_1$ *60*)
a-r-coset G ≡ *r-coset* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

a-l-coset :: [-, *'a*, *'a set*] ⇒ *'a set* (**infixl** <+$_1$ *60*)
a-l-coset G ≡ *l-coset* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

A-RCOSETS :: [-, *'a set*] ⇒ (*'a set*)*set* (*a'-rcosets*$_1$ - [*81*] *80*)
A-RCOSETS G H ≡ *RCOSETS* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|) *H*

set-add :: [-, *'a set* ,*'a set*] ⇒ *'a set* (**infixl** <+>$_1$ *60*)
set-add G ≡ *set-mult* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

A-SET-INV :: [-,*'a set*] ⇒ *'a set* (*a'-set'-inv*$_1$ - [*81*] *80*)

A-SET-INV G H ≡ *SET-INV* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|) *H*

constdefs (**structure** *G*)
a-r-congruent :: [('*a*,'*b*)*ring-scheme*, '*a set*] ⇒ ('*a*∗'*a*)*set*
(*racong*ı -)
a-r-congruent G ≡ *r-congruent* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

constdefs
A-FactGroup :: [('*a*,'*b*) *ring-scheme*, '*a set*] ⇒ ('*a set*) *monoid*
(**infixl** *A'-Mod 65*)
— Actually defined for groups rather than monoids
A-FactGroup G H ≡ *FactGroup* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|) *H*

constdefs
a-kernel :: ('*a*, '*m*) *ring-scheme* ⇒ ('*b*, '*n*) *ring-scheme* ⇒
('*a* ⇒ '*b*) ⇒ '*a set*
— the kernel of a homomorphism (additive)
a-kernel G H h ≡ *kernel* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)
(|*carrier* = *carrier H*, *mult* = *add H*, *one* = *zero H*|) *h*

locale *abelian-group-hom* = *abelian-group G* + *abelian-group H* + *var h* +
assumes *a-group-hom*: *group-hom* (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)
(| *carrier* = *carrier H*, *mult* = *add H*, *one* = *zero H* |) *h*

lemmas *a-r-coset-defs* =
a-r-coset-def r-coset-def

lemma *a-r-coset-def'*:
includes *struct G*
shows *H* +> *a* ≡ ⋃*h*∈*H*. {*h* ⊕ *a*}
unfolding *a-r-coset-defs*
by *simp*

lemmas *a-l-coset-defs* =
a-l-coset-def l-coset-def

lemma *a-l-coset-def'*:
includes *struct G*
shows *a* <+ *H* ≡ ⋃*h*∈*H*. {*a* ⊕ *h*}
unfolding *a-l-coset-defs*
by *simp*

lemmas *A-RCOSETS-defs* =
A-RCOSETS-def RCOSETS-def

lemma *A-RCOSETS-def'*:
includes *struct G*
shows *a-rcosets H* $\equiv \bigcup a \in$ *carrier G*. $\{H +> a\}$
unfolding *A-RCOSETS-defs*
by (*fold a-r-coset-def*, *simp*)

lemmas *set-add-defs* =
set-add-def set-mult-def

lemma *set-add-def'*:
includes *struct G*
shows *H* <+> *K* $\equiv \bigcup h \in H. \bigcup k \in K. \{h \oplus k\}$
unfolding *set-add-defs*
by *simp*

lemmas *A-SET-INV-defs* =
A-SET-INV-def SET-INV-def

lemma *A-SET-INV-def'*:
includes *struct G*
shows *a-set-inv H* $\equiv \bigcup h \in H. \{\ominus h\}$
unfolding *A-SET-INV-defs*
by (*fold a-inv-def*)

12.2 Cosets

lemma (**in** *abelian-group*) *a-coset-add-assoc*:
$[|\ M \subseteq$ *carrier G*; $g \in$ *carrier G*; $h \in$ *carrier G* $|]$
$==> (M +> g) +> h = M +> (g \oplus h)$
by (*rule group.coset-mult-assoc* [*OF a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-coset-add-zero* [*simp*]:
$M \subseteq$ *carrier G* $==> M +> \mathbf{0} = M$
by (*rule group.coset-mult-one* [*OF a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-coset-add-inv1*:
$[|\ M +> (x \oplus (\ominus y)) = M;\ x \in$ *carrier G* ; $y \in$ *carrier G*;
$M \subseteq$ *carrier G* $|] ==> M +> x = M +> y$
by (*rule group.coset-mult-inv1* [*OF a-group*,
folded a-r-coset-def a-inv-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-coset-add-inv2*:
$[|\ M +> x = M +> y;\ x \in$ *carrier G*; $y \in$ *carrier G*; $M \subseteq$ *carrier G* $|]$
$==> M +> (x \oplus (\ominus y)) = M$
by (*rule group.coset-mult-inv2* [*OF a-group*,
folded a-r-coset-def a-inv-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-coset-join1*:
[| H +> $x = H$; $x \in$ *carrier* G; *subgroup* H (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|) |] ==> $x \in H$
by (*rule group.coset-join1* [*OF a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-solve-equation*:
⟦*subgroup* H (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|); $x \in H$; $y \in H$⟧ ⟹ $\exists h{\in}H.\ y = h \oplus x$
by (*rule group.solve-equation* [*OF a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-repr-independence*:
⟦$y \in H$ +> x; $x \in$ *carrier* G; *subgroup* H (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|) ⟧ ⟹ H +> $x = H$ +> y
by (*rule group.repr-independence* [*OF a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-coset-join2*:
⟦$x \in$ *carrier* G; *subgroup* H (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|); $x{\in}H$⟧ ⟹ H +> $x = H$
by (*rule group.coset-join2* [*OF a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-monoid*) *a-r-coset-subset-G*:
[| $H \subseteq$ *carrier* G; $x \in$ *carrier* G |] ==> H +> $x \subseteq$ *carrier* G
by (*rule monoid.r-coset-subset-G* [*OF a-monoid*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-rcosI*:
[| $h \in H$; $H \subseteq$ *carrier* G; $x \in$ *carrier* G|] ==> $h \oplus x \in H$ +> x
by (*rule group.rcosI* [*OF a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-rcosetsI*:
⟦$H \subseteq$ *carrier* G; $x \in$ *carrier* G⟧ ⟹ H +> $x \in$ *a-rcosets* H
by (*rule group.rcosetsI* [*OF a-group*,
folded a-r-coset-def A-RCOSETS-def, *simplified monoid-record-simps*])

Really needed?

lemma (**in** *abelian-group*) *a-transpose-inv*:
[| $x \oplus y = z$; $x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* G |]
==> $(\ominus x) \oplus z = y$
by (*rule group.transpose-inv* [*OF a-group*,
folded a-r-coset-def a-inv-def, *simplified monoid-record-simps*])

12.3 Subgroups

locale *additive-subgroup* = *var* H + *struct* G +

assumes *a-subgroup*: *subgroup H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

lemma (**in** *additive-subgroup*) *is-additive-subgroup*:
shows *additive-subgroup H G*
by (*rule additive-subgroup-axioms*)

lemma *additive-subgroupI*:
includes *struct G*
assumes *a-subgroup*: *subgroup H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)
shows *additive-subgroup H G*
by (*rule additive-subgroup.intro*) (*rule a-subgroup*)

lemma (**in** *additive-subgroup*) *a-subset*:
$H \subseteq$ *carrier G*
by (*rule subgroup.subset*[*OF a-subgroup*,
simplified monoid-record-simps])

lemma (**in** *additive-subgroup*) *a-closed* [*intro*, *simp*]:
$[\![x \in H; y \in H]\!] \Longrightarrow x \oplus y \in H$
by (*rule subgroup.m-closed*[*OF a-subgroup*,
simplified monoid-record-simps])

lemma (**in** *additive-subgroup*) *zero-closed* [*simp*]:
$\mathbf{0} \in H$
by (*rule subgroup.one-closed*[*OF a-subgroup*,
simplified monoid-record-simps])

lemma (**in** *additive-subgroup*) *a-inv-closed* [*intro*,*simp*]:
$x \in H \Longrightarrow \ominus x \in H$
by (*rule subgroup.m-inv-closed*[*OF a-subgroup*,
folded a-inv-def, *simplified monoid-record-simps*])

12.4 Normal additive subgroups

12.4.1 Definition of *abelian-subgroup*

Every subgroup of an *abelian-group* is normal

locale *abelian-subgroup* = *additive-subgroup H G* + *abelian-group G* +
assumes *a-normal*: *normal H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

lemma (**in** *abelian-subgroup*) *is-abelian-subgroup*:
shows *abelian-subgroup H G*
by (*rule abelian-subgroup-axioms*)

lemma *abelian-subgroupI*:
assumes *a-normal*: *normal H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero*

```
G⦈
    and a-comm: !!x y. [| x ∈ carrier G; y ∈ carrier G |] ==> x ⊕_G y = y ⊕_G
x
  shows abelian-subgroup H G
proof −
  interpret normal [H (⦇carrier = carrier G, mult = add G, one = zero G⦈)]
  by (rule a-normal)

  show abelian-subgroup H G
  by (unfold-locales, simp add: a-comm)
qed

lemma abelian-subgroupI2:
  includes struct G
  assumes a-comm-group: comm-group (⦇carrier = carrier G, mult = add G, one
= zero G⦈)
    and a-subgroup: subgroup H (⦇carrier = carrier G, mult = add G, one = zero
G⦈)
  shows abelian-subgroup H G
proof −
  interpret comm-group [(⦇carrier = carrier G, mult = add G, one = zero G⦈)]
  by (rule a-comm-group)
  interpret subgroup [H (⦇carrier = carrier G, mult = add G, one = zero G⦈)]
  by (rule a-subgroup)

  show abelian-subgroup H G
  apply unfold-locales
  proof (simp add: r-coset-def l-coset-def, clarsimp)
    fix x
    assume xcarr: x ∈ carrier G
    from a-subgroup
      have Hcarr: H ⊆ carrier G by (unfold subgroup-def, simp)
    from xcarr Hcarr
      show (⋃h∈H. {h ⊕_G x}) = (⋃h∈H. {x ⊕_G h})
      using m-comm[simplified]
      by fast
  qed
qed

lemma abelian-subgroupI3:
  includes struct G
  assumes asg: additive-subgroup H G
    and ag: abelian-group G
  shows abelian-subgroup H G
apply (rule abelian-subgroupI2)
 apply (rule abelian-group.a-comm-group[OF ag])
apply (rule additive-subgroup.a-subgroup[OF asg])
done
```

lemma (**in** *abelian-subgroup*) *a-coset-eq*:
(∀ *x* ∈ *carrier G*. *H* +> *x* = *x* <+ *H*)
by (*rule normal.coset-eq*[*OF a-normal*,
folded a-r-coset-def a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-inv-op-closed1*:
shows ⟦*x* ∈ *carrier G*; *h* ∈ *H*⟧ ⟹ (⊖ *x*) ⊕ *h* ⊕ *x* ∈ *H*
by (*rule normal.inv-op-closed1* [*OF a-normal*,
folded a-inv-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-inv-op-closed2*:
shows ⟦*x* ∈ *carrier G*; *h* ∈ *H*⟧ ⟹ *x* ⊕ *h* ⊕ (⊖ *x*) ∈ *H*
by (*rule normal.inv-op-closed2* [*OF a-normal*,
folded a-inv-def, *simplified monoid-record-simps*])

Alternative characterization of normal subgroups

lemma (**in** *abelian-group*) *a-normal-inv-iff*:
(*N* ◁ (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)) =
(*subgroup N* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|) & (∀ *x* ∈ *carrier G*. ∀ *h* ∈ *N*. *x* ⊕ *h* ⊕ (⊖ *x*) ∈ *N*))
(**is** *- = ?rhs*)
by (*rule group.normal-inv-iff* [*OF a-group*,
folded a-inv-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-lcos-m-assoc*:
[| *M* ⊆ *carrier G*; *g* ∈ *carrier G*; *h* ∈ *carrier G* |]
==> *g* <+ (*h* <+ *M*) = (*g* ⊕ *h*) <+ *M*
by (*rule group.lcos-m-assoc* [*OF a-group*,
folded a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-lcos-mult-one*:
M ⊆ *carrier G* ==> **0** <+ *M* = *M*
by (*rule group.lcos-mult-one* [*OF a-group*,
folded a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-l-coset-subset-G*:
[| *H* ⊆ *carrier G*; *x* ∈ *carrier G* |] ==> *x* <+ *H* ⊆ *carrier G*
by (*rule group.l-coset-subset-G* [*OF a-group*,
folded a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-l-coset-swap*:
⟦*y* ∈ *x* <+ *H*; *x* ∈ *carrier G*; *subgroup H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)⟧ ⟹ *x* ∈ *y* <+ *H*
by (*rule group.l-coset-swap* [*OF a-group*,
folded a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-l-coset-carrier*:

[| $y \in x <+ H$; $x \in carrier\ G$; $subgroup\ H\ (\!|carrier = carrier\ G, mult = add\ G, one = zero\ G|\!)$ |] ==> $y \in carrier\ G$
by (*rule group.l-coset-carrier* [*OF a-group*,
folded a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-l-repr-imp-subset*:
assumes *y*: $y \in x <+ H$ **and** *x*: $x \in carrier\ G$ **and** *sb*: $subgroup\ H\ (\!|carrier = carrier\ G, mult = add\ G, one = zero\ G|\!)$
shows $y <+ H \subseteq x <+ H$
apply (*rule group.l-repr-imp-subset* [*OF a-group*,
folded a-l-coset-def, *simplified monoid-record-simps*])
apply (*rule y*)
apply (*rule x*)
apply (*rule sb*)
done

lemma (**in** *abelian-group*) *a-l-repr-independence*:
assumes *y*: $y \in x <+ H$ **and** *x*: $x \in carrier\ G$ **and** *sb*: $subgroup\ H\ (\!|carrier = carrier\ G, mult = add\ G, one = zero\ G|\!)$
shows $x <+ H = y <+ H$
apply (*rule group.l-repr-independence* [*OF a-group*,
folded a-l-coset-def, *simplified monoid-record-simps*])
apply (*rule y*)
apply (*rule x*)
apply (*rule sb*)
done

lemma (**in** *abelian-group*) *setadd-subset-G*:
$[\![H \subseteq carrier\ G; K \subseteq carrier\ G]\!] \Longrightarrow H <+> K \subseteq carrier\ G$
by (*rule group.setmult-subset-G* [*OF a-group*,
folded set-add-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *subgroup-add-id*: $subgroup\ H\ (\!|carrier = carrier\ G, mult = add\ G, one = zero\ G|\!) \Longrightarrow H <+> H = H$
by (*rule group.subgroup-mult-id* [*OF a-group*,
folded set-add-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcos-inv*:
assumes *x*: $x \in carrier\ G$
shows $a\text{-}set\text{-}inv\ (H +> x) = H +> (\ominus x)$
by (*rule normal.rcos-inv* [*OF a-normal*,
folded a-r-coset-def a-inv-def A-SET-INV-def, *simplified monoid-record-simps*])
(*rule x*)

lemma (**in** *abelian-group*) *a-setmult-rcos-assoc*:
$[\![H \subseteq carrier\ G; K \subseteq carrier\ G; x \in carrier\ G]\!]$
$\Longrightarrow H <+> (K +> x) = (H <+> K) +> x$
by (*rule group.setmult-rcos-assoc* [*OF a-group*,
folded set-add-def a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-rcos-assoc-lcos*:
$[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G;\ x \in carrier\ G]\!]$
$\Longrightarrow (H +> x) <+> K = H <+> (x <+ K)$
by (*rule group.rcos-assoc-lcos* [*OF a-group*,
folded set-add-def a-r-coset-def a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcos-sum*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H +> x) <+> (H +> y) = H +> (x \oplus y)$
by (*rule normal.rcos-sum* [*OF a-normal*,
folded set-add-def a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *rcosets-add-eq*:
$M \in a\text{-}rcosets\ H \Longrightarrow H <+> M = M$
— generalizes *subgroup-mult-id*
by (*rule normal.rcosets-mult-eq* [*OF a-normal*,
folded set-add-def A-RCOSETS-def, *simplified monoid-record-simps*])

12.5 Congruence Relation

lemma (**in** *abelian-subgroup*) *a-equiv-rcong*:
shows *equiv* (*carrier G*) (*racong H*)
by (*rule subgroup.equiv-rcong* [*OF a-subgroup a-group*,
folded a-r-congruent-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-l-coset-eq-rcong*:
assumes *a*: $a \in carrier\ G$
shows $a <+ H = racong\ H\ ``\ \{a\}$
by (*rule subgroup.l-coset-eq-rcong* [*OF a-subgroup a-group*,
folded a-r-congruent-def a-l-coset-def, *simplified monoid-record-simps*]) (*rule a*)

lemma (**in** *abelian-subgroup*) *a-rcos-equation*:
shows
$[\![ha \oplus a = h \oplus b;\ a \in carrier\ G;\ \ b \in carrier\ G;$
$h \in H;\ \ ha \in H;\ \ hb \in H]\!]$
$\Longrightarrow hb \oplus a \in (\bigcup h \in H.\ \{h \oplus b\})$
by (*rule group.rcos-equation* [*OF a-group a-subgroup*,
folded a-r-congruent-def a-l-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcos-disjoint*:
shows $[\![a \in a\text{-}rcosets\ H;\ b \in a\text{-}rcosets\ H;\ a \neq b]\!] \Longrightarrow a \cap b = \{\}$
by (*rule group.rcos-disjoint* [*OF a-group a-subgroup*,
folded A-RCOSETS-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcos-self*:
shows $x \in carrier\ G \Longrightarrow x \in H +> x$
by (*rule group.rcos-self* [*OF a-group - a-subgroup*,
folded a-r-coset-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcosets-part-G*:
shows $\bigcup$(*a-rcosets H*) = *carrier G*
by (*rule group.rcosets-part-G* [*OF a-group a-subgroup*,
folded A-RCOSETS-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-cosets-finite*:
⟦*c* ∈ *a-rcosets H*; *H* ⊆ *carrier G*; *finite* (*carrier G*)⟧ ⟹ *finite c*
by (*rule group.cosets-finite* [*OF a-group*,
folded A-RCOSETS-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *a-card-cosets-equal*:
⟦*c* ∈ *a-rcosets H*; *H* ⊆ *carrier G*; *finite*(*carrier G*)⟧
⟹ *card c* = *card H*
by (*rule group.card-cosets-equal* [*OF a-group*,
folded A-RCOSETS-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-group*) *rcosets-subset-PowG*:
additive-subgroup H G ⟹ *a-rcosets H* ⊆ *Pow*(*carrier G*)
by (*rule group.rcosets-subset-PowG* [*OF a-group*,
folded A-RCOSETS-def, *simplified monoid-record-simps*],
rule additive-subgroup.a-subgroup)

theorem (**in** *abelian-group*) *a-lagrange*:
⟦*finite*(*carrier G*); *additive-subgroup H G*⟧
⟹ *card*(*a-rcosets H*) ∗ *card*(*H*) = *order*(*G*)
by (*rule group.lagrange* [*OF a-group*,
folded A-RCOSETS-def, *simplified monoid-record-simps order-def*, *folded order-def*])
(*fast intro*!: *additive-subgroup.a-subgroup*)+

12.6 Factorization

lemmas *A-FactGroup-defs* = *A-FactGroup-def FactGroup-def*

lemma *A-FactGroup-def'*:
includes *struct G*
shows *G A-Mod H* ≡ (|*carrier* = *a-rcosets*$_G$ *H*, *mult* = *set-add G*, *one* = *H*|)
unfolding *A-FactGroup-defs*
by (*fold A-RCOSETS-def set-add-def*)

lemma (**in** *abelian-subgroup*) *a-setmult-closed*:
⟦*K1* ∈ *a-rcosets H*; *K2* ∈ *a-rcosets H*⟧ ⟹ *K1* <+> *K2* ∈ *a-rcosets H*
by (*rule normal.setmult-closed* [*OF a-normal*,
folded A-RCOSETS-def set-add-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-setinv-closed*:
K ∈ *a-rcosets H* ⟹ *a-set-inv K* ∈ *a-rcosets H*
by (*rule normal.setinv-closed* [*OF a-normal*,

folded A-RCOSETS-def A-SET-INV-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcosets-assoc*:
⟦*M1* ∈ *a-rcosets H*; *M2* ∈ *a-rcosets H*; *M3* ∈ *a-rcosets H*⟧
⟹ *M1* <+> *M2* <+> *M3* = *M1* <+> (*M2* <+> *M3*)
by (*rule normal.rcosets-assoc* [*OF a-normal*,
folded A-RCOSETS-def set-add-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-subgroup-in-rcosets*:
H ∈ *a-rcosets H*
by (*rule subgroup.subgroup-in-rcosets* [*OF a-subgroup a-group*,
folded A-RCOSETS-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcosets-inv-mult-group-eq*:
M ∈ *a-rcosets H* ⟹ *a-set-inv M* <+> *M* = *H*
by (*rule normal.rcosets-inv-mult-group-eq* [*OF a-normal*,
folded A-RCOSETS-def A-SET-INV-def set-add-def, *simplified monoid-record-simps*])

theorem (**in** *abelian-subgroup*) *a-factorgroup-is-group*:
group (*G A-Mod H*)
by (*rule normal.factorgroup-is-group* [*OF a-normal*,
folded A-FactGroup-def, *simplified monoid-record-simps*])

Since the Factorization is based on an *abelian* subgroup, is results in a commutative group

theorem (**in** *abelian-subgroup*) *a-factorgroup-is-comm-group*:
comm-group (*G A-Mod H*)
apply (*intro comm-group.intro comm-monoid.intro*) **prefer** *3*
apply (*rule a-factorgroup-is-group*)
apply (*rule group.axioms*[*OF a-factorgroup-is-group*])
apply (*rule comm-monoid-axioms.intro*)
apply (*unfold A-FactGroup-def FactGroup-def RCOSETS-def*, *fold set-add-def a-r-coset-def*, *clarsimp*)
apply (*simp add*: *a-rcos-sum a-comm*)
done

lemma *add-A-FactGroup* [*simp*]: *X* ⊗$_{(G\ A\text{-}Mod\ H)}$ *X'* = *X* <+>$_G$ *X'*
by (*simp add*: *A-FactGroup-def set-add-def*)

lemma (**in** *abelian-subgroup*) *a-inv-FactGroup*:
X ∈ *carrier* (*G A-Mod H*) ⟹ *inv*$_{G\ A\text{-}Mod\ H}$ *X* = *a-set-inv X*
by (*rule normal.inv-FactGroup* [*OF a-normal*,
folded A-FactGroup-def A-SET-INV-def, *simplified monoid-record-simps*])

The coset map is a homomorphism from G to the quotient group $G\ Mod\ H$

lemma (**in** *abelian-subgroup*) *a-r-coset-hom-A-Mod*:
(λ*a*. *H* +> *a*) ∈ *hom* (⌊*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*⌋) (*G A-Mod H*)
by (*rule normal.r-coset-hom-Mod* [*OF a-normal*,

folded A-FactGroup-def a-r-coset-def, *simplified monoid-record-simps*])

The isomorphism theorems have been omitted from lifting, at least for now

12.7 The First Isomorphism Theorem

The quotient by the kernel of a homomorphism is isomorphic to the range of that homomorphism.

lemmas *a-kernel-defs* =
a-kernel-def kernel-def

lemma *a-kernel-def'*:
a-kernel R S h ≡ {*x* ∈ *carrier R*. *h x* = $\mathbf{0}_S$}
by (*rule a-kernel-def*[*unfolded kernel-def*, *simplified ring-record-simps*])

12.8 Homomorphisms

lemma *abelian-group-homI*:
includes *abelian-group G*
includes *abelian-group H*
assumes *a-group-hom*: *group-hom* (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)
(| *carrier* = *carrier H*, *mult* = *add H*, *one* = *zero H* |) *h*
shows *abelian-group-hom G H h*
apply (*intro abelian-group-hom.intro abelian-group-hom-axioms.intro*)
apply (*rule G.abelian-group-axioms*)
apply (*rule H.abelian-group-axioms*)
apply (*rule a-group-hom*)
done

lemma (**in** *abelian-group-hom*) *is-abelian-group-hom*:
abelian-group-hom G H h
by (*unfold-locales*)

lemma (**in** *abelian-group-hom*) *hom-add* [*simp*]:
[| *x* : *carrier G*; *y* : *carrier G* |]
==> *h* (*x* $\oplus_G$ *y*) = *h x* $\oplus_H$ *h y*
by (*rule group-hom.hom-mult*[*OF a-group-hom*,
simplified ring-record-simps])

lemma (**in** *abelian-group-hom*) *hom-closed* [*simp*]:
x ∈ *carrier G* ⟹ *h x* ∈ *carrier H*
by (*rule group-hom.hom-closed*[*OF a-group-hom*,
simplified ring-record-simps])

lemma (**in** *abelian-group-hom*) *zero-closed* [*simp*]:
h **0** ∈ *carrier H*
by (*rule group-hom.one-closed*[*OF a-group-hom*,
simplified ring-record-simps])

lemma (**in** *abelian-group-hom*) *hom-zero* [*simp*]:
  $h\ \mathbf{0} = \mathbf{0}_H$
by (*rule group-hom.hom-one*[*OF a-group-hom*,
    simplified ring-record-simps])

lemma (**in** *abelian-group-hom*) *a-inv-closed* [*simp*]:
  $x \in$ *carrier* G ==> $h\ (\ominus x) \in$ *carrier* H
by (*rule group-hom.inv-closed*[*OF a-group-hom*,
    folded a-inv-def, *simplified ring-record-simps*])

lemma (**in** *abelian-group-hom*) *hom-a-inv* [*simp*]:
  $x \in$ *carrier* G ==> $h\ (\ominus x) = \ominus_H\ (h\ x)$
by (*rule group-hom.hom-inv*[*OF a-group-hom*,
    folded a-inv-def, *simplified ring-record-simps*])

lemma (**in** *abelian-group-hom*) *additive-subgroup-a-kernel*:
  additive-subgroup (*a-kernel* G H h) G
apply (*rule additive-subgroup.intro*)
apply (*rule group-hom.subgroup-kernel*[*OF a-group-hom*,
      folded a-kernel-def, *simplified ring-record-simps*])
done

The kernel of a homomorphism is an abelian subgroup

lemma (**in** *abelian-group-hom*) *abelian-subgroup-a-kernel*:
  abelian-subgroup (*a-kernel* G H h) G
apply (*rule abelian-subgroupI*)
apply (*rule group-hom.normal-kernel*[*OF a-group-hom*,
      folded a-kernel-def, *simplified ring-record-simps*])
apply (*simp add*: *G.a-comm*)
done

lemma (**in** *abelian-group-hom*) *A-FactGroup-nonempty*:
  assumes X: $X \in$ *carrier* (*G A-Mod a-kernel G H h*)
  shows $X \neq \{\}$
by (*rule group-hom.FactGroup-nonempty*[*OF a-group-hom*,
    folded a-kernel-def A-FactGroup-def, *simplified ring-record-simps*]) (*rule X*)

lemma (**in** *abelian-group-hom*) *FactGroup-contents-mem*:
  assumes X: $X \in$ *carrier* (*G A-Mod* (*a-kernel G H h*))
  shows *contents* (h‘X) $\in$ *carrier* H
by (*rule group-hom.FactGroup-contents-mem*[*OF a-group-hom*,
    folded a-kernel-def A-FactGroup-def, *simplified ring-record-simps*]) (*rule X*)

lemma (**in** *abelian-group-hom*) *A-FactGroup-hom*:
    ($\lambda X.$ *contents* (h‘X)) $\in$ *hom* (*G A-Mod* (*a-kernel G H h*))
        (|*carrier* = *carrier H*, *mult* = *add H*, *one* = *zero H*|)
by (*rule group-hom.FactGroup-hom*[*OF a-group-hom*,
    folded a-kernel-def A-FactGroup-def, *simplified ring-record-simps*])

lemma (**in** *abelian-group-hom*) *A-FactGroup-inj-on*:
inj-on (λ*X*. *contents* (*h* ‘ *X*)) (*carrier* (*G A-Mod a-kernel G H h*))
by (*rule group-hom.FactGroup-inj-on*[*OF a-group-hom*,
folded a-kernel-def A-FactGroup-def, *simplified ring-record-simps*])

If the homomorphism h is onto H, then so is the homomorphism from the quotient group

lemma (**in** *abelian-group-hom*) *A-FactGroup-onto*:
assumes *h*: *h* ‘ *carrier G* = *carrier H*
shows (λ*X*. *contents* (*h* ‘ *X*)) ‘ *carrier* (*G A-Mod a-kernel G H h*) = *carrier H*
by (*rule group-hom.FactGroup-onto*[*OF a-group-hom*,
folded a-kernel-def A-FactGroup-def, *simplified ring-record-simps*]) (*rule h*)

If h is a homomorphism from G onto H, then the quotient group G *Mod* *kernel* G H h is isomorphic to H.

theorem (**in** *abelian-group-hom*) *A-FactGroup-iso*:
h ‘ *carrier G* = *carrier H*
⟹ (λ*X*. *contents* (*h*‘*X*)) ∈ (*G A-Mod* (*a-kernel G H h*)) ≅
(| *carrier* = *carrier H*, *mult* = *add H*, *one* = *zero H* |)
by (*rule group-hom.FactGroup-iso*[*OF a-group-hom*,
folded a-kernel-def A-FactGroup-def, *simplified ring-record-simps*])

13 Lemmas Lifted from CosetExt.thy

Not eveything from `CosetExt.thy` is lifted here.

13.1 General Lemmas from `AlgebraExt.thy`

lemma (**in** *additive-subgroup*) *a-Hcarr* [*simp*]:
assumes *hH*: *h* ∈ *H*
shows *h* ∈ *carrier G*
by (*rule subgroup.mem-carrier* [*OF a-subgroup*,
simplified monoid-record-simps]) (*rule hH*)

13.2 Lemmas for Right Cosets

lemma (**in** *abelian-subgroup*) *a-elemrcos-carrier*:
assumes *acarr*: *a* ∈ *carrier G*
and *a′*: *a′* ∈ *H* +> *a*
shows *a′* ∈ *carrier G*
by (*rule subgroup.elemrcos-carrier* [*OF a-subgroup a-group*,
folded a-r-coset-def, *simplified monoid-record-simps*]) (*rule acarr*, *rule a′*)

lemma (**in** *abelian-subgroup*) *a-rcos-const*:
assumes *hH*: *h* ∈ *H*
shows *H* +> *h* = *H*
by (*rule subgroup.rcos-const* [*OF a-subgroup a-group*,

folded a-r-coset-def, *simplified monoid-record-simps*]) (*rule hH*)

lemma (**in** *abelian-subgroup*) *a-rcos-module-imp*:
assumes *xcarr*: $x \in carrier\ G$
and *x'cos*: $x' \in H +> x$
shows $(x' \oplus \ominus x) \in H$
by (*rule subgroup.rcos-module-imp* [*OF a-subgroup a-group*,
folded a-r-coset-def a-inv-def, *simplified monoid-record-simps*]) (*rule xcarr*, *rule x'cos*)

lemma (**in** *abelian-subgroup*) *a-rcos-module-rev*:
assumes $x \in carrier\ G\ x' \in carrier\ G$
and $(x' \oplus \ominus x) \in H$
shows $x' \in H +> x$
using *assms*
by (*rule subgroup.rcos-module-rev* [*OF a-subgroup a-group*,
folded a-r-coset-def a-inv-def, *simplified monoid-record-simps*])

lemma (**in** *abelian-subgroup*) *a-rcos-module*:
assumes $x \in carrier\ G\ x' \in carrier\ G$
shows $(x' \in H +> x) = (x' \oplus \ominus x \in H)$
using *assms*
by (*rule subgroup.rcos-module* [*OF a-subgroup a-group*,
folded a-r-coset-def a-inv-def, *simplified monoid-record-simps*])

— variant
lemma (**in** *abelian-subgroup*) *a-rcos-module-minus*:
includes *ring G*
assumes *carr*: $x \in carrier\ G\ x' \in carrier\ G$
shows $(x' \in H +> x) = (x' \ominus x \in H)$
proof −
from *carr*
have $(x' \in H +> x) = (x' \oplus \ominus x \in H)$ **by** (*rule a-rcos-module*)
with *carr*
show $(x' \in H +> x) = (x' \ominus x \in H)$
by (*simp add*: *minus-eq*)
qed

lemma (**in** *abelian-subgroup*) *a-repr-independence'*:
assumes *y*: $y \in H +> x$
and *xcarr*: $x \in carrier\ G$
shows $H +> x = H +> y$
apply (*rule a-repr-independence*)
apply (*rule y*)
apply (*rule xcarr*)
apply (*rule a-subgroup*)
done

lemma (**in** *abelian-subgroup*) *a-repr-independenceD*:

assumes *ycarr*: $y \in carrier\ G$
and *repr*: $H +> x = H +> y$
shows $y \in H +> x$
by (*rule group.repr-independenceD* [*OF a-group a-subgroup*,
folded a-r-coset-def, *simplified monoid-record-simps*]) (*rule ycarr*, *rule repr*)

13.3 Lemmas for the Set of Right Cosets

lemma (**in** *abelian-subgroup*) *a-rcosets-carrier*:
$X \in a\text{-}rcosets\ H \Longrightarrow X \subseteq carrier\ G$
by (*rule subgroup.rcosets-carrier* [*OF a-subgroup a-group*,
folded A-RCOSETS-def, *simplified monoid-record-simps*])

13.4 Addition of Subgroups

lemma (**in** *abelian-monoid*) *set-add-closed*:
assumes *Acarr*: $A \subseteq carrier\ G$
and *Bcarr*: $B \subseteq carrier\ G$
shows $A <+> B \subseteq carrier\ G$
by (*rule monoid.set-mult-closed* [*OF a-monoid*,
folded set-add-def, *simplified monoid-record-simps*]) (*rule Acarr*, *rule Bcarr*)

lemma (**in** *abelian-group*) *add-additive-subgroups*:
assumes *subH*: *additive-subgroup H G*
and *subK*: *additive-subgroup K G*
shows *additive-subgroup* $(H <+> K)\ G$
apply (*rule additive-subgroup.intro*)
apply (*unfold set-add-def*)
apply (*intro comm-group.mult-subgroups*)
apply (*rule a-comm-group*)
apply (*rule additive-subgroup.a-subgroup*[*OF subH*])
apply (*rule additive-subgroup.a-subgroup*[*OF subK*])
done

end

theory *Ideal*
imports *Ring AbelCoset*
begin

14 Ideals

14.1 General definition

locale *ideal* = *additive-subgroup I R* + *ring R* +
assumes *I-l-closed*: $[\![a \in I;\ x \in carrier\ R]\!] \Longrightarrow x \otimes a \in I$
and *I-r-closed*: $[\![a \in I;\ x \in carrier\ R]\!] \Longrightarrow a \otimes x \in I$

interpretation *ideal* $\subseteq$ *abelian-subgroup I R*
apply (*intro abelian-subgroupI3 abelian-group.intro*)
apply (*rule ideal.axioms*, *rule ideal-axioms*)
apply (*rule abelian-group.axioms*, *rule ring.axioms*, *rule ideal.axioms*, *rule ideal-axioms*)
apply (*rule abelian-group.axioms*, *rule ring.axioms*, *rule ideal.axioms*, *rule ideal-axioms*)
done

lemma (**in** *ideal*) *is-ideal*:
ideal I R
by (*rule ideal-axioms*)

lemma *idealI*:
includes *ring*
assumes *a-subgroup*: *subgroup I* (|*carrier* = *carrier R*, *mult* = *add R*, *one* = *zero R*|)
and *I-l-closed*: $\bigwedge a\ x.\ [\![a \in I;\ x \in carrier\ R]\!] \Longrightarrow x \otimes a \in I$
and *I-r-closed*: $\bigwedge a\ x.\ [\![a \in I;\ x \in carrier\ R]\!] \Longrightarrow a \otimes x \in I$
shows *ideal I R*
apply (*intro ideal.intro ideal-axioms.intro additive-subgroupI*)
apply (*rule a-subgroup*)
apply (*rule is-ring*)
apply (*erule* (*1*) *I-l-closed*)
apply (*erule* (*1*) *I-r-closed*)
done

14.2 Ideals Generated by a Subset of *carrier R*

constdefs (**structure** *R*)
genideal :: ('*a*, '*b*) *ring-scheme* $\Rightarrow$ '*a set* $\Rightarrow$ '*a set* (*Idl*$_1$ - [*80*] *79*)
genideal R S $\equiv$ *Inter* $\{I.\ ideal\ I\ R \wedge S \subseteq I\}$

14.3 Principal Ideals

locale *principalideal* = *ideal* +
assumes *generate*: $\exists i \in carrier\ R.\ I = Idl\ \{i\}$

lemma (**in** *principalideal*) *is-principalideal*:
shows *principalideal I R*
by (*rule principalideal-axioms*)

lemma *principalidealI*:
includes *ideal*
assumes *generate*: $\exists i \in carrier\ R.\ I = Idl\ \{i\}$
shows *principalideal I R*
by (*intro principalideal.intro principalideal-axioms.intro*) (*rule is-ideal*, *rule generate*)

14.4 Maximal Ideals

locale *maximalideal* = *ideal* +
  assumes *I-notcarr*: *carrier R* ≠ *I*
    and *I-maximal*: ⟦*ideal J R*; *I* ⊆ *J*; *J* ⊆ *carrier R*⟧ ⟹ *J* = *I* ∨ *J* = *carrier R*

lemma (**in** *maximalideal*) *is-maximalideal*:
 shows *maximalideal I R*
by (*rule maximalideal-axioms*)

lemma *maximalidealI*:
  includes *ideal*
  assumes *I-notcarr*: *carrier R* ≠ *I*
    and *I-maximal*: ⋀*J*. ⟦*ideal J R*; *I* ⊆ *J*; *J* ⊆ *carrier R*⟧ ⟹ *J* = *I* ∨ *J* = *carrier R*
  shows *maximalideal I R*
  by (*intro maximalideal.intro maximalideal-axioms.intro*)
    (*rule is-ideal*, *rule I-notcarr*, *rule I-maximal*)

14.5 Prime Ideals

locale *primeideal* = *ideal* + *cring* +
  assumes *I-notcarr*: *carrier R* ≠ *I*
    and *I-prime*: ⟦*a* ∈ *carrier R*; *b* ∈ *carrier R*; *a* ⊗ *b* ∈ *I*⟧ ⟹ *a* ∈ *I* ∨ *b* ∈ *I*

lemma (**in** *primeideal*) *is-primeideal*:
 shows *primeideal I R*
by (*rule primeideal-axioms*)

lemma *primeidealI*:
  includes *ideal*
  includes *cring*
  assumes *I-notcarr*: *carrier R* ≠ *I*
    and *I-prime*: ⋀*a b*. ⟦*a* ∈ *carrier R*; *b* ∈ *carrier R*; *a* ⊗ *b* ∈ *I*⟧ ⟹ *a* ∈ *I* ∨ *b* ∈ *I*
  shows *primeideal I R*
  by (*intro primeideal.intro primeideal-axioms.intro*)
    (*rule is-ideal*, *rule is-cring*, *rule I-notcarr*, *rule I-prime*)

lemma *primeidealI2*:
  includes *additive-subgroup I R*
  includes *cring*
  assumes *I-l-closed*: ⋀*a x*. ⟦*a* ∈ *I*; *x* ∈ *carrier R*⟧ ⟹ *x* ⊗ *a* ∈ *I*
    and *I-r-closed*: ⋀*a x*. ⟦*a* ∈ *I*; *x* ∈ *carrier R*⟧ ⟹ *a* ⊗ *x* ∈ *I*
    and *I-notcarr*: *carrier R* ≠ *I*
    and *I-prime*: ⋀*a b*. ⟦*a* ∈ *carrier R*; *b* ∈ *carrier R*; *a* ⊗ *b* ∈ *I*⟧ ⟹ *a* ∈ *I* ∨ *b* ∈ *I*
  shows *primeideal I R*
apply (*intro-locales*)

apply (*intro ideal-axioms.intro*)
apply (*erule* (*1*) *I-l-closed*)
apply (*erule* (*1*) *I-r-closed*)
apply (*intro primeideal-axioms.intro*)
apply (*rule I-notcarr*)
apply (*erule* (*2*) *I-prime*)
done

15 Properties of Ideals

15.1 Special Ideals

lemma (**in** *ring*) *zeroideal*:
shows *ideal* {**0**} *R*
apply (*intro idealI subgroup.intro*)
apply (*rule is-ring*)
apply *simp+*
apply (*fold a-inv-def*, *simp*)
apply *simp+*
done

lemma (**in** *ring*) *oneideal*:
shows *ideal* (*carrier R*) *R*
apply (*intro idealI subgroup.intro*)
apply (*rule is-ring*)
apply *simp+*
apply (*fold a-inv-def*, *simp*)
apply *simp+*
done

lemma (**in** *domain*) *zeroprimeideal*:
shows *primeideal* {**0**} *R*
apply (*intro primeidealI*)
apply (*rule zeroideal*)
apply (*rule domain.axioms*, *rule domain-axioms*)
defer *1*
apply (*simp add*: *integral*)
proof (*rule ccontr*, *simp*)
assume *carrier R* = {**0**}
from *this* **have** **1** = **0** **by** (*rule one-zeroI*)
from *this* **and** *one-not-zero*
show *False* **by** *simp*
qed

15.2 General Ideal Properies

lemma (**in** *ideal*) *one-imp-carrier*:
assumes *I-one-closed*: **1** ∈ *I*
shows *I* = *carrier R*

apply (*rule*)
apply (*rule*)
apply (*rule a-Hcarr*, *simp*)
proof
  fix *x*
  assume *xcarr*: $x \in$ *carrier R*
  from *I-one-closed* **and** *this*
    have $x \otimes \mathbf{1} \in I$ **by** (*intro I-l-closed*)
  from *this* **and** *xcarr*
    show $x \in I$ **by** *simp*
qed

lemma (**in** *ideal*) *Icarr*:
  assumes *iI*: $i \in I$
  shows $i \in$ *carrier R*
using *iI* **by** (*rule a-Hcarr*)

15.3 Intersection of Ideals

Intersection of two ideals The intersection of any two ideals is again an ideal in R

lemma (**in** *ring*) *i-intersect*:
  includes *ideal I R*
  includes *ideal J R*
  shows *ideal* $(I \cap J)$ *R*
apply (*intro idealI subgroup.intro*)
    apply (*rule is-ring*)
   apply (*force simp add*: *a-subset*)
  apply (*simp add*: *a-inv-def*[*symmetric*])
 apply *simp*
  apply (*simp add*: *a-inv-def*[*symmetric*])
 apply (*clarsimp*, *rule*)
  apply (*fast intro*: *ideal.I-l-closed ideal.intro prems*)+
apply (*clarsimp*, *rule*)
 apply (*fast intro*: *ideal.I-r-closed ideal.intro prems*)+
done

15.3.1 Intersection of a Set of Ideals

The intersection of any Number of Ideals is again an Ideal in R

lemma (**in** *ring*) *i-Intersect*:
  assumes *Sideals*: $\bigwedge I.\ I \in S \Longrightarrow$ *ideal I R*
    and *notempty*: $S \neq \{\}$
  shows *ideal* (*Inter S*) *R*
apply (*unfold-locales*)
apply (*simp-all add*: *Inter-def INTER-def*)
    apply (*rule*, *simp*) **defer** *1*
    apply *rule* **defer** *1*

```
      apply rule defer 1
      apply (fold a-inv-def, rule) defer 1
      apply rule defer 1
      apply rule defer 1
proof −
  fix x
  assume ∀ I∈S. x ∈ I
  hence xI: ⋀I. I ∈ S ⟹ x ∈ I by simp

  from notempty have ∃ I0. I0 ∈ S by blast
  from this obtain I0 where I0S: I0 ∈ S by auto

  interpret ideal [I0 R] by (rule Sideals[OF I0S])

  from xI[OF I0S] have x ∈ I0 .
  from this and a-subset show x ∈ carrier R by fast
next
  fix x y
  assume ∀ I∈S. x ∈ I
  hence xI: ⋀I. I ∈ S ⟹ x ∈ I by simp
  assume ∀ I∈S. y ∈ I
  hence yI: ⋀I. I ∈ S ⟹ y ∈ I by simp

  fix J
  assume JS: J ∈ S
  interpret ideal [J R] by (rule Sideals[OF JS])
  from xI[OF JS] and yI[OF JS]
     show x ⊕ y ∈ J by (rule a-closed)
next
  fix J
  assume JS: J ∈ S
  interpret ideal [J R] by (rule Sideals[OF JS])
  show 0 ∈ J by simp
next
  fix x
  assume ∀ I∈S. x ∈ I
  hence xI: ⋀I. I ∈ S ⟹ x ∈ I by simp

  fix J
  assume JS: J ∈ S
  interpret ideal [J R] by (rule Sideals[OF JS])

  from xI[OF JS]
     show ⊖ x ∈ J by (rule a-inv-closed)
next
  fix x y
  assume ∀ I∈S. x ∈ I
  hence xI: ⋀I. I ∈ S ⟹ x ∈ I by simp
  assume ycarr: y ∈ carrier R
```

fix J
assume JS: $J \in S$
interpret *ideal* $[J\ R]$ **by** (*rule Sideals*[*OF JS*])

from xI[*OF JS*] **and** *ycarr*
show $y \otimes x \in J$ **by** (*rule I-l-closed*)
next
fix $x\ y$
assume $\forall I{\in}S.\ x \in I$
hence xI: $\bigwedge I.\ I \in S \Longrightarrow x \in I$ **by** *simp*
assume *ycarr*: $y \in$ *carrier* R

fix J
assume JS: $J \in S$
interpret *ideal* $[J\ R]$ **by** (*rule Sideals*[*OF JS*])

from xI[*OF JS*] **and** *ycarr*
show $x \otimes y \in J$ **by** (*rule I-r-closed*)
qed

15.4 Addition of Ideals

lemma (**in** *ring*) *add-ideals*:
assumes *idealI*: *ideal* I R
and *idealJ*: *ideal* J R
shows *ideal* $(I <+> J)$ R
apply (*rule ideal.intro*)
apply (*rule add-additive-subgroups*)
apply (*intro ideal.axioms*[*OF idealI*])
apply (*intro ideal.axioms*[*OF idealJ*])
apply (*rule is-ring*)
apply (*rule ideal-axioms.intro*)
apply (*simp add*: *set-add-defs*, *clarsimp*) **defer** *1*
apply (*simp add*: *set-add-defs*, *clarsimp*) **defer** *1*
proof −
fix $x\ i\ j$
assume *xcarr*: $x \in$ *carrier* R
and *iI*: $i \in I$
and *jJ*: $j \in J$
from *xcarr ideal.Icarr*[*OF idealI iI*] *ideal.Icarr*[*OF idealJ jJ*]
have *c*: $(i \oplus j) \otimes x = (i \otimes x) \oplus (j \otimes x)$ **by** *algebra*
from *xcarr* **and** *iI*
have *a*: $i \otimes x \in I$ **by** (*simp add*: *ideal.I-r-closed*[*OF idealI*])
from *xcarr* **and** *jJ*
have *b*: $j \otimes x \in J$ **by** (*simp add*: *ideal.I-r-closed*[*OF idealJ*])
from *a b c*
show $\exists ha{\in}I.\ \exists ka{\in}J.\ (i \oplus j) \otimes x = ha \oplus ka$ **by** *fast*
next

fix x i j
assume *xcarr*: $x \in$ *carrier* R
  and *iI*: $i \in I$
  and *jJ*: $j \in J$
from *xcarr ideal.Icarr*[*OF idealI iI*] *ideal.Icarr*[*OF idealJ jJ*]
  have *c*: $x \otimes (i \oplus j) = (x \otimes i) \oplus (x \otimes j)$ **by** *algebra*
from *xcarr* **and** *iI*
  have *a*: $x \otimes i \in I$ **by** (*simp add*: *ideal.I-l-closed*[*OF idealI*])
from *xcarr* **and** *jJ*
  have *b*: $x \otimes j \in J$ **by** (*simp add*: *ideal.I-l-closed*[*OF idealJ*])
from *a b c*
  show $\exists ha{\in}I.\ \exists ka{\in}J.\ x \otimes (i \oplus j) = ha \oplus ka$ **by** *fast*
qed

15.5 Ideals generated by a subset of *carrier* R

15.5.1 Generation of Ideals in General Rings

genideal generates an ideal

lemma (**in** *ring*) *genideal-ideal*:
  assumes *Scarr*: $S \subseteq$ *carrier* R
  shows *ideal* (*Idl* S) R
unfolding *genideal-def*
proof (*rule i-Intersect*, *fast*, *simp*)
  from *oneideal* **and** *Scarr*
  show $\exists I.$ *ideal* I $R \wedge S \leq I$ **by** *fast*
qed

lemma (**in** *ring*) *genideal-self*:
  assumes $S \subseteq$ *carrier* R
  shows $S \subseteq$ *Idl* S
unfolding *genideal-def*
by *fast*

lemma (**in** *ring*) *genideal-self'*:
  assumes *carr*: $i \in$ *carrier* R
  shows $i \in$ *Idl* $\{i\}$
proof −
  from *carr*
    have $\{i\} \subseteq$ *Idl* $\{i\}$ **by** (*fast intro*!: *genideal-self*)
  thus $i \in$ *Idl* $\{i\}$ **by** *fast*
qed

genideal generates the minimal ideal

lemma (**in** *ring*) *genideal-minimal*:
  assumes *a*: *ideal* I R
    and *b*: $S \subseteq I$
  shows *Idl* $S \subseteq I$
unfolding *genideal-def*

by (*rule*, *elim InterD*, *simp add*: *a b*)

Generated ideals and subsets

lemma (**in** *ring*) *Idl-subset-ideal*:
  assumes *Iideal*: *ideal I R*
    and *Hcarr*: $H \subseteq$ *carrier R*
  shows (*Idl H* $\subseteq$ *I*) = (*H* $\subseteq$ *I*)
proof
  assume *a*: *Idl H* $\subseteq$ *I*
  from *Hcarr* **have** *H* $\subseteq$ *Idl H* **by** (*rule genideal-self*)
  from *this* **and** *a*
    show *H* $\subseteq$ *I* **by** *simp*
next
  fix *x*
  assume *HI*: *H* $\subseteq$ *I*

  from *Iideal* **and** *HI*
    have *I* $\in$ {*I*. *ideal I R* $\wedge$ *H* $\subseteq$ *I*} **by** *fast*
  from *this*
    show *Idl H* $\subseteq$ *I*
    unfolding *genideal-def*
    by *fast*
qed

lemma (**in** *ring*) *subset-Idl-subset*:
  assumes *Icarr*: *I* $\subseteq$ *carrier R*
    and *HI*: *H* $\subseteq$ *I*
  shows *Idl H* $\subseteq$ *Idl I*
proof −
  from *HI* **and** *genideal-self*[*OF Icarr*]
    have *HIdlI*: *H* $\subseteq$ *Idl I* **by** *fast*

  from *Icarr*
    have *Iideal*: *ideal* (*Idl I*) *R* **by** (*rule genideal-ideal*)
  from *HI* **and** *Icarr*
    have *H* $\subseteq$ *carrier R* **by** *fast*
  from *Iideal* **and** *this*
    have (*H* $\subseteq$ *Idl I*) = (*Idl H* $\subseteq$ *Idl I*)
    by (*rule Idl-subset-ideal*[*symmetric*])

  from *HIdlI* **and** *this*
    show *Idl H* $\subseteq$ *Idl I* **by** *simp*
qed

lemma (**in** *ring*) *Idl-subset-ideal′*:
  assumes *acarr*: *a* $\in$ *carrier R* **and** *bcarr*: *b* $\in$ *carrier R*
  shows (*Idl* {*a*} $\subseteq$ *Idl* {*b*}) = (*a* $\in$ *Idl* {*b*})
apply (*subst Idl-subset-ideal*[*OF genideal-ideal*[*of* {*b*}], *of* {*a*}])
  apply (*fast intro*: *bcarr*, *fast intro*: *acarr*)

apply *fast*
done

lemma (**in** *ring*) *genideal-zero*:
Idl {**0**} = {**0**}
apply *rule*
apply (*rule genideal-minimal*[*OF zeroideal*], *simp*)
apply (*simp add*: *genideal-self'*)
done

lemma (**in** *ring*) *genideal-one*:
Idl {**1**} = *carrier R*
proof −
interpret *ideal* [*Idl* {**1**} *R*] **by** (*rule genideal-ideal*, *fast intro*: *one-closed*)
show *Idl* {**1**} = *carrier R*
apply (*rule*, *rule a-subset*)
apply (*simp add*: *one-imp-carrier genideal-self'*)
done
qed

15.5.2 Generation of Principal Ideals in Commutative Rings

constdefs (**structure** *R*)
cgenideal :: ('*a*, '*b*) *monoid-scheme* ⇒ '*a* ⇒ '*a set* (*PIdl*₁ - [*80*] *79*)
cgenideal R a ≡ { *x* ⊗ *a* | *x*. *x* ∈ *carrier R* }

genhideal (?) really generates an ideal

lemma (**in** *cring*) *cgenideal-ideal*:
assumes *acarr*: *a* ∈ *carrier R*
shows *ideal* (*PIdl a*) *R*
apply (*unfold cgenideal-def*)
apply (*rule idealI*[*OF is-ring*])
apply (*rule subgroup.intro*)
apply (*simp-all add*: *monoid-record-simps*)
apply (*blast intro*: *acarr m-closed*)
apply *clarsimp* **defer** *1*
defer *1*
apply (*fold a-inv-def*, *clarsimp*) **defer** *1*
apply *clarsimp* **defer** *1*
apply *clarsimp* **defer** *1*
proof −
fix *x y*
assume *xcarr*: *x* ∈ *carrier R*
and *ycarr*: *y* ∈ *carrier R*
note *carr* = *acarr xcarr ycarr*

from *carr*
have *x* ⊗ *a* ⊕ *y* ⊗ *a* = (*x* ⊕ *y*) ⊗ *a* **by** (*simp add*: *l-distr*)
from *this* **and** *carr*

```
    show ∃z. x ⊗ a ⊕ y ⊗ a = z ⊗ a ∧ z ∈ carrier R by fast
next
  from l-null[OF acarr, symmetric] and zero-closed
    show ∃x. 0 = x ⊗ a ∧ x ∈ carrier R by fast
next
  fix x
  assume xcarr: x ∈ carrier R
  note carr = acarr xcarr

  from carr
    have ⊖ (x ⊗ a) = (⊖ x) ⊗ a by (simp add: l-minus)
  from this and carr
    show ∃z. ⊖ (x ⊗ a) = z ⊗ a ∧ z ∈ carrier R by fast
next
  fix x y
  assume xcarr: x ∈ carrier R
    and ycarr: y ∈ carrier R
  note carr = acarr xcarr ycarr

  from carr
    have y ⊗ a ⊗ x = (y ⊗ x) ⊗ a by (simp add: m-assoc, simp add: m-comm)
  from this and carr
    show ∃z. y ⊗ a ⊗ x = z ⊗ a ∧ z ∈ carrier R by fast
next
  fix x y
  assume xcarr: x ∈ carrier R
    and ycarr: y ∈ carrier R
  note carr = acarr xcarr ycarr

  from carr
    have x ⊗ (y ⊗ a) = (x ⊗ y) ⊗ a by (simp add: m-assoc)
  from this and carr
    show ∃z. x ⊗ (y ⊗ a) = z ⊗ a ∧ z ∈ carrier R by fast
qed

lemma (in ring) cgenideal-self:
  assumes icarr: i ∈ carrier R
  shows i ∈ PIdl i
unfolding cgenideal-def
proof simp
  from icarr
    have i = 1 ⊗ i by simp
  from this and icarr
    show ∃x. i = x ⊗ i ∧ x ∈ carrier R by fast
qed
```

cgenideal is minimal

```
lemma (in ring) cgenideal-minimal:
  includes ideal J R
```

assumes *aJ*: $a \in J$
shows *PIdl a* $\subseteq J$
unfolding *cgenideal-def*
apply *rule*
apply *clarify*
using *aJ*
apply (*erule I-l-closed*)
done

lemma (**in** *cring*) *cgenideal-eq-genideal*:
assumes *icarr*: $i \in$ *carrier R*
shows *PIdl i* = *Idl* $\{i\}$
apply *rule*
apply (*intro cgenideal-minimal*)
apply (*rule genideal-ideal*, *fast intro*: *icarr*)
apply (*rule genideal-self'*, *fast intro*: *icarr*)
apply (*intro genideal-minimal*)
apply (*rule cgenideal-ideal* [*OF icarr*])
apply (*simp*, *rule cgenideal-self* [*OF icarr*])
done

lemma (**in** *cring*) *cgenideal-eq-rcos*:
PIdl i = *carrier R #> i*
unfolding *cgenideal-def r-coset-def*
by *fast*

lemma (**in** *cring*) *cgenideal-is-principalideal*:
assumes *icarr*: $i \in$ *carrier R*
shows *principalideal* (*PIdl i*) *R*
apply (*rule principalidealI*)
apply (*rule cgenideal-ideal* [*OF icarr*])
proof −
from *icarr*
have *PIdl i* = *Idl* $\{i\}$ **by** (*rule cgenideal-eq-genideal*)
from *icarr* **and** *this*
show $\exists i' \in$*carrier R*. *PIdl i* = *Idl* $\{i'\}$ **by** *fast*
qed

15.6 Union of Ideals

lemma (**in** *ring*) *union-genideal*:
assumes *idealI*: *ideal I R*
and *idealJ*: *ideal J R*
shows *Idl* $(I \cup J)$ = *I* <+> *J*
apply *rule*
apply (*rule ring.genideal-minimal*)
apply (*rule R.is-ring*)
apply (*rule add-ideals*[*OF idealI idealJ*])
apply (*rule*)

apply (*simp add*: *set-add-defs*) **apply** (*elim disjE*) **defer** *1* **defer** *1*
apply (*rule*) **apply** (*simp add*: *set-add-defs genideal-def*) **apply** *clarsimp* **defer** *1*
proof −
fix x
assume *xI*: $x \in I$
have *ZJ*: $\mathbf{0} \in J$
by (*intro additive-subgroup.zero-closed*, *rule ideal.axioms*[*OF idealJ*])
from *ideal.Icarr*[*OF idealI xI*]
have $x = x \oplus \mathbf{0}$ **by** *algebra*
from *xI* **and** *ZJ* **and** *this*
show $\exists\, h{\in}I.\ \exists\, k{\in}J.\ x = h \oplus k$ **by** *fast*
next
fix x
assume *xJ*: $x \in J$
have *ZI*: $\mathbf{0} \in I$
by (*intro additive-subgroup.zero-closed*, *rule ideal.axioms*[*OF idealI*])
from *ideal.Icarr*[*OF idealJ xJ*]
have $x = \mathbf{0} \oplus x$ **by** *algebra*
from *ZI* **and** *xJ* **and** *this*
show $\exists\, h{\in}I.\ \exists\, k{\in}J.\ x = h \oplus k$ **by** *fast*
next
fix *i j K*
assume *iI*: $i \in I$
and *jJ*: $j \in J$
and *idealK*: *ideal K R*
and *IK*: $I \subseteq K$
and *JK*: $J \subseteq K$
from *iI* **and** *IK*
have *iK*: $i \in K$ **by** *fast*
from *jJ* **and** *JK*
have *jK*: $j \in K$ **by** *fast*
from *iK* **and** *jK*
show $i \oplus j \in K$ **by** (*intro additive-subgroup.a-closed*) (*rule ideal.axioms*[*OF idealK*])
qed

15.7 Properties of Principal Ideals

0 generates the zero ideal

lemma (**in** *ring*) *zero-genideal*:
shows *Idl* $\{\mathbf{0}\} = \{\mathbf{0}\}$
apply *rule*
apply (*simp add*: *genideal-minimal zeroideal*)
apply (*fast intro*!: *genideal-self*)
done

1 generates the unit ideal

lemma (**in** *ring*) *one-genideal*:

```
  shows Idl {1} = carrier R
proof −
  have 1 ∈ Idl {1} by (simp add: genideal-self')
 thus Idl {1} = carrier R by (intro ideal.one-imp-carrier, fast intro: genideal-ideal)
qed
```

The zero ideal is a principal ideal

```
corollary (in ring) zeropideal:
  shows principalideal {0} R
apply (rule principalidealI)
 apply (rule zeroideal)
apply (blast intro!: zero-closed zero-genideal[symmetric])
done
```

The unit ideal is a principal ideal

```
corollary (in ring) onepideal:
  shows principalideal (carrier R) R
apply (rule principalidealI)
 apply (rule oneideal)
apply (blast intro!: one-closed one-genideal[symmetric])
done
```

Every principal ideal is a right coset of the carrier

```
lemma (in principalideal) rcos-generate:
  includes cring
  shows ∃x∈I. I = carrier R #> x
proof −
  from generate
     obtain i
       where icarr: i ∈ carrier R
       and I1: I = Idl {i}
     by fast+

  from icarr and genideal-self[of {i}]
     have i ∈ Idl {i} by fast
  hence iI: i ∈ I by (simp add: I1)

  from I1 icarr
     have I2: I = PIdl i by (simp add: cgenideal-eq-genideal)

  have PIdl i = carrier R #> i
     unfolding cgenideal-def r-coset-def
     by fast

  from I2 and this
     have I = carrier R #> i by simp

  from iI and this
     show ∃x∈I. I = carrier R #> x by fast
```

qed

15.8 Prime Ideals

lemma (**in** *ideal*) *primeidealCD*:
includes *cring*
assumes *notprime*: ¬ *primeideal I R*
shows *carrier R = I* ∨ (∃ *a b*. *a* ∈ *carrier R* ∧ *b* ∈ *carrier R* ∧ *a* ⊗ *b* ∈ *I* ∧ *a* ∉ *I* ∧ *b* ∉ *I*)
proof (*rule ccontr*, *clarsimp*)
assume *InR*: *carrier R* ≠ *I*
and ∀ *a*. *a* ∈ *carrier R* ⟶ (∀ *b*. *a* ⊗ *b* ∈ *I* ⟶ *b* ∈ *carrier R* ⟶ *a* ∈ *I* ∨ *b* ∈ *I*)
hence *I-prime*: ⋀ *a b*. ⟦*a* ∈ *carrier R*; *b* ∈ *carrier R*; *a* ⊗ *b* ∈ *I*⟧ ⟹ *a* ∈ *I* ∨ *b* ∈ *I* **by** *simp*
have *primeideal I R*
apply (*rule primeideal.intro* [*OF is-ideal is-cring*])
apply (*rule primeideal-axioms.intro*)
apply (*rule InR*)
apply (*erule* (*2*) *I-prime*)
done
from *this* **and** *notprime*
show *False* **by** *simp*
qed

lemma (**in** *ideal*) *primeidealCE*:
includes *cring*
assumes *notprime*: ¬ *primeideal I R*
obtains *carrier R = I*
| ∃ *a b*. *a* ∈ *carrier R* ∧ *b* ∈ *carrier R* ∧ *a* ⊗ *b* ∈ *I* ∧ *a* ∉ *I* ∧ *b* ∉ *I*
using *primeidealCD* [*OF R.is-cring notprime*] **by** *blast*

If {**0**} is a prime ideal of a commutative ring, the ring is a domain

lemma (**in** *cring*) *zeroprimeideal-domainI*:
assumes *pi*: *primeideal* {**0**} *R*
shows *domain R*
apply (*rule domain.intro*, *rule is-cring*)
apply (*rule domain-axioms.intro*)
proof (*rule ccontr*, *simp*)
interpret *primeideal* [{**0**} *R*] **by** (*rule pi*)
assume 1 = 0
hence *carrier R* = {**0**} **by** (*rule one-zeroD*)
from *this*[*symmetric*] **and** *I-notcarr*
show *False* **by** *simp*
next
interpret *primeideal* [{**0**} *R*] **by** (*rule pi*)
fix *a b*
assume *ab*: *a* ⊗ *b* = **0**
and *carr*: *a* ∈ *carrier R b* ∈ *carrier R*

```
  from ab
    have abI: a ⊗ b ∈ {0} by fast
  from carr and this
    have a ∈ {0} ∨ b ∈ {0} by (rule I-prime)
  thus a = 0 ∨ b = 0 by simp
qed

corollary (in cring) domain-eq-zeroprimeideal:
  shows domain R = primeideal {0} R
apply rule
 apply (erule domain.zeroprimeideal)
apply (erule zeroprimeideal-domainI)
done
```

15.9 Maximal Ideals

```
lemma (in ideal) helper-I-closed:
  assumes carr: a ∈ carrier R x ∈ carrier R y ∈ carrier R
    and axI: a ⊗ x ∈ I
  shows a ⊗ (x ⊗ y) ∈ I
proof −
  from axI and carr
    have (a ⊗ x) ⊗ y ∈ I by (simp add: I-r-closed)
  also from carr
    have (a ⊗ x) ⊗ y = a ⊗ (x ⊗ y) by (simp add: m-assoc)
  finally
    show a ⊗ (x ⊗ y) ∈ I .
qed

lemma (in ideal) helper-max-prime:
  includes cring
  assumes acarr: a ∈ carrier R
  shows ideal {x∈carrier R. a ⊗ x ∈ I} R
apply (rule idealI)
   apply (rule cring.axioms[OF is-cring])
  apply (rule subgroup.intro)
     apply (simp, fast)
    apply clarsimp apply (simp add: r-distr acarr)
   apply (simp add: acarr)
  apply (simp add: a-inv-def[symmetric], clarify) defer 1
  apply clarsimp defer 1
  apply (fast intro!: helper-I-closed acarr)
proof −
  fix x
  assume xcarr: x ∈ carrier R
    and ax: a ⊗ x ∈ I
  from ax and acarr xcarr
    have ⊖(a ⊗ x) ∈ I by simp
  also from acarr xcarr
```

```
    have ⊖(a ⊗ x) = a ⊗ (⊖x) by algebra
  finally
    show a ⊗ (⊖x) ∈ I .
  from acarr
    have a ⊗ 0 = 0 by simp
next
  fix x y
  assume xcarr: x ∈ carrier R
    and ycarr: y ∈ carrier R
    and ayI: a ⊗ y ∈ I
  from ayI and acarr xcarr ycarr
    have a ⊗ (y ⊗ x) ∈ I by (simp add: helper-I-closed)
  moreover from xcarr ycarr
    have y ⊗ x = x ⊗ y by (simp add: m-comm)
  ultimately
    show a ⊗ (x ⊗ y) ∈ I by simp
qed
```

In a cring every maximal ideal is prime

```
lemma (in cring) maximalideal-is-prime:
  includes maximalideal
  shows primeideal I R
apply (rule ccontr)
apply (rule primeidealCE)
  apply (rule is-cring)
  apply assumption
 apply (simp add: I-notcarr)
proof −
  assume ∃ a b. a ∈ carrier R ∧ b ∈ carrier R ∧ a ⊗ b ∈ I ∧ a ∉ I ∧ b ∉ I
  from this
    obtain a b
      where acarr: a ∈ carrier R
      and bcarr: b ∈ carrier R
      and abI: a ⊗ b ∈ I
      and anI: a ∉ I
      and bnI: b ∉ I
    by fast
  def J ≡ {x∈carrier R. a ⊗ x ∈ I}

  from R.is-cring and acarr
  have idealJ: ideal J R unfolding J-def by (rule helper-max-prime)

  have IsubJ: I ⊆ J
  proof
    fix x
    assume xI: x ∈ I
    from this and acarr
    have a ⊗ x ∈ I by (intro I-l-closed)
    from xI[THEN a-Hcarr] this
```

```
    show x ∈ J unfolding J-def by fast
  qed

  from abI and acarr bcarr
      have b ∈ J unfolding J-def by fast
  from bnI and this
      have JnI: J ≠ I by fast
  from acarr
      have a = a ⊗ 1 by algebra
  from this and anI
      have a ⊗ 1 ∉ I by simp
  from one-closed and this
      have 1 ∉ J unfolding J-def by fast
  hence Jncarr: J ≠ carrier R by fast

  interpret ideal [J R] by (rule idealJ)

  have J = I ∨ J = carrier R
    apply (intro I-maximal)
    apply (rule idealJ)
    apply (rule IsubJ)
    apply (rule a-subset)
    done

  from this and JnI and Jncarr
      show False by simp
qed
```

15.10 Derived Theorems Involving Ideals

— A non-zero cring that has only the two trivial ideals is a field

```
lemma (in cring) trivialideals-fieldI:
  assumes carrnzero: carrier R ≠ {0}
      and haveideals: {I. ideal I R} = {{0}, carrier R}
  shows field R
apply (rule cring-fieldI)
apply (rule, rule, rule)
 apply (erule Units-closed)
defer 1
  apply rule
defer 1
proof (rule ccontr, simp)
  assume zUnit: 0 ∈ Units R
  hence a: 0 ⊗ inv 0 = 1 by (rule Units-r-inv)
  from zUnit
      have 0 ⊗ inv 0 = 0 by (intro l-null, rule Units-inv-closed)
  from a[symmetric] and this
      have 1 = 0 by simp
  hence carrier R = {0} by (rule one-zeroD)
```

```
  from this and carrnzero
    show False by simp
next
  fix x
  assume xcarr': x ∈ carrier R − {0}
  hence xcarr: x ∈ carrier R by fast
  from xcarr'
    have xnZ: x ≠ 0 by fast
  from xcarr
    have xIdl: ideal (PIdl x) R by (intro cgenideal-ideal, fast)

  from xcarr
    have x ∈ PIdl x by (intro cgenideal-self, fast)
  from this and xnZ
    have PIdl x ≠ {0} by fast
  from haveideals and this
    have PIdl x = carrier R
    by (blast intro!: xIdl)
  hence 1 ∈ PIdl x by simp
  hence ∃ y. 1 = y ⊗ x ∧ y ∈ carrier R unfolding cgenideal-def by blast
  from this
    obtain y
      where ycarr:  y ∈ carrier R
      and ylinv: 1 = y ⊗ x
    by fast+
  from ylinv and xcarr ycarr
    have yrinv: 1 = x ⊗ y by (simp add: m-comm)
  from ycarr and ylinv[symmetric] and yrinv[symmetric]
    have ∃ y ∈ carrier R. y ⊗ x = 1 ∧ x ⊗ y = 1 by fast
  from this and xcarr
    show x ∈ Units R
    unfolding Units-def
    by fast
qed

lemma (in field) all-ideals:
  shows {I. ideal I R} = {{0}, carrier R}
apply (rule, rule)
proof −
  fix I
  assume a: I ∈ {I. ideal I R}
  with this
    interpret ideal [I R] by simp

  show I ∈ {{0}, carrier R}
  proof (cases ∃ a. a ∈ I − {0})
    assume ∃ a. a ∈ I − {0}
    from this
      obtain a
```

```
        where aI: a ∈ I
        and anZ: a ≠ 0
      by fast+
    from aI[THEN a-Hcarr] anZ
      have aUnit: a ∈ Units R by (simp add: field-Units)
    hence a: a ⊗ inv a = 1 by (rule Units-r-inv)
    from aI and aUnit
      have a ⊗ inv a ∈ I by (simp add: I-r-closed)
    hence oneI: 1 ∈ I by (simp add: a[symmetric])

    have carrier R ⊆ I
    proof
      fix x
      assume xcarr: x ∈ carrier R
      from oneI and this
        have 1 ⊗ x ∈ I by (rule I-r-closed)
      from this and xcarr
        show x ∈ I by simp
    qed
    from this and a-subset
      have I = carrier R by fast
    thus I ∈ {{0}, carrier R} by fast
  next
    assume ¬ (∃ a. a ∈ I − {0})
    hence IZ: ⋀a. a ∈ I ⟹ a = 0 by simp

    have a: I ⊆ {0}
    proof
      fix x
      assume x ∈ I
      hence x = 0 by (rule IZ)
      thus x ∈ {0} by fast
    qed

    have 0 ∈ I by simp
    hence {0} ⊆ I by fast

    from this and a
      have I = {0} by fast
    thus I ∈ {{0}, carrier R} by fast
  qed
qed (simp add: zeroideal oneideal)

— Jacobson Theorem 2.2
lemma (in cring) trivialideals-eq-field:
  assumes carrnzero: carrier R ≠ {0}
  shows ({I. ideal I R} = {{0}, carrier R}) = field R
by (fast intro!: trivialideals-fieldI[OF carrnzero] field.all-ideals)
```

Like zeroprimeideal for domains

lemma (**in** *field*) *zeromaximalideal*:
  maximalideal {**0**} *R*
apply (*rule maximalidealI*)
  apply (*rule zeroideal*)
proof−
  from *one-not-zero*
    have 1 ∉ {**0**} **by** *simp*
  from *this* **and** *one-closed*
    show *carrier R* ≠ {**0**} **by** *fast*
next
  fix *J*
  assume *Jideal*: *ideal J R*
  hence *J* ∈ {*I*. *ideal I R*}
    by *fast*

  from *this* **and** *all-ideals*
    show *J* = {**0**} ∨ *J* = *carrier R* **by** *simp*
qed

lemma (**in** *cring*) *zeromaximalideal-fieldI*:
  assumes *zeromax*: *maximalideal* {**0**} *R*
  shows *field R*
apply (*rule trivialideals-fieldI*, *rule maximalideal.I-notcarr*[*OF zeromax*])
apply *rule* **apply** *clarsimp* **defer** *1*
 apply (*simp add*: *zeroideal oneideal*)
proof −
  fix *J*
  assume *Jn0*: *J* ≠ {**0**}
    and *idealJ*: *ideal J R*
  interpret *ideal* [*J R*] **by** (*rule idealJ*)
  have {**0**} ⊆ *J* **by** (*rule ccontr*, *simp*)
  from *zeromax* **and** *idealJ* **and** *this* **and** *a-subset*
    have *J* = {**0**} ∨ *J* = *carrier R* **by** (*rule maximalideal.I-maximal*)
  from *this* **and** *Jn0*
    show *J* = *carrier R* **by** *simp*
qed

lemma (**in** *cring*) *zeromaximalideal-eq-field*:
  maximalideal {**0**} *R* = *field R*
apply *rule*
 apply (*erule zeromaximalideal-fieldI*)
apply (*erule field.zeromaximalideal*)
done

end

theory *RingHom*

imports *Ideal*
begin

16 Homomorphisms of Non-Commutative Rings

Lifting existing lemmas in a *ring-hom-ring* locale

locale *ring-hom-ring* = *ring R* + *ring S* + *var h* +
  assumes *homh*: *h* ∈ *ring-hom R S*
  notes *hom-mult* [*simp*] = *ring-hom-mult* [*OF homh*]
    and *hom-one* [*simp*] = *ring-hom-one* [*OF homh*]

interpretation *ring-hom-cring* ⊆ *ring-hom-ring*
  by (*unfold-locales*, *rule homh*)

interpretation *ring-hom-ring* ⊆ *abelian-group-hom R S*
apply (*rule abelian-group-homI*)
  apply (*rule R.is-abelian-group*)
 apply (*rule S.is-abelian-group*)
apply (*intro group-hom.intro group-hom-axioms.intro*)
  apply (*rule R.a-group*)
 apply (*rule S.a-group*)
apply (*insert homh*, *unfold hom-def ring-hom-def*)
apply *simp*
done

lemma (**in** *ring-hom-ring*) *is-ring-hom-ring*:
  includes *struct R* + *struct S*
  shows *ring-hom-ring R S h*
by (*rule ring-hom-ring-axioms*)

lemma *ring-hom-ringI*:
  includes *ring R* + *ring S*
  assumes
        hom-closed: !!*x*. *x* ∈ *carrier R* ==> *h x* ∈ *carrier S*
      and *compatible-mult*: !!*x y*. [| *x* : *carrier R*; *y* : *carrier R* |] ==> *h* (*x* ⊗ *y*) = *h x* ⊗$_S$ *h y*
      and *compatible-add*: !!*x y*. [| *x* : *carrier R*; *y* : *carrier R* |] ==> *h* (*x* ⊕ *y*) = *h x* ⊕$_S$ *h y*
      and *compatible-one*: *h* **1** = **1**$_S$
  shows *ring-hom-ring R S h*
apply *unfold-locales*
apply (*unfold ring-hom-def*, *safe*)
   apply (*simp add*: *hom-closed Pi-def*)
  apply (*erule* (*1*) *compatible-mult*)
 apply (*erule* (*1*) *compatible-add*)
apply (*rule compatible-one*)
done

lemma *ring-hom-ringI2*:
includes *ring R* + *ring S*
assumes *h*: $h \in$ *ring-hom R S*
shows *ring-hom-ring R S h*
apply (*intro ring-hom-ring.intro ring-hom-ring-axioms.intro*)
apply (*rule R.is-ring*)
apply (*rule S.is-ring*)
apply (*rule h*)
done

lemma *ring-hom-ringI3*:
includes *abelian-group-hom R S* + *ring R* + *ring S*
assumes *compatible-mult*: !!*x y*. [| *x* : *carrier R*; *y* : *carrier R* |] ==> *h* (*x* $\otimes$ *y*) = *h x* $\otimes_S$ *h y*
and *compatible-one*: *h* $\mathbf{1} = \mathbf{1}_S$
shows *ring-hom-ring R S h*
apply (*intro ring-hom-ring.intro ring-hom-ring-axioms.intro*, *rule R.is-ring*, *rule S.is-ring*)
apply (*insert group-hom.homh*[*OF a-group-hom*])
apply (*unfold hom-def ring-hom-def*, *simp*)
apply *safe*
apply (*erule* (*1*) *compatible-mult*)
apply (*rule compatible-one*)
done

lemma *ring-hom-cringI*:
includes *ring-hom-ring R S h* + *cring R* + *cring S*
shows *ring-hom-cring R S h*
by (*intro ring-hom-cring.intro ring-hom-cring-axioms.intro*)
(*rule R.is-cring*, *rule S.is-cring*, *rule homh*)

16.1 The kernel of a ring homomorphism

— the kernel of a ring homomorphism is an ideal
lemma (**in** *ring-hom-ring*) *kernel-is-ideal*:
shows *ideal* (*a-kernel R S h*) *R*
apply (*rule idealI*)
apply (*rule R.is-ring*)
apply (*rule additive-subgroup.a-subgroup*[*OF additive-subgroup-a-kernel*])
apply (*unfold a-kernel-def'*, *simp*+)
done

Elements of the kernel are mapped to zero

lemma (**in** *abelian-group-hom*) *kernel-zero* [*simp*]:
$i \in$ *a-kernel R S h* $\Longrightarrow$ *h i* = $\mathbf{0}_S$
by (*simp add*: *a-kernel-defs*)

16.2 Cosets

Cosets of the kernel correspond to the elements of the image of the homomorphism

lemma (**in** *ring-hom-ring*) *rcos-imp-homeq*:
  assumes *acarr*: $a \in carrier\ R$
    and *xrcos*: $x \in a\text{-}kernel\ R\ S\ h +> a$
  shows $h\ x = h\ a$
proof −
  interpret *ideal* [*a-kernel R S h R*] **by** (*rule kernel-is-ideal*)

  from *xrcos*
    have $\exists i \in a\text{-}kernel\ R\ S\ h.\ x = i \oplus a$ **by** (*simp add*: *a-r-coset-defs*)
  from *this* **obtain** *i*
    where *iker*: $i \in a\text{-}kernel\ R\ S\ h$
      and *x*: $x = i \oplus a$
    by *fast+*
  note *carr* = *acarr iker*[*THEN a-Hcarr*]

  from *x*
    have $h\ x = h\ (i \oplus a)$ **by** *simp*
  also from *carr*
    have $\ldots = h\ i \oplus_S h\ a$ **by** *simp*
  also from *iker*
    have $\ldots = \mathbf{0}_S \oplus_S h\ a$ **by** *simp*
  also from *carr*
    have $\ldots = h\ a$ **by** *simp*
  finally
    show $h\ x = h\ a$ **.**
qed

lemma (**in** *ring-hom-ring*) *homeq-imp-rcos*:
  assumes *acarr*: $a \in carrier\ R$
    and *xcarr*: $x \in carrier\ R$
    and *hx*: $h\ x = h\ a$
  shows $x \in a\text{-}kernel\ R\ S\ h +> a$
proof −
  interpret *ideal* [*a-kernel R S h R*] **by** (*rule kernel-is-ideal*)

  note *carr* = *acarr xcarr*
  note *hcarr* = *acarr*[*THEN hom-closed*] *xcarr*[*THEN hom-closed*]

  from *hx* **and** *hcarr*
    have *a*: $h\ x \oplus_S \ominus_S h\ a = \mathbf{0}_S$ **by** *algebra*
  from *carr*
    have $h\ x \oplus_S \ominus_S h\ a = h\ (x \oplus \ominus a)$ **by** *simp*
  from *a* **and** *this*
    have *b*: $h\ (x \oplus \ominus a) = \mathbf{0}_S$ **by** *simp*

```
  from carr have x ⊕ ⊖a ∈ carrier R by simp
  from this and b
      have x ⊕ ⊖a ∈ a-kernel R S h
      unfolding a-kernel-def'
      by fast

  from this and carr
      show x ∈ a-kernel R S h +> a by (simp add: a-rcos-module-rev)
qed

corollary (in ring-hom-ring) rcos-eq-homeq:
  assumes acarr: a ∈ carrier R
  shows (a-kernel R S h) +> a = {x ∈ carrier R. h x = h a}
apply rule defer 1
apply clarsimp defer 1
proof
  interpret ideal [a-kernel R S h R] by (rule kernel-is-ideal)

  fix x
  assume xrcos: x ∈ a-kernel R S h +> a
  from acarr and this
      have xcarr: x ∈ carrier R
      by (rule a-elemrcos-carrier)

  from xrcos
      have h x = h a by (rule rcos-imp-homeq[OF acarr])
  from xcarr and this
      show x ∈ {x ∈ carrier R. h x = h a} by fast
next
  interpret ideal [a-kernel R S h R] by (rule kernel-is-ideal)

  fix x
  assume xcarr: x ∈ carrier R
    and hx: h x = h a
  from acarr xcarr hx
      show x ∈ a-kernel R S h +> a by (rule homeq-imp-rcos)
qed

end
```

17 QuotRing: Quotient Rings

```
theory QuotRing
imports RingHom
begin
```

17.1 Multiplication on Cosets

constdefs (**structure** *R*)
rcoset-mult :: [('*a*, -) *ring-scheme*, '*a set*, '*a set*, '*a set*] ⇒ '*a set*
([*mod* -:] - ⨂₁ - [*81*,*81*,*81*] *80*)
rcoset-mult R I A B ≡ ⋃ *a*∈*A*. ⋃ *b*∈*B*. *I* +> (*a* ⊗ *b*)

rcoset-mult fulfils the properties required by congruences

lemma (**in** *ideal*) *rcoset-mult-add*:
⟦*x* ∈ *carrier R*; *y* ∈ *carrier R*⟧ ⟹ [*mod I*:] (*I* +> *x*) ⨂ (*I* +> *y*) = *I* +> (*x* ⊗ *y*)
apply *rule*
apply (*rule*, *simp add*: *rcoset-mult-def*, *clarsimp*)
defer *1*
apply (*rule*, *simp add*: *rcoset-mult-def*)
defer *1*
proof −
fix *z x' y'*
assume *carr*: *x* ∈ *carrier R y* ∈ *carrier R*
and *x'rcos*: *x'* ∈ *I* +> *x*
and *y'rcos*: *y'* ∈ *I* +> *y*
and *zrcos*: *z* ∈ *I* +> *x'* ⊗ *y'*

from *x'rcos*
have ∃ *h*∈*I*. *x'* = *h* ⊕ *x* **by** (*simp add*: *a-r-coset-def r-coset-def*)
from *this* **obtain** *hx*
where *hxI*: *hx* ∈ *I*
and *x'*: *x'* = *hx* ⊕ *x*
by *fast+*

from *y'rcos*
have ∃ *h*∈*I*. *y'* = *h* ⊕ *y* **by** (*simp add*: *a-r-coset-def r-coset-def*)
from *this*
obtain *hy*
where *hyI*: *hy* ∈ *I*
and *y'*: *y'* = *hy* ⊕ *y*
by *fast+*

from *zrcos*
have ∃ *h*∈*I*. *z* = *h* ⊕ (*x'* ⊗ *y'*) **by** (*simp add*: *a-r-coset-def r-coset-def*)
from *this*
obtain *hz*
where *hzI*: *hz* ∈ *I*
and *z*: *z* = *hz* ⊕ (*x'* ⊗ *y'*)
by *fast+*

note *carr* = *carr hxI*[*THEN a-Hcarr*] *hyI*[*THEN a-Hcarr*] *hzI*[*THEN a-Hcarr*]

from *z* **have** *z* = *hz* ⊕ (*x'* ⊗ *y'*) .
also from *x' y'*

have ... = $hz \oplus ((hx \oplus x) \otimes (hy \oplus y))$ **by** *simp*
also from *carr*
have ... = $(hz \oplus (hx \otimes (hy \oplus y)) \oplus x \otimes hy) \oplus x \otimes y$ **by** *algebra*
finally
have *z2*: $z = (hz \oplus (hx \otimes (hy \oplus y)) \oplus x \otimes hy) \oplus x \otimes y$ **.**

from *hxI hyI hzI carr*
have $hz \oplus (hx \otimes (hy \oplus y)) \oplus x \otimes hy \in I$ **by** (*simp add*: *I-l-closed I-r-closed*)

from *this* **and** *z2*
have $\exists h{\in}I.\ z = h \oplus x \otimes y$ **by** *fast*
thus $z \in I +> x \otimes y$ **by** (*simp add*: *a-r-coset-def r-coset-def*)
next
fix *z*
assume *xcarr*: $x \in carrier\ R$
and *ycarr*: $y \in carrier\ R$
and *zrcos*: $z \in I +> x \otimes y$
from *xcarr*
have *xself*: $x \in I +> x$ **by** (*intro a-rcos-self*)
from *ycarr*
have *yself*: $y \in I +> y$ **by** (*intro a-rcos-self*)

from *xself* **and** *yself* **and** *zrcos*
show $\exists a{\in}I +> x.\ \exists b{\in}I +> y.\ z \in I +> a \otimes b$ **by** *fast*
qed

17.2 Quotient Ring Definition

constdefs (**structure** *R*)
FactRing :: [('*a*,'*b*) *ring-scheme*, '*a set*] ⇒ ('*a set*) *ring*
(**infixl** *Quot 65*)
FactRing R I ≡
(|*carrier* = *a-rcosets I*, *mult* = *rcoset-mult R I*, *one* = (*I* +> **1**), *zero* = *I*, *add* = *set-add R*|)

17.3 Factorization over General Ideals

The quotient is a ring

lemma (**in** *ideal*) *quotient-is-ring*:
shows *ring* (*R Quot I*)
apply (*rule ringI*)
— abelian group
apply (*rule comm-group-abelian-groupI*)
apply (*simp add*: *FactRing-def*)
apply (*rule a-factorgroup-is-comm-group*[*unfolded A-FactGroup-def'*])
— mult monoid
apply (*rule monoidI*)
apply (*simp-all add*: *FactRing-def A-RCOSETS-def RCOSETS-def*
a-r-coset-def[*symmetric*])

```
      — mult closed
      apply (clarify)
      apply (simp add: rcoset-mult-add, fast)
     — mult one-closed
     apply (force intro: one-closed)
    — mult assoc
    apply clarify
    apply (simp add: rcoset-mult-add m-assoc)
   — mult one
   apply clarify
   apply (simp add: rcoset-mult-add l-one)
  apply clarify
  apply (simp add: rcoset-mult-add r-one)
 — distr
 apply clarify
 apply (simp add: rcoset-mult-add a-rcos-sum l-distr)
apply clarify
apply (simp add: rcoset-mult-add a-rcos-sum r-distr)
done
```

This is a ring homomorphism

```
lemma (in ideal) rcos-ring-hom:
  (op +> I) ∈ ring-hom R (R Quot I)
apply (rule ring-hom-memI)
   apply (simp add: FactRing-def a-rcosetsI[OF a-subset])
  apply (simp add: FactRing-def rcoset-mult-add)
 apply (simp add: FactRing-def a-rcos-sum)
apply (simp add: FactRing-def)
done

lemma (in ideal) rcos-ring-hom-ring:
  ring-hom-ring R (R Quot I) (op +> I)
apply (rule ring-hom-ringI)
     apply (rule is-ring, rule quotient-is-ring)
   apply (simp add: FactRing-def a-rcosetsI[OF a-subset])
  apply (simp add: FactRing-def rcoset-mult-add)
 apply (simp add: FactRing-def a-rcos-sum)
apply (simp add: FactRing-def)
done
```

The quotient of a cring is also commutative

```
lemma (in ideal) quotient-is-cring:
  includes cring
  shows cring (R Quot I)
apply (intro cring.intro comm-monoid.intro comm-monoid-axioms.intro)
  apply (rule quotient-is-ring)
 apply (rule ring.axioms[OF quotient-is-ring])
apply (simp add: FactRing-def A-RCOSETS-defs a-r-coset-def[symmetric])
apply clarify
```

apply (*simp add*: *rcoset-mult-add m-comm*)
done

Cosets as a ring homomorphism on crings

lemma (**in** *ideal*) *rcos-ring-hom-cring*:
includes *cring*
shows *ring-hom-cring R* (*R Quot I*) (*op +> I*)
apply (*rule ring-hom-cringI*)
apply (*rule rcos-ring-hom-ring*)
apply (*rule R.is-cring*)
apply (*rule quotient-is-cring*)
apply (*rule R.is-cring*)
done

17.4 Factorization over Prime Ideals

The quotient ring generated by a prime ideal is a domain

lemma (**in** *primeideal*) *quotient-is-domain*:
shows *domain* (*R Quot I*)
apply (*rule domain.intro*)
apply (*rule quotient-is-cring*, *rule is-cring*)
apply (*rule domain-axioms.intro*)
apply (*simp add*: *FactRing-def*) **defer** *1*
apply (*simp add*: *FactRing-def A-RCOSETS-defs a-r-coset-def*[*symmetric*], *clarify*)
apply (*simp add*: *rcoset-mult-add*) **defer** *1*
proof (*rule ccontr*, *clarsimp*)
assume $I +> \mathbf{1} = I$
hence $\mathbf{1} \in I$ **by** (*simp only*: *a-coset-join1 one-closed a-subgroup*)
hence *carrier R* $\subseteq I$ **by** (*subst one-imp-carrier*, *simp*, *fast*)
from *this* **and** *a-subset*
have $I =$ *carrier R* **by** *fast*
from *this* **and** *I-notcarr*
show *False* **by** *fast*
next
fix *x y*
assume *carr*: $x \in$ *carrier R* $y \in$ *carrier R*
and *a*: $I +> x \otimes y = I$
and *b*: $I +> y \neq I$

have *ynI*: $y \notin I$
proof (*rule ccontr*, *simp*)
assume $y \in I$
hence $I +> y = I$ **by** (*rule a-rcos-const*)
from *this* **and** *b*
show *False* **by** *simp*
qed

from *carr*

```
    have x ⊗ y ∈ I +> x ⊗ y by (simp add: a-rcos-self)
  from this
    have xyI: x ⊗ y ∈ I by (simp add: a)

  from xyI and carr
    have xI: x ∈ I ∨ y ∈ I by (simp add: I-prime)
  from this and ynI
    have x ∈ I by fast
  thus I +> x = I by (rule a-rcos-const)
qed
```

Generating right cosets of a prime ideal is a homomorphism on commutative rings

```
lemma (in primeideal) rcos-ring-hom-cring:
  shows ring-hom-cring R (R Quot I) (op +> I)
by (rule rcos-ring-hom-cring, rule is-cring)
```

17.5 Factorization over Maximal Ideals

In a commutative ring, the quotient ring over a maximal ideal is a field. The proof follows "W. Adkins, S. Weintraub: Algebra – An Approach via Module Theory"

```
lemma (in maximalideal) quotient-is-field:
  includes cring
  shows field (R Quot I)
apply (intro cring.cring-fieldI2)
  apply (rule quotient-is-cring, rule is-cring)
 defer 1
 apply (simp add: FactRing-def A-RCOSETS-defs a-r-coset-def[symmetric], clar-
simp)
 apply (simp add: rcoset-mult-add) defer 1
proof (rule ccontr, simp)
  — Quotient is not empty
  assume 0_(R Quot I) = 1_(R Quot I)
  hence II1: I = I +> 1 by (simp add: FactRing-def)
  from a-rcos-self[OF one-closed]
  have 1 ∈ I by (simp add: II1[symmetric])
  hence I = carrier R by (rule one-imp-carrier)
  from this and I-notcarr
  show False by simp
next
  — Existence of Inverse
  fix a
  assume IanI: I +> a ≠ I
    and acarr: a ∈ carrier R

  — Helper ideal J
  def J ≡ (carrier R #> a) <+> I :: 'a set
```

have *idealJ*: *ideal J R*
apply (*unfold J-def*, *rule add-ideals*)
apply (*simp only*: *cgenideal-eq-rcos*[*symmetric*], *rule cgenideal-ideal*, *rule acarr*)
apply (*rule is-ideal*)
done

— Showing J not smaller than I
have *IinJ*: $I \subseteq J$
proof (*rule*, *simp add*: *J-def r-coset-def set-add-defs*)
fix x
assume *xI*: $x \in I$
have *Zcarr*: $\mathbf{0} \in$ *carrier R* **by** *fast*
from *xI*[*THEN a-Hcarr*] *acarr*
have $x = \mathbf{0} \otimes a \oplus x$ **by** *algebra*

from *Zcarr* **and** *xI* **and** *this*
show $\exists\, xa \in$ *carrier R*. $\exists\, k \in I.\ x = xa \otimes a \oplus k$ **by** *fast*
qed

— Showing $J \neq I$
have *anI*: $a \notin I$
proof (*rule ccontr*, *simp*)
assume $a \in I$
hence $I +> a = I$ **by** (*rule a-rcos-const*)
from *this* **and** *IanI*
show *False* **by** *simp*
qed

have *aJ*: $a \in J$
proof (*simp add*: *J-def r-coset-def set-add-defs*)
from *acarr*
have $a = \mathbf{1} \otimes a \oplus \mathbf{0}$ **by** *algebra*
from *one-closed* **and** *additive-subgroup.zero-closed*[*OF is-additive-subgroup*]
and *this*
show $\exists\, x \in$ *carrier R*. $\exists\, k \in I.\ a = x \otimes a \oplus k$ **by** *fast*
qed

from *aJ* **and** *anI*
have *JnI*: $J \neq I$ **by** *fast*

— Deducing $J =$ *carrier R* because I is maximal
from *idealJ* **and** *IinJ*
have $J = I \vee J =$ *carrier R*
proof (*rule I-maximal*, *unfold J-def*)
have *carrier R #> a* $\subseteq$ *carrier R*
using *subset-refl acarr*
by (*rule r-coset-subset-G*)
from *this* **and** *a-subset*
show *carrier R #> a <+> I* $\subseteq$ *carrier R* **by** (*rule set-add-closed*)

qed

from *this* **and** *JnI*
have *Jcarr*: $J = carrier\ R$ **by** *simp*

— Calculating an inverse for a
from *one-closed*[*folded Jcarr*]
have $\exists r{\in}carrier\ R.\ \exists i{\in}I.\ \mathbf{1} = r \otimes a \oplus i$
by (*simp add*: *J-def r-coset-def set-add-defs*)
from *this*
obtain $r\ i$
where *rcarr*: $r \in carrier\ R$
and *iI*: $i \in I$
and *one*: $\mathbf{1} = r \otimes a \oplus i$
by *fast*
from *one* **and** *rcarr* **and** *acarr* **and** *iI*[*THEN a-Hcarr*]
have *rai1*: $a \otimes r = \ominus i \oplus \mathbf{1}$ **by** *algebra*

— Lifting to cosets
from *iI*
have $\ominus i \oplus \mathbf{1} \in I +> \mathbf{1}$
by (*intro a-rcosI*, *simp*, *intro a-subset*, *simp*)
from *this* **and** *rai1*
have $a \otimes r \in I +> \mathbf{1}$ **by** *simp*
from *this* **have** $I +> \mathbf{1} = I +> a \otimes r$
by (*rule a-repr-independence*, *simp*) (*rule a-subgroup*)

from *rcarr* **and** *this*[*symmetric*]
show $\exists r{\in}carrier\ R.\ I +> a \otimes r = I +> \mathbf{1}$ **by** *fast*
qed

end

theory *IntRing*
imports *QuotRing Int*
begin

18 The Ring of Integers

18.1 Some properties of *int*

lemma *dvds-imp-abseq*:
$[\![l\ dvd\ k;\ k\ dvd\ l]\!] \Longrightarrow abs\ l = abs\ (k::int)$
apply (*subst abs-split*, *rule conjI*)
apply (*clarsimp*, *subst abs-split*, *rule conjI*)
apply (*clarsimp*)
apply (*cases k=0*, *simp*)

apply (*cases l=0*, *simp*)
apply (*simp add*: *zdvd-anti-sym*)
apply *clarsimp*
apply (*cases k=0*, *simp*)
apply (*simp add*: *zdvd-anti-sym*)
apply (*clarsimp*, *subst abs-split*, *rule conjI*)
apply (*clarsimp*)
apply (*cases l=0*, *simp*)
apply (*simp add*: *zdvd-anti-sym*)
apply (*clarsimp*)
apply (*subgoal-tac* $-l = -k$, *simp*)
apply (*intro zdvd-anti-sym*, *simp+*)
done

lemma *abseq-imp-dvd*:
assumes *a-lk*: *abs l* = *abs* (*k*::*int*)
shows *l dvd k*
proof −
from *a-lk*
have *nat* (*abs l*) = *nat* (*abs k*) **by** *simp*
hence *nat* (*abs l*) *dvd nat* (*abs k*) **by** *simp*
hence *int* (*nat* (*abs l*)) *dvd k* **by** (*subst int-dvd-iff*)
hence *abs l dvd k* **by** *simp*
thus *l dvd k*
apply (*unfold dvd-def*, *cases l<0*)
defer *1* **apply** *clarsimp*
proof (*clarsimp*)
fix *k*
assume *l0*: $l < 0$
have $-(l * k) = l * (-k)$ **by** *simp*
thus $\exists ka. -(l * k) = l * ka$ **by** *fast*
qed
qed

lemma *dvds-eq-abseq*:
(*l dvd k* ∧ *k dvd l*) = (*abs l* = *abs* (*k*::*int*))
apply *rule*
apply (*simp add*: *dvds-imp-abseq*)
apply (*rule conjI*)
apply (*simp add*: *abseq-imp-dvd*)+
done

18.2 The Set of Integers as Algebraic Structure

18.2.1 Definition of $\mathcal{Z}$

constdefs
int-ring :: *int ring* ($\mathcal{Z}$)
int-ring ≡ (|*carrier* = *UNIV*, *mult* = *op* ∗, *one* = *1*, *zero* = *0*, *add* = *op* +|)

lemma *int-Zcarr* [*intro*!, *simp*]:
  $k \in$ *carrier* $\mathcal{Z}$
  by (*simp add*: *int-ring-def*)

lemma *int-is-cring*:
  cring $\mathcal{Z}$
unfolding *int-ring-def*
apply (*rule cringI*)
  apply (*rule abelian-groupI*, *simp-all*)
  defer *1*
  apply (*rule comm-monoidI*, *simp-all*)
 apply (*rule zadd-zmult-distrib*)
apply (*fast intro*: *zadd-zminus-inverse2*)
done

18.2.2 Interpretations

Since definitions of derived operations are global, their interpretation needs to be done as early as possible — that is, with as few assumptions as possible.

interpretation *int*: *monoid* [$\mathcal{Z}$]
  where *carrier* $\mathcal{Z} = UNIV$
    and *mult* $\mathcal{Z}$ x $y = x * y$
    and *one* $\mathcal{Z} = 1$
    and *pow* $\mathcal{Z}$ x $n = x\hat{}n$
proof −
  — Specification
  show *monoid* $\mathcal{Z}$ **by** (*unfold-locales*) (*auto simp*: *int-ring-def*)
  then interpret *int*: *monoid* [$\mathcal{Z}$] **.**

  — Carrier
  show *carrier* $\mathcal{Z} = UNIV$ **by** (*simp add*: *int-ring-def*)

  — Operations
  { fix x y **show** *mult* $\mathcal{Z}$ x $y = x * y$ **by** (*simp add*: *int-ring-def*) **}**
  note *mult* = *this*
  show *one*: *one* $\mathcal{Z} = 1$ **by** (*simp add*: *int-ring-def*)
  show *pow* $\mathcal{Z}$ x $n = x\hat{}n$ **by** (*induct n*) (*simp*, *simp add*: *int-ring-def*)+
qed

interpretation *int*: *comm-monoid* [$\mathcal{Z}$]
  where *finprod* $\mathcal{Z}$ f A = (*if finite A then setprod f A else arbitrary*)
proof −
  — Specification
  show *comm-monoid* $\mathcal{Z}$ **by** (*unfold-locales*) (*auto simp*: *int-ring-def*)
  then interpret *int*: *comm-monoid* [$\mathcal{Z}$] **.**

  — Operations
  { fix x y **have** *mult* $\mathcal{Z}$ x $y = x * y$ **by** (*simp add*: *int-ring-def*) **}**
  note *mult* = *this*

```
  have one: one ℤ = 1 by (simp add: int-ring-def)
  show finprod ℤ f A = (if finite A then setprod f A else arbitrary)
  proof (cases finite A)
    case True then show ?thesis proof induct
      case empty show ?case by (simp add: one)
    next
      case insert then show ?case by (simp add: Pi-def mult)
    qed
  next
    case False then show ?thesis by (simp add: finprod-def)
  qed
qed

interpretation int: abelian-monoid [ℤ]
  where zero ℤ = 0
    and add ℤ x y = x + y
    and finsum ℤ f A = (if finite A then setsum f A else arbitrary)
proof −
  — Specification
  show abelian-monoid ℤ by (unfold-locales) (auto simp: int-ring-def)
  then interpret int: abelian-monoid [ℤ] .

  — Operations
  { fix x y show add ℤ x y = x + y by (simp add: int-ring-def) }
  note add = this
  show zero: zero ℤ = 0 by (simp add: int-ring-def)
  show finsum ℤ f A = (if finite A then setsum f A else arbitrary)
  proof (cases finite A)
    case True then show ?thesis proof induct
      case empty show ?case by (simp add: zero)
    next
      case insert then show ?case by (simp add: Pi-def add)
    qed
  next
    case False then show ?thesis by (simp add: finsum-def finprod-def)
  qed
qed

interpretation int: abelian-group [ℤ]
  where a-inv ℤ x = − x
    and a-minus ℤ x y = x − y
proof −
  — Specification
  show abelian-group ℤ
  proof (rule abelian-groupI)
    show !!x. x ∈ carrier ℤ ==> EX y : carrier ℤ. y ⊕ℤ x = 0ℤ
      by (simp add: int-ring-def) arith
  qed (auto simp: int-ring-def)
  then interpret int: abelian-group [ℤ] .
```

— Operations
{ fix *x y* **have** *add* $\mathcal{Z}$ *x y* = *x* + *y* **by** (*simp add*: *int-ring-def*) **}**
note *add* = *this*
have *zero*: *zero* $\mathcal{Z}$ = *0* **by** (*simp add*: *int-ring-def*)
{ fix *x*
have *add* $\mathcal{Z}$ (−*x*) *x* = *zero* $\mathcal{Z}$ **by** (*simp add*: *add zero*)
then show *a-inv* $\mathcal{Z}$ *x* = − *x* **by** (*simp add*: *int.minus-equality*) **}**
note *a-inv* = *this*
show *a-minus* $\mathcal{Z}$ *x y* = *x* − *y* **by** (*simp add*: *int.minus-eq add a-inv*)
qed

interpretation *int*: *domain* [$\mathcal{Z}$]
by (*unfold-locales*) (*auto simp*: *int-ring-def left-distrib right-distrib*)

Removal of occurrences of *UNIV* in interpretation result — experimental.

lemma *UNIV*:
x ∈ *UNIV* = *True*
A ⊆ *UNIV* = *True*
(*ALL x* : *UNIV*. *P x*) = (*ALL x*. *P x*)
(*EX x* : *UNIV*. *P x*) = (*EX x*. *P x*)
(*True* −−> *Q*) = *Q*
(*True* ==> *PROP R*) == *PROP R*
by *simp-all*

interpretation *int* [*unfolded UNIV*]:
partial-order [(| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |)]
where *carrier* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) = *UNIV*
and *le* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) *x y* = (*x* ≤ *y*)
and *lless* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) *x y* = (*x* < *y*)
proof −
show *partial-order* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |)
by *unfold-locales simp-all*
show *carrier* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) = *UNIV*
by *simp*
show *le* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) *x y* = (*x* ≤ *y*)
by *simp*
show *lless* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) *x y* = (*x* < *y*)
by (*simp add*: *lless-def*) *auto*
qed

interpretation *int* [*unfolded UNIV*]:
lattice [(| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |)]
where *join* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) *x y* = *max x y*
and *meet* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) *x y* = *min x y*
proof −
let *?Z* = (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |)
show *lattice ?Z*
apply *unfold-locales*

apply (*simp add*: *least-def Upper-def*)
apply *arith*
apply (*simp add*: *greatest-def Lower-def*)
apply *arith*
done
then interpret *int*: *lattice* [*?Z*] **.**
show *join ?Z x y* = *max x y*
apply (*rule int.joinI*)
apply (*simp-all add*: *least-def Upper-def*)
apply *arith*
done
show *meet ?Z x y* = *min x y*
apply (*rule int.meetI*)
apply (*simp-all add*: *greatest-def Lower-def*)
apply *arith*
done
qed

interpretation *int* [*unfolded UNIV*]:
total-order [(| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |)]
by *unfold-locales clarsimp*

18.2.3 Generated Ideals of $\mathcal{Z}$

lemma *int-Idl*:
$Idl_{\mathcal{Z}}\ \{a\} = \{x * a \mid x.\ True\}$
apply (*subst int.cgenideal-eq-genideal*[*symmetric*]) **apply** (*simp add*: *int-ring-def*)
apply (*simp add*: *cgenideal-def int-ring-def*)
done

lemma *multiples-principalideal*:
principalideal $\{x * a \mid x.\ True\}\ \mathcal{Z}$
apply (*subst int-Idl*[*symmetric*], *rule principalidealI*)
apply (*rule int.genideal-ideal*, *simp*)
apply *fast*
done

lemma *prime-primeideal*:
assumes *prime*: *prime* (*nat p*)
shows *primeideal* ($Idl_{\mathcal{Z}}\ \{p\}$) $\mathcal{Z}$
apply (*rule primeidealI*)
apply (*rule int.genideal-ideal*, *simp*)
apply (*rule int-is-cring*)
apply (*simp add*: *int.cgenideal-eq-genideal*[*symmetric*] *cgenideal-def*)
apply (*simp add*: *int-ring-def*)
apply *clarsimp* **defer** *1*
apply (*simp add*: *int.cgenideal-eq-genideal*[*symmetric*] *cgenideal-def*)
apply (*simp add*: *int-ring-def*)
apply (*elim exE*)

proof −
fix *a b x*

from *prime*
have *ppos*: *0 <= p* **by** (*simp add*: *prime-def*)
have *unnat*: !!*x*. *nat p dvd nat* (*abs x*) ==> *p dvd x*
proof −
fix *x*
assume *nat p dvd nat* (*abs x*)
hence *int* (*nat p*) *dvd x* **by** (*simp add*: *int-dvd-iff*[*symmetric*])
thus *p dvd x* **by** (*simp add*: *ppos*)
qed

assume *a * b = x * p*
hence *p dvd a * b* **by** *simp*
hence *nat p dvd nat* (*abs* (*a * b*))
apply (*subst nat-dvd-iff*, *clarsimp*)
apply (*rule conjI*, *clarsimp*, *simp add*: *zabs-def*)
proof (*clarsimp*)
assume *a*: ~ *0 <= p*
from *prime*
have *0 < p* **by** (*simp add*: *prime-def*)
from *a* **and** *this*
have *False* **by** *simp*
thus *nat* (*abs* (*a * b*)) = *0* **..**
qed
hence *nat p dvd* (*nat* (*abs a*) * *nat* (*abs b*)) **by** (*simp add*: *nat-abs-mult-distrib*)
hence *nat p dvd nat* (*abs a*) | *nat p dvd nat* (*abs b*) **by** (*rule prime-dvd-mult*[*OF prime*])
hence *p dvd a* | *p dvd b* **by** (*fast intro*: *unnat*)
thus (*EX x*. *a = x * p*) | (*EX x*. *b = x * p*)
proof
assume *p dvd a*
hence *EX x*. *a = p * x* **by** (*simp add*: *dvd-def*)
from *this* **obtain** *x*
where *a = p * x* **by** *fast*
hence *a = x * p* **by** *simp*
hence *EX x*. *a = x * p* **by** *simp*
thus (*EX x*. *a = x * p*) | (*EX x*. *b = x * p*) **..**
next
assume *p dvd b*
hence *EX x*. *b = p * x* **by** (*simp add*: *dvd-def*)
from *this* **obtain** *x*
where *b = p * x* **by** *fast*
hence *b = x * p* **by** *simp*
hence *EX x*. *b = x * p* **by** *simp*
thus (*EX x*. *a = x * p*) | (*EX x*. *b = x * p*) **..**
qed

next
assume *UNIV* = {*uu*. *EX x*. *uu* = *x* ∗ *p*}
from *this* **obtain** *x*
where *1* = *x* ∗ *p* **by** *fast*
from *this* [*symmetric*]
have *p* ∗ *x* = *1* **by** (*subst zmult-commute*)
hence |*p* ∗ *x*| = *1* **by** *simp*
hence |*p*| = *1* **by** (*rule abs-zmult-eq-1*)
from *this* **and** *prime*
show *False* **by** (*simp add*: *prime-def*)
qed

18.2.4 Ideals and Divisibility

lemma *int-Idl-subset-ideal*:
Idl$_{\mathcal{Z}}$ {*k*} ⊆ *Idl*$_{\mathcal{Z}}$ {*l*} = (*k* ∈ *Idl*$_{\mathcal{Z}}$ {*l*})
by (*rule int.Idl-subset-ideal'*, *simp+*)

lemma *Idl-subset-eq-dvd*:
(*Idl*$_{\mathcal{Z}}$ {*k*} ⊆ *Idl*$_{\mathcal{Z}}$ {*l*}) = (*l dvd k*)
apply (*subst int-Idl-subset-ideal*, *subst int-Idl*, *simp*)
apply (*rule*, *clarify*)
apply (*simp add*: *dvd-def*, *clarify*)
apply (*simp add*: *int.m-comm*)
done

lemma *dvds-eq-Idl*:
(*l dvd k* ∧ *k dvd l*) = (*Idl*$_{\mathcal{Z}}$ {*k*} = *Idl*$_{\mathcal{Z}}$ {*l*})
proof −
have *a*: *l dvd k* = (*Idl*$_{\mathcal{Z}}$ {*k*} ⊆ *Idl*$_{\mathcal{Z}}$ {*l*}) **by** (*rule Idl-subset-eq-dvd*[*symmetric*])
have *b*: *k dvd l* = (*Idl*$_{\mathcal{Z}}$ {*l*} ⊆ *Idl*$_{\mathcal{Z}}$ {*k*}) **by** (*rule Idl-subset-eq-dvd*[*symmetric*])

have (*l dvd k* ∧ *k dvd l*) = ((*Idl*$_{\mathcal{Z}}$ {*k*} ⊆ *Idl*$_{\mathcal{Z}}$ {*l*}) ∧ (*Idl*$_{\mathcal{Z}}$ {*l*} ⊆ *Idl*$_{\mathcal{Z}}$ {*k*}))
by (*subst a*, *subst b*, *simp*)
also have ((*Idl*$_{\mathcal{Z}}$ {*k*} ⊆ *Idl*$_{\mathcal{Z}}$ {*l*}) ∧ (*Idl*$_{\mathcal{Z}}$ {*l*} ⊆ *Idl*$_{\mathcal{Z}}$ {*k*})) = (*Idl*$_{\mathcal{Z}}$ {*k*} = *Idl*$_{\mathcal{Z}}$ {*l*}) **by** (*rule*, *fast+*)
finally
show *?thesis* **.**
qed

lemma *Idl-eq-abs*:
(*Idl*$_{\mathcal{Z}}$ {*k*} = *Idl*$_{\mathcal{Z}}$ {*l*}) = (*abs l* = *abs k*)
apply (*subst dvds-eq-abseq*[*symmetric*])
apply (*rule dvds-eq-Idl*[*symmetric*])
done

18.2.5 Ideals and the Modulus

constdefs
ZMod :: *int* => *int* => *int set*

ZMod k r == (Idl$_{\mathcal{Z}}$ *{k}) +>*$_{\mathcal{Z}}$ *r*

lemmas *ZMod-defs =*
ZMod-def genideal-def

lemma *rcos-zfact*:
assumes *kIl*: $k \in$ *ZMod l r*
shows *EX x*. $k = x * l + r$
proof −
from *kIl*[*unfolded ZMod-def*]
have $\exists\, xl \in$*Idl*$_{\mathcal{Z}}$ *{l}*. $k = xl + r$ **by** (*simp add*: *a-r-coset-defs int-ring-def*)
from *this* **obtain** *xl*
where *xl*: $xl \in$ *Idl*$_{\mathcal{Z}}$ *{l}*
and *k*: $k = xl + r$
by *auto*
from *xl* **obtain** *x*
where $xl = x * l$
by (*simp add*: *int-Idl*, *fast*)
from *k* **and** *this*
have $k = x * l + r$ **by** *simp*
thus $\exists\, x.\ k = x * l + r$ **..**
qed

lemma *ZMod-imp-zmod*:
assumes *zmods*: *ZMod m a = ZMod m b*
shows *a mod m = b mod m*
proof −
interpret *ideal* [*Idl*$_{\mathcal{Z}}$ *{m}* $\mathcal{Z}$] **by** (*rule int.genideal-ideal*, *fast*)
from *zmods*
have $b \in$ *ZMod m a*
unfolding *ZMod-def*
by (*simp add*: *a-repr-independenceD*)
from *this*
have *EX x*. $b = x * m + a$ **by** (*rule rcos-zfact*)
from *this* **obtain** *x*
where $b = x * m + a$
by *fast*

hence *b mod m* = $(x * m + a)$ *mod m* **by** *simp*
also
have $\ldots = ((x * m)\ mod\ m) + (a\ mod\ m)$ **by** (*simp add*: *zmod-zadd1-eq*)
also
have $\ldots = a\ mod\ m$ **by** *simp*
finally
have *b mod m = a mod m* **.**
thus *a mod m = b mod m* **..**
qed

lemma *ZMod-mod*:

shows *ZMod m a = ZMod m (a mod m)*
proof −
interpret *ideal* [$Idl_{\mathcal{Z}}$ {*m*} $\mathcal{Z}$] **by** (*rule int.genideal-ideal*, *fast*)
show *?thesis*
unfolding *ZMod-def*
apply (*rule a-repr-independence'*[*symmetric*])
apply (*simp add*: *int-Idl a-r-coset-defs*)
apply (*simp add*: *int-ring-def*)
proof −
have *a = m * (a div m) + (a mod m)* **by** (*simp add*: *zmod-zdiv-equality*)
hence *a = (a div m) * m + (a mod m)* **by** *simp*
thus ∃ *h*. (∃ *x*. *h = x * m*) ∧ *a = h + a mod m* **by** *fast*
qed *simp*
qed

lemma *zmod-imp-ZMod*:
assumes *modeq*: *a mod m = b mod m*
shows *ZMod m a = ZMod m b*
proof −
have *ZMod m a = ZMod m (a mod m)* **by** (*rule ZMod-mod*)
also have ... = *ZMod m (b mod m)* **by** (*simp add*: *modeq*[*symmetric*])
also have ... = *ZMod m b* **by** (*rule ZMod-mod*[*symmetric*])
finally show *?thesis* **.**
qed

corollary *ZMod-eq-mod*:
shows (*ZMod m a = ZMod m b*) = (*a mod m = b mod m*)
by (*rule*, *erule ZMod-imp-zmod*, *erule zmod-imp-ZMod*)

18.2.6 Factorization

constdefs
ZFact :: *int* ⇒ *int set ring*
ZFact k == $\mathcal{Z}$ *Quot* ($Idl_{\mathcal{Z}}$ {*k*})

lemmas *ZFact-defs = ZFact-def FactRing-def*

lemma *ZFact-is-cring*:
shows *cring* (*ZFact k*)
apply (*unfold ZFact-def*)
apply (*rule ideal.quotient-is-cring*)
apply (*intro ring.genideal-ideal*)
apply (*simp add*: *cring.axioms*[*OF int-is-cring*] *ring.intro*)
apply *simp*
apply (*rule int-is-cring*)
done

lemma *ZFact-zero*:
carrier (*ZFact 0*) = (⋃ *a*. {{*a*}})

apply (*insert int.genideal-zero*)
apply (*simp add*: *ZFact-defs A-RCOSETS-defs r-coset-def int-ring-def ring-record-simps*)
done

lemma *ZFact-one*:
  carrier (*ZFact 1*) = {*UNIV*}
apply (*simp only*: *ZFact-defs A-RCOSETS-defs r-coset-def int-ring-def ring-record-simps*)
apply (*subst int.genideal-one*[*unfolded int-ring-def*, *simplified ring-record-simps*])
apply (*rule*, *rule*, *clarsimp*)
 apply (*rule*, *rule*, *clarsimp*)
 apply (*rule*, *clarsimp*, *arith*)
apply (*rule*, *clarsimp*)
apply (*rule exI*[*of - 0*], *clarsimp*)
done

lemma *ZFact-prime-is-domain*:
  assumes *pprime*: *prime* (*nat p*)
  shows *domain* (*ZFact p*)
apply (*unfold ZFact-def*)
apply (*rule primeideal.quotient-is-domain*)
apply (*rule prime-primeideal*[*OF pprime*])
done

end

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