Examples for program extraction in Higher-Order Logic

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Contents

1 Auxiliary lemmas used in program extraction examples

theory *Util*
imports *Main*
begin

Decidability of equality on natural numbers.

lemma *nat-eq-dec*: $\bigwedge n{::}nat.\ m = n \vee m \neq n$
⟨*proof*⟩

Well-founded induction on natural numbers, derived using the standard structural induction rule.

lemma *nat-wf-ind*:
assumes *R*: $\bigwedge x{::}nat.\ (\bigwedge y.\ y < x \Longrightarrow P\ y) \Longrightarrow P\ x$

shows $P\ z$
⟨*proof*⟩

Bounded search for a natural number satisfying a decidable predicate.

lemma *search*:
assumes *dec*: $\bigwedge x{::}nat.\ P\ x \vee \neg\ P\ x$
shows $(\exists\, x{<}y.\ P\ x) \vee \neg\ (\exists\, x{<}y.\ P\ x)$
⟨*proof*⟩

end

2 Quotient and remainder

theory *QuotRem* **imports** *Util* **begin**

Derivation of quotient and remainder using program extraction.

theorem *division*: $\exists\, r\ q.\ a = Suc\ b * q + r \wedge r \leq b$
⟨*proof*⟩

extract *division*

The program extracted from the above proof looks as follows

```
division ≡
λx xa.
  nat-rec (0, 0)
   (λa H. let (x, y) = H
          in case nat-eq-dec x xa of Left ⇒ (0, Suc y)
             | Right ⇒ (Suc x, y))
   x
```

The corresponding correctness theorem is

$a = Suc\ b * snd\ (division\ a\ b) + fst\ (division\ a\ b) \wedge fst\ (division\ a\ b) \leq b$

code-module *Div*
contains
test = *division 9 2*

export-code *division* **in** *SML*

end

3 Greatest common divisor

theory *Greatest-Common-Divisor*

imports *QuotRem*
begin

theorem *greatest-common-divisor*:
$\bigwedge n{::}nat.\ Suc\ m < n \Longrightarrow \exists\, k\ n1\ m1.\ k * n1 = n \wedge k * m1 = Suc\ m \wedge$
$(\forall\, l\ l1\ l2.\ l * l1 = n \longrightarrow l * l2 = Suc\ m \longrightarrow l \leq k)$
⟨*proof*⟩

extract *greatest-common-divisor*

The extracted program for computing the greatest common divisor is

```
greatest-common-divisor ≡
λx. nat-wf-ind-P x
      (λx H2 xa.
         let (xa, y) = division xa x
         in case xa of 0 ⇒ (Suc x, y, 1)
            | Suc nat ⇒
               let (x, ya) = H2 nat (Suc x); (xa, ya) = ya
               in (x, xa * y + ya, xa))
```

consts-code
arbitrary ((*error arbitrary*))

code-module *GCD*
contains
test = *greatest-common-divisor 7 12*

⟨*ML*⟩

end

4 Warshall's algorithm

theory *Warshall*
imports *Main*
begin

Derivation of Warshall's algorithm using program extraction, based on Berger, Schwichtenberg and Seisenberger [1].

datatype $b = T \mid F$

primrec
is-path' :: $('a \Rightarrow 'a \Rightarrow b) \Rightarrow 'a \Rightarrow 'a\ list \Rightarrow 'a \Rightarrow bool$
where
is-path' r x [] *z* = (*r x z* = *T*)
| *is-path' r x* (*y* # *ys*) *z* = (*r x y* = *T* ∧ *is-path' r y ys z*)

definition
is-path :: $(nat \Rightarrow nat \Rightarrow b) \Rightarrow (nat * nat\ list * nat) \Rightarrow$
$nat \Rightarrow nat \Rightarrow nat \Rightarrow bool$
where
$is\text{-}path\ r\ p\ i\ j\ k \longleftrightarrow fst\ p = j \wedge snd\ (snd\ p) = k \wedge$
$list\text{-}all\ (\lambda x.\ x < i)\ (fst\ (snd\ p)) \wedge$
$is\text{-}path'\ r\ (fst\ p)\ (fst\ (snd\ p))\ (snd\ (snd\ p))$

definition
conc :: $('a * 'a\ list * 'a) \Rightarrow ('a * 'a\ list * 'a) \Rightarrow ('a * 'a\ list * 'a)$
where
$conc\ p\ q = (fst\ p,\ fst\ (snd\ p)\ @\ fst\ q\ \#\ fst\ (snd\ q),\ snd\ (snd\ q))$

theorem *is-path'-snoc* [*simp*]:
$\bigwedge x.\ is\text{-}path'\ r\ x\ (ys\ @\ [y])\ z = (is\text{-}path'\ r\ x\ ys\ y \wedge r\ y\ z = T)$
⟨*proof*⟩

theorem *list-all-scoc* [*simp*]: $list\text{-}all\ P\ (xs\ @\ [x]) = (P\ x \wedge list\text{-}all\ P\ xs)$
⟨*proof*⟩

theorem *list-all-lemma*:
$list\text{-}all\ P\ xs \Longrightarrow (\bigwedge x.\ P\ x \Longrightarrow Q\ x) \Longrightarrow list\text{-}all\ Q\ xs$
⟨*proof*⟩

theorem *lemma1*: $\bigwedge p.\ is\text{-}path\ r\ p\ i\ j\ k \Longrightarrow is\text{-}path\ r\ p\ (Suc\ i)\ j\ k$
⟨*proof*⟩

theorem *lemma2*: $\bigwedge p.\ is\text{-}path\ r\ p\ 0\ j\ k \Longrightarrow r\ j\ k = T$
⟨*proof*⟩

theorem *is-path'-conc*: $is\text{-}path'\ r\ j\ xs\ i \Longrightarrow is\text{-}path'\ r\ i\ ys\ k \Longrightarrow$
$is\text{-}path'\ r\ j\ (xs\ @\ i\ \#\ ys)\ k$
⟨*proof*⟩

theorem *lemma3*:
$\bigwedge p\ q.\ is\text{-}path\ r\ p\ i\ j\ i \Longrightarrow is\text{-}path\ r\ q\ i\ i\ k \Longrightarrow$
$is\text{-}path\ r\ (conc\ p\ q)\ (Suc\ i)\ j\ k$
⟨*proof*⟩

theorem *lemma5*:
$\bigwedge p.\ is\text{-}path\ r\ p\ (Suc\ i)\ j\ k \Longrightarrow \sim is\text{-}path\ r\ p\ i\ j\ k \Longrightarrow$
$(\exists q.\ is\text{-}path\ r\ q\ i\ j\ i) \wedge (\exists q'.\ is\text{-}path\ r\ q'\ i\ i\ k)$
⟨*proof*⟩

theorem *lemma5'*:
$\bigwedge p.\ is\text{-}path\ r\ p\ (Suc\ i)\ j\ k \Longrightarrow \neg\ is\text{-}path\ r\ p\ i\ j\ k \Longrightarrow$
$\neg\ (\forall q.\ \neg\ is\text{-}path\ r\ q\ i\ j\ i) \wedge \neg\ (\forall q'.\ \neg\ is\text{-}path\ r\ q'\ i\ i\ k)$
⟨*proof*⟩

theorem *warshall*:
$\bigwedge j\ k.\ \neg\ (\exists\, p.\ \mathit{is\text{-}path}\ r\ p\ i\ j\ k) \vee (\exists\, p.\ \mathit{is\text{-}path}\ r\ p\ i\ j\ k)$
⟨*proof*⟩

extract *warshall*

The program extracted from the above proof looks as follows

```
warshall ≡
λx xa xb xc.
  nat-rec (λxa xb. case x xa xb of T ⇒ Some (xa, [], xb) | F ⇒ None)
   (λx H2 xa xb.
      case H2 xa xb of
      None ⇒
        case H2 xa x of None ⇒ None
        | Some q ⇒
            case H2 x xb of None ⇒ None | Some qa ⇒ Some (conc q qa)
      | Some q ⇒ Some q)
   xa xb xc
```

The corresponding correctness theorem is

case warshall r i j k of None ⇒ ∀ *x*. ¬ *is-path r x i j k*
| *Some q* ⇒ *is-path r q i j k*

end

5 Higman's lemma

theory *Higman*
imports *Main*
begin

Formalization by Stefan Berghofer and Monika Seisenberger, based on Coquand and Fridlender [2].

datatype *letter* = *A* | *B*

inductive *emb* :: *letter list* ⇒ *letter list* ⇒ *bool*
where
  emb0 [*Pure.intro*]: *emb* [] *bs*
| *emb1* [*Pure.intro*]: *emb as bs* ⟹ *emb as* (*b* # *bs*)
| *emb2* [*Pure.intro*]: *emb as bs* ⟹ *emb* (*a* # *as*) (*a* # *bs*)

inductive *L* :: *letter list* ⇒ *letter list list* ⇒ *bool*
  for *v* :: *letter list*
where
  L0 [*Pure.intro*]: *emb w v* ⟹ *L v* (*w* # *ws*)
| *L1* [*Pure.intro*]: *L v ws* ⟹ *L v* (*w* # *ws*)

inductive *good* :: *letter list list* ⇒ *bool*
where
  good0 [*Pure.intro*]: *L w ws* ⟹ *good* (*w* # *ws*)
| *good1* [*Pure.intro*]: *good ws* ⟹ *good* (*w* # *ws*)

inductive *R* :: *letter* ⇒ *letter list list* ⇒ *letter list list* ⇒ *bool*
  for *a* :: *letter*
where
  R0 [*Pure.intro*]: *R a* [] []
| *R1* [*Pure.intro*]: *R a vs ws* ⟹ *R a* (*w* # *vs*) ((*a* # *w*) # *ws*)

inductive *T* :: *letter* ⇒ *letter list list* ⇒ *letter list list* ⇒ *bool*
  for *a* :: *letter*
where
  T0 [*Pure.intro*]: *a* ≠ *b* ⟹ *R b ws zs* ⟹ *T a* (*w* # *zs*) ((*a* # *w*) # *zs*)
| *T1* [*Pure.intro*]: *T a ws zs* ⟹ *T a* (*w* # *ws*) ((*a* # *w*) # *zs*)
| *T2* [*Pure.intro*]: *a* ≠ *b* ⟹ *T a ws zs* ⟹ *T a ws* ((*b* # *w*) # *zs*)

inductive *bar* :: *letter list list* ⇒ *bool*
where
  bar1 [*Pure.intro*]: *good ws* ⟹ *bar ws*
| *bar2* [*Pure.intro*]: (⋀*w*. *bar* (*w* # *ws*)) ⟹ *bar ws*

theorem *prop1*: *bar* ([] # *ws*) ⟨*proof*⟩

theorem *lemma1*: *L as ws* ⟹ *L* (*a* # *as*) *ws*
  ⟨*proof*⟩

lemma *lemma2′*: *R a vs ws* ⟹ *L as vs* ⟹ *L* (*a* # *as*) *ws*
  ⟨*proof*⟩

lemma *lemma2*: *R a vs ws* ⟹ *good vs* ⟹ *good ws*
  ⟨*proof*⟩

lemma *lemma3′*: *T a vs ws* ⟹ *L as vs* ⟹ *L* (*a* # *as*) *ws*
  ⟨*proof*⟩

lemma *lemma3*: *T a ws zs* ⟹ *good ws* ⟹ *good zs*
  ⟨*proof*⟩

lemma *lemma4*: *R a ws zs* ⟹ *ws* ≠ [] ⟹ *T a ws zs*
  ⟨*proof*⟩

lemma *letter-neq*: (*a*::*letter*) ≠ *b* ⟹ *c* ≠ *a* ⟹ *c* = *b*
  ⟨*proof*⟩

lemma *letter-eq-dec*: (*a*::*letter*) = *b* ∨ *a* ≠ *b*
  ⟨*proof*⟩

theorem *prop2*:
assumes *ab*: $a \neq b$ **and** *bar*: *bar xs*
shows $\bigwedge ys\ zs.\ bar\ ys \Longrightarrow T\ a\ xs\ zs \Longrightarrow T\ b\ ys\ zs \Longrightarrow bar\ zs\ \langle proof \rangle$

theorem *prop3*:
assumes *bar*: *bar xs*
shows $\bigwedge zs.\ xs \neq [] \Longrightarrow R\ a\ xs\ zs \Longrightarrow bar\ zs\ \langle proof \rangle$

theorem *higman*: *bar* []
$\langle proof \rangle$

primrec
is-prefix :: $'a\ list \Rightarrow (nat \Rightarrow 'a) \Rightarrow bool$
where
is-prefix [] *f* = *True*
| *is-prefix* (*x* # *xs*) *f* = (*x* = *f* (*length xs*) ∧ *is-prefix xs f*)

theorem *L-idx*:
assumes *L*: *L w ws*
shows *is-prefix ws f* $\Longrightarrow \exists i.\ emb\ (f\ i)\ w \wedge i < length\ ws\ \langle proof \rangle$

theorem *good-idx*:
assumes *good*: *good ws*
shows *is-prefix ws f* $\Longrightarrow \exists i\ j.\ emb\ (f\ i)\ (f\ j) \wedge i < j\ \langle proof \rangle$

theorem *bar-idx*:
assumes *bar*: *bar ws*
shows *is-prefix ws f* $\Longrightarrow \exists i\ j.\ emb\ (f\ i)\ (f\ j) \wedge i < j\ \langle proof \rangle$

Strong version: yields indices of words that can be embedded into each other.

theorem *higman-idx*: $\exists (i::nat)\ j.\ emb\ (f\ i)\ (f\ j) \wedge i < j$
$\langle proof \rangle$

Weak version: only yield sequence containing words that can be embedded into each other.

theorem *good-prefix-lemma*:
assumes *bar*: *bar ws*
shows *is-prefix ws f* $\Longrightarrow \exists vs.$ *is-prefix vs f* ∧ *good vs* $\langle proof \rangle$

theorem *good-prefix*: $\exists vs.$ *is-prefix vs f* ∧ *good vs*
$\langle proof \rangle$

5.1 Extracting the program

declare *R.induct* [*ind-realizer*]
declare *T.induct* [*ind-realizer*]
declare *L.induct* [*ind-realizer*]

declare *good.induct* [*ind-realizer*]
declare *bar.induct* [*ind-realizer*]

extract *higman-idx*

Program extracted from the proof of *higman-idx*:

$$higman\text{-}idx \equiv \lambda x.\ bar\text{-}idx\ x\ higman$$

Corresponding correctness theorem:

$$emb\ (f\ (fst\ (higman\text{-}idx\ f)))\ (f\ (snd\ (higman\text{-}idx\ f))) \wedge fst\ (higman\text{-}idx\ f) < snd\ (higman\text{-}idx\ f)$$

Program extracted from the proof of *higman*:

$$higman \equiv bar2\ []\ (list\text{-}rec\ (prop1\ [])\ (\lambda a\ w\ H.\ prop3\ a\ [a \#\ w]\ H\ (R1\ []\ []\ w\ R0)))$$

Program extracted from the proof of *prop1*:

$$prop1 \equiv \lambda x.\ bar2\ ([] \#\ x)\ (\lambda w.\ bar1\ (w \#\ [] \#\ x)\ (good0\ w\ ([] \#\ x)\ (L0\ []\ x)))$$

Program extracted from the proof of *prop2*:

```
prop2 ≡
λx xa xb xc H.
  barT-rec (λws xa xb xc H Ha Hb. bar1 xc (lemma3 x Ha xa))
   (λws xb r xc xd H.
      barT-rec (λws x xb H Ha. bar1 xb (lemma3 xa Ha x))
       (λwsa xb ra xc H Ha.
          bar2 xc
           (list-case (prop1 xc)
             (λa list.
                case letter-eq-dec a x of
                Left ⇒
                  r list wsa ((x # list) # xc) (bar2 wsa xb)
                   (T1 ws xc list H) (T2 x wsa xc list Ha)
                | Right ⇒
                    ra list ((xa # list) # xc) (T2 xa ws xc list H)
                     (T1 wsa xc list Ha))))
       H xd)
   H xb xc
```

Program extracted from the proof of *prop3*:

```
prop3 ≡
λx xa H.
  barT-rec (λws xa xb H. bar1 xb (lemma2 x H xa))
```

```
(λws xa r xb H.
    bar2 xb
     (list-rec (prop1 xb)
       (λa w Ha.
           case letter-eq-dec a x of
           Left ⇒ r w ((x # w) # xb) (R1 ws xb w H)
           | Right ⇒
              prop2 a x ws ((a # w) # xb) Ha (bar2 ws xa)
               (T0 x ws xb w H) (T2 a ws xb w (lemma4 x H)))))
H xa
```

5.2 Some examples

consts-code
  arbitrary :: *LT* (({∗ *L0* [] [] ∗}))
  arbitrary :: *TT* (({∗ *T0 A* [] [] [] *R0* ∗}))

code-module *Higman*
contains
  higman = *higman-idx*

⟨*ML*⟩

definition
  arbitrary-LT :: *LT* **where**
  [*symmetric*, *code inline*]: *arbitrary-LT* = *arbitrary*

definition
  arbitrary-TT :: *TT* **where**
  [*symmetric*, *code inline*]: *arbitrary-TT* = *arbitrary*

code-datatype *L0 L1 arbitrary-LT*
code-datatype *T0 T1 T2 arbitrary-TT*

export-code *higman-idx* **in** *SML* **module-name** *Higman*

⟨*ML*⟩

end

6 The pigeonhole principle

theory *Pigeonhole*
imports *Util Efficient-Nat*
begin

We formalize two proofs of the pigeonhole principle, which lead to extracted

programs of quite different complexity. The original formalization of these proofs in NUPRL is due to Aleksey Nogin [3].

This proof yields a polynomial program.

theorem *pigeonhole*:
$\bigwedge f.\ (\bigwedge i.\ i \leq Suc\ n \Longrightarrow f\ i \leq n) \Longrightarrow \exists\, i\ j.\ i \leq Suc\ n \wedge j < i \wedge f\ i = f\ j$
⟨*proof*⟩

The following proof, although quite elegant from a mathematical point of view, leads to an exponential program:

theorem *pigeonhole-slow*:
$\bigwedge f.\ (\bigwedge i.\ i \leq Suc\ n \Longrightarrow f\ i \leq n) \Longrightarrow \exists\, i\ j.\ i \leq Suc\ n \wedge j < i \wedge f\ i = f\ j$
⟨*proof*⟩

extract *pigeonhole pigeonhole-slow*

The programs extracted from the above proofs look as follows:

```
pigeonhole ≡
nat-rec (λx. (Suc 0, 0))
 (λx H2 xa.
    nat-rec arbitrary
     (λx H2.
        case search (Suc x) (λxb. nat-eq-dec (xa (Suc x)) (xa xb)) of
        None ⇒ let (x, y) = H2 in (x, y) | Some p ⇒ (Suc x, p))
     (Suc (Suc x)))

pigeonhole-slow ≡
nat-rec (λx. (Suc 0, 0))
 (λx H2 xa.
    case search (Suc (Suc x))
          (λxb. nat-eq-dec (xa (Suc (Suc x))) (xa xb)) of
    None ⇒
      let (x, y) = H2 (λi. if xa i = Suc x then xa (Suc (Suc x)) else xa i)
      in (x, y)
    | Some p ⇒ (Suc (Suc x), p))
```

The program for searching for an element in an array is

```
search ≡
λx H. nat-rec None
        (λy Ha.
           case Ha of None ⇒ case H y of Left ⇒ Some y | Right ⇒ None
           | Some p ⇒ Some p)
        x
```

The correctness statement for *pigeonhole* is

$(\bigwedge i.\ i \leq Suc\ n \Longrightarrow f\ i \leq n) \Longrightarrow$

fst (*pigeonhole n f*) ≤ *Suc n* ∧
snd (*pigeonhole n f*) < *fst* (*pigeonhole n f*) ∧
f (*fst* (*pigeonhole n f*)) = *f* (*snd* (*pigeonhole n f*))

In order to analyze the speed of the above programs, we generate ML code from them.

definition
test n u = *pigeonhole n* (λ*m*. *m* − *1*)
definition
test′ n u = *pigeonhole-slow n* (λ*m*. *m* − *1*)
definition
test″ u = *pigeonhole 8* (*op* ! [*0*, *1*, *2*, *3*, *4*, *5*, *6*, *3*, *7*, *8*])

consts-code
arbitrary :: *nat* ({∗ *0*::*nat* ∗})
arbitrary :: *nat* × *nat* ({∗ (*0*::*nat*, *0*::*nat*) ∗})

definition
arbitrary-nat-pair :: *nat* × *nat* **where**
[*symmetric*, *code inline*]: *arbitrary-nat-pair* = *arbitrary*

definition
arbitrary-nat :: *nat* **where**
[*symmetric*, *code inline*]: *arbitrary-nat* = *arbitrary*

code-const *arbitrary-nat-pair* (*SML* (~*1*, ~*1*))

code-const *arbitrary-nat* (*SML* ~*1*)

code-module *PH1*
contains
test = *test*
test′ = *test′*
test″ = *test″*

export-code *test test′ test″* **in** *SML* **module-name** *PH2*

⟨*ML*⟩

end

7 Euclid's theorem

theory *Euclid*
imports ~~/*src*/*HOL*/*NumberTheory*/*Factorization Efficient-Nat Util*

begin

A constructive version of the proof of Euclid's theorem by Markus Wenzel and Freek Wiedijk [4].

lemma *prime-eq*: *prime p* = (*1* < *p* ∧ (∀ *m*. *m dvd p* ⟶ *1* < *m* ⟶ *m* = *p*))
⟨*proof*⟩

lemma *prime-eq'*: *prime p* = (*1* < *p* ∧ (∀ *m k*. *p* = *m* ∗ *k* ⟶ *1* < *m* ⟶ *m* = *p*))
⟨*proof*⟩

lemma *factor-greater-one1*: *n* = *m* ∗ *k* ⟹ *m* < *n* ⟹ *k* < *n* ⟹ *Suc 0* < *m*
⟨*proof*⟩

lemma *factor-greater-one2*: *n* = *m* ∗ *k* ⟹ *m* < *n* ⟹ *k* < *n* ⟹ *Suc 0* < *k*
⟨*proof*⟩

lemma *not-prime-ex-mk*:
assumes *n*: *Suc 0* < *n*
shows (∃ *m k*. *Suc 0* < *m* ∧ *Suc 0* < *k* ∧ *m* < *n* ∧ *k* < *n* ∧ *n* = *m* ∗ *k*) ∨ *prime n*
⟨*proof*⟩

lemma *factor-exists*: *Suc 0* < *n* ⟹ (∃ *l*. *primel l* ∧ *prod l* = *n*)
⟨*proof*⟩

lemma *dvd-prod* [*iff*]: *n dvd prod* (*n* # *ns*)
⟨*proof*⟩

primrec *fact* :: *nat* ⇒ *nat* ((-!) [*1000*] *999*)
where
0! = *1*
| (*Suc n*)! = *n*! ∗ *Suc n*

lemma *fact-greater-0* [*iff*]: *0* < *n*!
⟨*proof*⟩

lemma *dvd-factorial*: *0* < *m* ⟹ *m* ≤ *n* ⟹ *m dvd n*!
⟨*proof*⟩

lemma *prime-factor-exists*:
assumes *N*: (*1*::*nat*) < *n*
shows ∃ *p*. *prime p* ∧ *p dvd n*
⟨*proof*⟩

Euclid's theorem: there are infinitely many primes.

lemma *Euclid*: ∃ *p*. *prime p* ∧ *n* < *p*
⟨*proof*⟩

extract *Euclid*

The program extracted from the proof of Euclid's theorem looks as follows.

Euclid $\equiv \lambda x.$ *prime-factor-exists* $(x! + 1)$

The program corresponding to the proof of the factorization theorem is

```
factor-exists ≡
λx. nat-wf-ind-P x
      (λx H2.
          case not-prime-ex-mk x of None ⇒ [x]
          | Some p ⇒ let (x, y) = p in split-primel (H2 x) (H2 y))
```

consts-code
  arbitrary ((*error arbitrary*))

code-module *Prime*
contains *Euclid*

⟨*ML*⟩

end

References

[1] U. Berger, H. Schwichtenberg, and M. Seisenberger. The Warshall algorithm and Dickson's lemma: Two examples of realistic program extraction. *Journal of Automated Reasoning*, 26:205–221, 2001.

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[4] M. Wenzel and F. Wiedijk. A comparison of the mathematical proof languages Mizar and Isar. *Journal of Automated Reasoning*, 29(3-4):389–411, 2002.