# The UNITY Formalism

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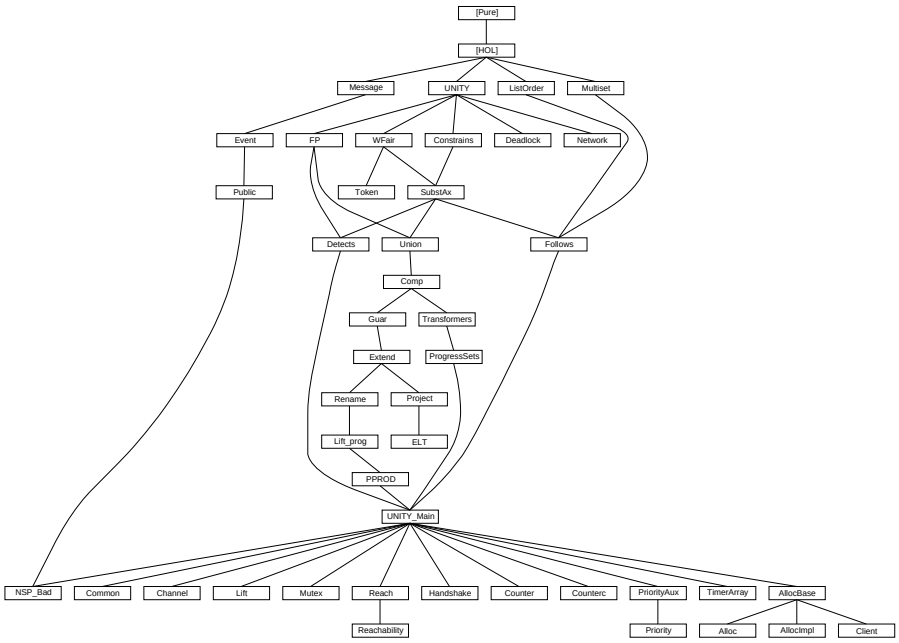
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# 1 The Basic UNITY Theory

**theory** *UNITY* **imports** *Main* **begin**

**typedef** *(Program)*
```
  'a program = "{(init:: 'a set, acts :: ('a * 'a)set set,
                  allowed :: ('a * 'a)set set). Id ∈ acts & Id: allowed}"
```

⟨*proof*⟩

**constdefs**
```
  Acts :: "'a program => ('a * 'a)set set"
    "Acts F == (%(init, acts, allowed). acts) (Rep_Program F)"

  "constrains" :: "['a set, 'a set] => 'a program set"  (infixl "co"     60)
    "A co B == {F. ∀act ∈ Acts F. act``A ⊆ B}"

  unless  :: "['a set, 'a set] => 'a program set"  (infixl "unless" 60)
    "A unless B == (A-B) co (A ∪ B)"

  mk_program :: "('a set * ('a * 'a)set set * ('a * 'a)set set)
                  => 'a program"
    "mk_program == %(init, acts, allowed).
                      Abs_Program (init, insert Id acts, insert Id allowed)"

  Init :: "'a program => 'a set"
    "Init F == (%(init, acts, allowed). init) (Rep_Program F)"

  AllowedActs :: "'a program => ('a * 'a)set set"
    "AllowedActs F == (%(init, acts, allowed). allowed) (Rep_Program F)"

  Allowed :: "'a program => 'a program set"
    "Allowed F == {G. Acts G ⊆ AllowedActs F}"

  stable     :: "'a set => 'a program set"
    "stable A == A co A"

  strongest_rhs :: "['a program, 'a set] => 'a set"
    "strongest_rhs F A == Inter {B. F ∈ A co B}"

  invariant :: "'a set => 'a program set"
    "invariant A == {F. Init F ⊆ A} ∩ stable A"

  increasing :: "['a => 'b::{order}] => 'a program set"
```
— Polymorphic in both states and the meaning of $\leq$
```
    "increasing f == ⋂z. stable {s. z ≤ f s}"
```

Perhaps HOL shouldn't add this in the first place!

**declare** *image_Collect [simp del]*

### 1.0.1 The abstract type of programs

**lemmas** *program_typedef =*
*Rep_Program Rep_Program_inverse Abs_Program_inverse*

```
Program_def Init_def Acts_def AllowedActs_def mk_program_def
```

**lemma** *Id_in_Acts [iff]: "Id ∈ Acts F"*
⟨*proof*⟩

**lemma** *insert_Id_Acts [iff]: "insert Id (Acts F) = Acts F"*
⟨*proof*⟩

**lemma** *Acts_nonempty [simp]: "Acts F ≠ {}"*
⟨*proof*⟩

**lemma** *Id_in_AllowedActs [iff]: "Id ∈ AllowedActs F"*
⟨*proof*⟩

**lemma** *insert_Id_AllowedActs [iff]: "insert Id (AllowedActs F) = AllowedActs F"*
⟨*proof*⟩

### 1.0.2 Inspectors for type "program"

**lemma** *Init_eq [simp]: "Init (mk_program (init,acts,allowed)) = init"*
⟨*proof*⟩

**lemma** *Acts_eq [simp]: "Acts (mk_program (init,acts,allowed)) = insert Id acts"*
⟨*proof*⟩

**lemma** *AllowedActs_eq [simp]:*
*"AllowedActs (mk_program (init,acts,allowed)) = insert Id allowed"*
⟨*proof*⟩

### 1.0.3 Equality for UNITY programs

**lemma** *surjective_mk_program [simp]:*
*"mk_program (Init F, Acts F, AllowedActs F) = F"*
⟨*proof*⟩

**lemma** *program_equalityI:*
*"[| Init F = Init G; Acts F = Acts G; AllowedActs F = AllowedActs G |]*

*==> F = G"*
⟨*proof*⟩

**lemma** *program_equalityE:*
*"[| F = G;*
*[| Init F = Init G; Acts F = Acts G; AllowedActs F = AllowedActs G |]*
*==> P |] ==> P"*
⟨*proof*⟩

**lemma** *program_equality_iff:*
*"(F=G) =*
*(Init F = Init G & Acts F = Acts G &AllowedActs F = AllowedActs G)"*
⟨*proof*⟩

### 1.0.4 co

**lemma** *constrainsI:*
```
"(!!act s s'. [| act: Acts F;  (s,s') ∈ act;  s ∈ A |] ==> s': A')
 ==> F ∈ A co A'"
```
⟨*proof*⟩

**lemma** *constrainsD:*
```
"[| F ∈ A co A'; act: Acts F;  (s,s'): act;  s ∈ A |] ==> s': A'"
```
⟨*proof*⟩

**lemma** *constrains_empty [iff]: "F ∈ {} co B"*
⟨*proof*⟩

**lemma** *constrains_empty2 [iff]: "(F ∈ A co {}) = (A={})"*
⟨*proof*⟩

**lemma** *constrains_UNIV [iff]: "(F ∈ UNIV co B) = (B = UNIV)"*
⟨*proof*⟩

**lemma** *constrains_UNIV2 [iff]: "F ∈ A co UNIV"*
⟨*proof*⟩

monotonic in 2nd argument

**lemma** *constrains_weaken_R:*
```
"[| F ∈ A co A'; A'<=B' |] ==> F ∈ A co B'"
```
⟨*proof*⟩

anti-monotonic in 1st argument

**lemma** *constrains_weaken_L:*
```
"[| F ∈ A co A'; B ⊆ A |] ==> F ∈ B co A'"
```
⟨*proof*⟩

**lemma** *constrains_weaken:*
```
"[| F ∈ A co A'; B ⊆ A; A'<=B' |] ==> F ∈ B co B'"
```
⟨*proof*⟩

### 1.0.5 Union

**lemma** *constrains_Un:*
```
"[| F ∈ A co A'; F ∈ B co B' |] ==> F ∈ (A ∪ B) co (A' ∪ B')"
```
⟨*proof*⟩

**lemma** *constrains_UN:*
```
"(!!i. i ∈ I ==> F ∈ (A i) co (A' i))
 ==> F ∈ (⋃i ∈ I. A i) co (⋃i ∈ I. A' i)"
```
⟨*proof*⟩

**lemma** *constrains_Un_distrib: "(A ∪ B) co C = (A co C) ∩ (B co C)"*
⟨*proof*⟩

**lemma** *constrains_UN_distrib: "(⋃i ∈ I. A i) co B = (⋂i ∈ I. A i co B)"*
⟨*proof*⟩

**lemma** *constrains_Int_distrib: "C co (A ∩ B) = (C co A) ∩ (C co B)"*
⟨*proof*⟩

**lemma** *constrains_INT_distrib: "A co (⋂i ∈ I. B i) = (⋂i ∈ I. A co B i)"*
⟨*proof*⟩

### 1.0.6 Intersection

**lemma** *constrains_Int:*
*"[| F ∈ A co A'; F ∈ B co B' |] ==> F ∈ (A ∩ B) co (A' ∩ B')"*
⟨*proof*⟩

**lemma** *constrains_INT:*
*"(!!i. i ∈ I ==> F ∈ (A i) co (A' i))*
*==> F ∈ (⋂i ∈ I. A i) co (⋂i ∈ I. A' i)"*
⟨*proof*⟩

**lemma** *constrains_imp_subset: "F ∈ A co A' ==> A ⊆ A'"*
⟨*proof*⟩

The reasoning is by subsets since "co" refers to single actions only. So this rule isn't that useful.

**lemma** *constrains_trans:*
*"[| F ∈ A co B; F ∈ B co C |] ==> F ∈ A co C"*
⟨*proof*⟩

**lemma** *constrains_cancel:*
*"[| F ∈ A co (A' ∪ B); F ∈ B co B' |] ==> F ∈ A co (A' ∪ B')"*
⟨*proof*⟩

### 1.0.7 unless

**lemma** *unlessI: "F ∈ (A-B) co (A ∪ B) ==> F ∈ A unless B"*
⟨*proof*⟩

**lemma** *unlessD: "F ∈ A unless B ==> F ∈ (A-B) co (A ∪ B)"*
⟨*proof*⟩

### 1.0.8 stable

**lemma** *stableI: "F ∈ A co A ==> F ∈ stable A"*
⟨*proof*⟩

**lemma** *stableD: "F ∈ stable A ==> F ∈ A co A"*
⟨*proof*⟩

**lemma** *stable_UNIV [simp]: "stable UNIV = UNIV"*
⟨*proof*⟩

### 1.0.9 Union

**lemma** *stable_Un:*
*"[| F ∈ stable A; F ∈ stable A' |] ==> F ∈ stable (A ∪ A')"*

⟨*proof*⟩

**lemma** *stable_UN:*
```
"(!!i. i ∈ I ==> F ∈ stable (A i)) ==> F ∈ stable (⋃i ∈ I. A i)"
```
⟨*proof*⟩

**lemma** *stable_Union:*
```
"(!!A. A ∈ X ==> F ∈ stable A) ==> F ∈ stable (⋃X)"
```
⟨*proof*⟩

### 1.0.10 Intersection

**lemma** *stable_Int:*
```
"[| F ∈ stable A;  F ∈ stable A' |] ==> F ∈ stable (A ∩ A')"
```
⟨*proof*⟩

**lemma** *stable_INT:*
```
"(!!i. i ∈ I ==> F ∈ stable (A i)) ==> F ∈ stable (⋂i ∈ I. A i)"
```
⟨*proof*⟩

**lemma** *stable_Inter:*
```
"(!!A. A ∈ X ==> F ∈ stable A) ==> F ∈ stable (⋂X)"
```
⟨*proof*⟩

**lemma** *stable_constrains_Un:*
```
"[| F ∈ stable C; F ∈ A co (C ∪ A') |] ==> F ∈ (C ∪ A) co (C ∪ A')"
```
⟨*proof*⟩

**lemma** *stable_constrains_Int:*
```
"[| F ∈ stable C; F ∈  (C ∩ A) co A' |] ==> F ∈ (C ∩ A) co (C ∩ A')"
```
⟨*proof*⟩

**lemmas** *stable_constrains_stable = stable_constrains_Int[THEN stableI, standard]*

### 1.0.11 invariant

**lemma** *invariantI: "[| Init F ⊆ A;  F ∈ stable A |] ==> F ∈ invariant A"*
⟨*proof*⟩

Could also say *invariant A ∩ invariant B ⊆ invariant (A ∩ B)*

**lemma** *invariant_Int:*
```
"[| F ∈ invariant A;  F ∈ invariant B |] ==> F ∈ invariant (A ∩ B)"
```
⟨*proof*⟩

### 1.0.12 increasing

**lemma** *increasingD:*
```
"F ∈ increasing f ==> F ∈ stable {s. z ⊆ f s}"
```
⟨*proof*⟩

**lemma** *increasing_constant [iff]: "F ∈ increasing (%s. c)"*
⟨*proof*⟩

**lemma** *mono_increasing_o:*
```
"mono g ==> increasing f ⊆ increasing (g o f)"
```

⟨*proof*⟩

**lemma** *strict_increasingD:*
"!!z::nat. F ∈ increasing f ==> F ∈ stable {s. z < f s}"
⟨*proof*⟩

**lemma** *elimination:*
"[| ∀m ∈ M. F ∈ {s. s x = m} co (B m) |]
==> F ∈ {s. s x ∈ M} co (⋃m ∈ M. B m)"
⟨*proof*⟩

As above, but for the trivial case of a one-variable state, in which the state is identified with its one variable.

**lemma** *elimination_sing:*
"(∀m ∈ M. F ∈ {m} co (B m)) ==> F ∈ M co (⋃m ∈ M. B m)"
⟨*proof*⟩

### 1.0.13 Theoretical Results from Section 6

**lemma** *constrains_strongest_rhs:*
"F ∈ A co (strongest_rhs F A )"
⟨*proof*⟩

**lemma** *strongest_rhs_is_strongest:*
"F ∈ A co B ==> strongest_rhs F A ⊆ B"
⟨*proof*⟩

### 1.0.14 Ad-hoc set-theory rules

**lemma** *Un_Diff_Diff [simp]: "A ∪ B - (A - B) = B"*
⟨*proof*⟩

**lemma** *Int_Union_Union: "Union(B) ∩ A = Union((%C. C ∩ A)‘B)"*
⟨*proof*⟩

Needed for WF reasoning in WFair.thy

**lemma** *Image_less_than [simp]: "less_than `` {k} = greaterThan k"*
⟨*proof*⟩

**lemma** *Image_inverse_less_than [simp]: "less_than^-1 `` {k} = lessThan k"*
⟨*proof*⟩

## 1.1 Partial versus Total Transitions

**constdefs**
```
  totalize_act :: "('a * 'a)set => ('a * 'a)set"
    "totalize_act act == act ∪ diag (-(Domain act))"

  totalize :: "'a program => 'a program"
    "totalize F == mk_program (Init F,
```

```
                              totalize_act ‘ Acts F,
                              AllowedActs F)"

  mk_total_program :: "('a set * ('a * 'a)set set * ('a * 'a)set set)
                   => 'a program"
    "mk_total_program args == totalize (mk_program args)"

  all_total :: "'a program => bool"
    "all_total F == ∀act ∈ Acts F. Domain act = UNIV"
```

**lemma** *insert_Id_image_Acts: "f Id = Id ==> insert Id (f‘Acts F) = f ‘ Acts F"*
⟨*proof*⟩

### 1.1.1 Basic properties

**lemma** *totalize_act_Id [simp]: "totalize_act Id = Id"*
⟨*proof*⟩

**lemma** *Domain_totalize_act [simp]: "Domain (totalize_act act) = UNIV"*
⟨*proof*⟩

**lemma** *Init_totalize [simp]: "Init (totalize F) = Init F"*
⟨*proof*⟩

**lemma** *Acts_totalize [simp]: "Acts (totalize F) = (totalize_act ‘ Acts F)"*
⟨*proof*⟩

**lemma** *AllowedActs_totalize [simp]: "AllowedActs (totalize F) = AllowedActs F"*
⟨*proof*⟩

**lemma** *totalize_constrains_iff [simp]: "(totalize F ∈ A co B) = (F ∈ A co B)"*
⟨*proof*⟩

**lemma** *totalize_stable_iff [simp]: "(totalize F ∈ stable A) = (F ∈ stable A)"*
⟨*proof*⟩

**lemma** *totalize_invariant_iff [simp]:*
*"(totalize F ∈ invariant A) = (F ∈ invariant A)"*
⟨*proof*⟩

**lemma** *all_total_totalize: "all_total (totalize F)"*
⟨*proof*⟩

**lemma** *Domain_iff_totalize_act: "(Domain act = UNIV) = (totalize_act act = act)"*
⟨*proof*⟩

**lemma** *all_total_imp_totalize: "all_total F ==> (totalize F = F)"*
⟨*proof*⟩

**lemma** *all_total_iff_totalize: "all_total F = (totalize F = F)"*
⟨*proof*⟩

**lemma** *mk_total_program_constrains_iff [simp]:*
*"(mk_total_program args ∈ A co B) = (mk_program args ∈ A co B)"*
⟨*proof*⟩

## 1.2 Rules for Lazy Definition Expansion

They avoid expanding the full program, which is a large expression

**lemma** *def_prg_Init:*
*"F == mk_total_program (init,acts,allowed) ==> Init F = init"*
⟨*proof*⟩

**lemma** *def_prg_Acts:*
*"F == mk_total_program (init,acts,allowed)*
*==> Acts F = insert Id (totalize_act ‘ acts)"*
⟨*proof*⟩

**lemma** *def_prg_AllowedActs:*
*"F == mk_total_program (init,acts,allowed)*
*==> AllowedActs F = insert Id allowed"*
⟨*proof*⟩

An action is expanded if a pair of states is being tested against it

**lemma** *def_act_simp:*
*"act == {(s,s’). P s s’} ==> ((s,s’) ∈ act) = P s s’"*
⟨*proof*⟩

A set is expanded only if an element is being tested against it

**lemma** *def_set_simp: "A == B ==> (x ∈ A) = (x ∈ B)"*
⟨*proof*⟩

### 1.2.1 Inspectors for type ”program”

**lemma** *Init_total_eq [simp]:*
*"Init (mk_total_program (init,acts,allowed)) = init"*
⟨*proof*⟩

**lemma** *Acts_total_eq [simp]:*
*"Acts(mk_total_program(init,acts,allowed)) = insert Id (totalize_act‘acts)"*
⟨*proof*⟩

**lemma** *AllowedActs_total_eq [simp]:*
*"AllowedActs (mk_total_program (init,acts,allowed)) = insert Id allowed"*
⟨*proof*⟩

**end**

# 2 Fixed Point of a Program

**theory** *FP* **imports** *UNITY* **begin**

**constdefs**

```
  FP_Orig :: "'a program => 'a set"
    "FP_Orig F == Union{A. ALL B. F : stable (A Int B)}"

  FP :: "'a program => 'a set"
    "FP F == {s. F : stable {s}}"
```

**lemma** *stable_FP_Orig_Int: "F : stable (FP_Orig F Int B)"*
⟨*proof*⟩

**lemma** *FP_Orig_weakest:*
*"(!!B. F : stable (A Int B)) ==> A <= FP_Orig F"*
⟨*proof*⟩

**lemma** *stable_FP_Int: "F : stable (FP F Int B)"*
⟨*proof*⟩

**lemma** *FP_equivalence: "FP F = FP_Orig F"*
⟨*proof*⟩

**lemma** *FP_weakest:*
*"(!!B. F : stable (A Int B)) ==> A <= FP F"*
⟨*proof*⟩

**lemma** *Compl_FP:*
*"-(FP F) = (UN act: Acts F. -{s. act``{s} <= {s}})"*
⟨*proof*⟩

**lemma** *Diff_FP: "A - (FP F) = (UN act: Acts F. A - {s. act``{s} <= {s}})"*
⟨*proof*⟩

**lemma** *totalize_FP [simp]: "FP (totalize F) = FP F"*
⟨*proof*⟩

**end**

# 3 Progress

**theory** *WFair* **imports** *UNITY* **begin**

The original version of this theory was based on weak fairness. (Thus, the entire UNITY development embodied this assumption, until February 2003.) Weak fairness states that if a command is enabled continuously, then it is eventually executed. Ernie Cohen suggested that I instead adopt unconditional fairness: every command is executed infinitely often.

In fact, Misra's paper on "Progress" seems to be ambiguous about the correct interpretation, and says that the two forms of fairness are equivalent. They differ only on their treatment of partial transitions, which under unconditional fairness behave magically. That is because if there are partial transitions then there may be no fair executions, making all leads-to properties hold vacuously.

Unconditional fairness has some great advantages. By distinguishing partial transitions from total ones that are the identity on part of their domain, it is more expressive. Also, by simplifying the definition of the transient property, it simplifies many proofs. A drawback is that some laws only hold under the assumption that all transitions are total. The best-known of these is the impossibility law for leads-to.

**constdefs**

— This definition specifies conditional fairness. The rest of the theory is generic to all forms of fairness. To get weak fairness, conjoin the inclusion below with `A ⊆ Domain act`, which specifies that the action is enabled over all of `A`.

```
  transient :: "'a set => 'a program set"
    "transient A == {F. ∃act∈Acts F. act``A ⊆ -A}"

  ensures :: "['a set, 'a set] => 'a program set"       (infixl "ensures" 60)
    "A ensures B == (A-B co A ∪ B) ∩ transient (A-B)"
```

**inductive_set**

```
  leads :: "'a program => ('a set * 'a set) set"
```

— LEADS-TO constant for the inductive definition

```
  for F :: "'a program"
  where

    Basis:  "F ∈ A ensures B ==> (A,B) ∈ leads F"

  | Trans:  "[| (A,B) ∈ leads F;  (B,C) ∈ leads F |] ==> (A,C) ∈ leads F"

  | Union:  "∀A ∈ S. (A,B) ∈ leads F ==> (Union S, B) ∈ leads F"
```

**constdefs**

```
  leadsTo :: "['a set, 'a set] => 'a program set"    (infixl "leadsTo" 60)
```

— visible version of the LEADS-TO relation

```
    "A leadsTo B == {F. (A,B) ∈ leads F}"

  wlt :: "['a program, 'a set] => 'a set"
```

— predicate transformer: the largest set that leads to `B`

```
    "wlt F B == Union {A. F ∈ A leadsTo B}"
```

**syntax** *(xsymbols)*

```
  "op leadsTo" :: "['a set, 'a set] => 'a program set" (infixl "⟼" 60)
```

## 3.1 transient

**lemma** *stable_transient:*

```
    "[| F ∈ stable A; F ∈ transient A |] ==> ∃act∈Acts F. A ⊆ - (Domain
act)"
```

⟨*proof*⟩

**lemma** *stable_transient_empty:*

```
    "[| F ∈ stable A; F ∈ transient A; all_total F |] ==> A = {}"
```

⟨*proof*⟩

**lemma** *transient_strengthen:*
```
"[| F ∈ transient A; B ⊆ A |] ==> F ∈ transient B"
```
⟨*proof*⟩

**lemma** *transientI:*
```
"[| act: Acts F;  act``A ⊆ -A |] ==> F ∈ transient A"
```
⟨*proof*⟩

**lemma** *transientE:*
```
"[| F ∈ transient A;
    !!act. [| act: Acts F;  act``A ⊆ -A |] ==> P |]
 ==> P"
```
⟨*proof*⟩

**lemma** *transient_empty [simp]: "transient {} = UNIV"*
⟨*proof*⟩

This equation recovers the notion of weak fairness. A totalized program satisfies a transient assertion just if the original program contains a suitable action that is also enabled.

**lemma** *totalize_transient_iff:*
```
"(totalize F ∈ transient A) = (∃act∈Acts F. A ⊆ Domain act & act``A ⊆
-A)"
```
⟨*proof*⟩

**lemma** *totalize_transientI:*
```
"[| act: Acts F;  A ⊆ Domain act;  act``A ⊆ -A |]
 ==> totalize F ∈ transient A"
```
⟨*proof*⟩

## 3.2 ensures

**lemma** *ensuresI:*
```
"[| F ∈ (A-B) co (A ∪ B); F ∈ transient (A-B) |] ==> F ∈ A ensures B"
```
⟨*proof*⟩

**lemma** *ensuresD:*
```
"F ∈ A ensures B ==> F ∈ (A-B) co (A ∪ B) & F ∈ transient (A-B)"
```
⟨*proof*⟩

**lemma** *ensures_weaken_R:*
```
"[| F ∈ A ensures A'; A'<=B' |] ==> F ∈ A ensures B'"
```
⟨*proof*⟩

The L-version (precondition strengthening) fails, but we have this

**lemma** *stable_ensures_Int:*
```
"[| F ∈ stable C;  F ∈ A ensures B |]
==> F ∈ (C ∩ A) ensures (C ∩ B)"
```
⟨*proof*⟩

**lemma** *stable_transient_ensures:*

"[| F ∈ stable A;  F ∈ transient C;  A ⊆ B ∪ C |] ==> F ∈ A ensures B"
⟨*proof*⟩

**lemma** *ensures_eq*: "(A ensures B) = (A unless B) ∩ transient (A-B)"
⟨*proof*⟩

## 3.3 leadsTo

**lemma** *leadsTo_Basis [intro]*: "F ∈ A ensures B ==> F ∈ A leadsTo B"
⟨*proof*⟩

**lemma** *leadsTo_Trans*:
"[| F ∈ A leadsTo B;  F ∈ B leadsTo C |] ==> F ∈ A leadsTo C"
⟨*proof*⟩

**lemma** *leadsTo_Basis'*:
"[| F ∈ A co A ∪ B; F ∈ transient A |] ==> F ∈ A leadsTo B"
⟨*proof*⟩

**lemma** *transient_imp_leadsTo*: "F ∈ transient A ==> F ∈ A leadsTo (-A)"
⟨*proof*⟩

Useful with cancellation, disjunction

**lemma** *leadsTo_Un_duplicate*: "F ∈ A leadsTo (A' ∪ A') ==> F ∈ A leadsTo A'"
⟨*proof*⟩

**lemma** *leadsTo_Un_duplicate2*:
"F ∈ A leadsTo (A' ∪ C ∪ C) ==> F ∈ A leadsTo (A' ∪ C)"
⟨*proof*⟩

The Union introduction rule as we should have liked to state it

**lemma** *leadsTo_Union*:
"(!!A. A ∈ S ==> F ∈ A leadsTo B) ==> F ∈ (Union S) leadsTo B"
⟨*proof*⟩

**lemma** *leadsTo_Union_Int*:
"(!!A. A ∈ S ==> F ∈ (A ∩ C) leadsTo B) ==> F ∈ (Union S ∩ C) leadsTo B"
⟨*proof*⟩

**lemma** *leadsTo_UN*:
"(!!i. i ∈ I ==> F ∈ (A i) leadsTo B) ==> F ∈ (⋃i ∈ I. A i) leadsTo B"
⟨*proof*⟩

Binary union introduction rule

**lemma** *leadsTo_Un*:
"[| F ∈ A leadsTo C; F ∈ B leadsTo C |] ==> F ∈ (A ∪ B) leadsTo C"
⟨*proof*⟩

**lemma** *single_leadsTo_I*:
"(!!x. x ∈ A ==> F ∈ {x} leadsTo B) ==> F ∈ A leadsTo B"

⟨*proof*⟩

The INDUCTION rule as we should have liked to state it

**lemma** *leadsTo_induct:*
```
  "[| F ∈ za leadsTo zb;
      !!A B. F ∈ A ensures B ==> P A B;
      !!A B C. [| F ∈ A leadsTo B; P A B; F ∈ B leadsTo C; P B C |]
               ==> P A C;
      !!B S. ∀A ∈ S. F ∈ A leadsTo B & P A B ==> P (Union S) B
   |] ==> P za zb"
```
⟨*proof*⟩

**lemma** *subset_imp_ensures: "A ⊆ B ==> F ∈ A ensures B"*
⟨*proof*⟩

**lemmas** *subset_imp_leadsTo = subset_imp_ensures [THEN leadsTo_Basis, standard]*

**lemmas** *leadsTo_refl = subset_refl [THEN subset_imp_leadsTo, standard]*

**lemmas** *empty_leadsTo = empty_subsetI [THEN subset_imp_leadsTo, standard, simp]*

**lemmas** *leadsTo_UNIV = subset_UNIV [THEN subset_imp_leadsTo, standard, simp]*

Lemma is the weak version: can't see how to do it in one step

**lemma** *leadsTo_induct_pre_lemma:*
```
  "[| F ∈ za leadsTo zb;
      P zb;
      !!A B. [| F ∈ A ensures B;  P B |] ==> P A;
      !!S. ∀A ∈ S. P A ==> P (Union S)
   |] ==> P za"
```
⟨*proof*⟩

**lemma** *leadsTo_induct_pre:*
```
  "[| F ∈ za leadsTo zb;
      P zb;
      !!A B. [| F ∈ A ensures B;  F ∈ B leadsTo zb;  P B |] ==> P A;
      !!S. ∀A ∈ S. F ∈ A leadsTo zb & P A ==> P (Union S)
   |] ==> P za"
```
⟨*proof*⟩

**lemma** *leadsTo_weaken_R: "[| F ∈ A leadsTo A'; A'<=B' |] ==> F ∈ A leadsTo B'"*
⟨*proof*⟩

**lemma** *leadsTo_weaken_L [rule_format]:*
```
     "[| F ∈ A leadsTo A'; B ⊆ A |] ==> F ∈ B leadsTo A'"
```
⟨*proof*⟩

Distributes over binary unions

**lemma** *leadsTo_Un_distrib:*
```
     "F ∈ (A ∪ B) leadsTo C  =  (F ∈ A leadsTo C & F ∈ B leadsTo C)"
```

⟨*proof*⟩

**lemma** *leadsTo_UN_distrib:*
```
"F ∈ (⋃i ∈ I. A i) leadsTo B  =  (∀i ∈ I. F ∈ (A i) leadsTo B)"
```
⟨*proof*⟩

**lemma** *leadsTo_Union_distrib:*
```
"F ∈ (Union S) leadsTo B  =  (∀A ∈ S. F ∈ A leadsTo B)"
```
⟨*proof*⟩

**lemma** *leadsTo_weaken:*
```
"[| F ∈ A leadsTo A'; B ⊆ A; A'<=B' |] ==> F ∈ B leadsTo B'"
```
⟨*proof*⟩

Set difference: maybe combine with `leadsTo_weaken_L`??

**lemma** *leadsTo_Diff:*
```
"[| F ∈ (A-B) leadsTo C; F ∈ B leadsTo C |]   ==> F ∈ A leadsTo C"
```
⟨*proof*⟩

**lemma** *leadsTo_UN_UN:*
```
"(!! i. i ∈ I ==> F ∈ (A i) leadsTo (A' i))
 ==> F ∈ (⋃i ∈ I. A i) leadsTo (⋃i ∈ I. A' i)"
```
⟨*proof*⟩

Binary union version

**lemma** *leadsTo_Un_Un:*
```
"[| F ∈ A leadsTo A'; F ∈ B leadsTo B' |]
 ==> F ∈ (A ∪ B) leadsTo (A' ∪ B')"
```
⟨*proof*⟩

**lemma** *leadsTo_cancel2:*
```
"[| F ∈ A leadsTo (A' ∪ B); F ∈ B leadsTo B' |]
 ==> F ∈ A leadsTo (A' ∪ B')"
```
⟨*proof*⟩

**lemma** *leadsTo_cancel_Diff2:*
```
"[| F ∈ A leadsTo (A' ∪ B); F ∈ (B-A') leadsTo B' |]
 ==> F ∈ A leadsTo (A' ∪ B')"
```
⟨*proof*⟩

**lemma** *leadsTo_cancel1:*
```
"[| F ∈ A leadsTo (B ∪ A'); F ∈ B leadsTo B' |]
==> F ∈ A leadsTo (B' ∪ A')"
```
⟨*proof*⟩

**lemma** *leadsTo_cancel_Diff1:*
```
"[| F ∈ A leadsTo (B ∪ A'); F ∈ (B-A') leadsTo B' |]
==> F ∈ A leadsTo (B' ∪ A')"
```
⟨*proof*⟩

The impossibility law

**lemma** *leadsTo_empty: "[|F ∈ A leadsTo {}; all_total F|] ==> A={}"*
⟨*proof*⟩

## 3.4 PSP: Progress-Safety-Progress

Special case of PSP: Misra's "stable conjunction"

**lemma** *psp_stable:*
```
"[| F ∈ A leadsTo A'; F ∈ stable B |]
 ==> F ∈ (A ∩ B) leadsTo (A' ∩ B)"
```
⟨*proof*⟩

**lemma** *psp_stable2:*
```
"[| F ∈ A leadsTo A'; F ∈ stable B |] ==> F ∈ (B ∩ A) leadsTo (B ∩ A')"
```
⟨*proof*⟩

**lemma** *psp_ensures:*
```
"[| F ∈ A ensures A'; F ∈ B co B' |]
 ==> F ∈ (A ∩ B') ensures ((A' ∩ B) ∪ (B' - B))"
```
⟨*proof*⟩

**lemma** *psp:*
```
"[| F ∈ A leadsTo A'; F ∈ B co B' |]
 ==> F ∈ (A ∩ B') leadsTo ((A' ∩ B) ∪ (B' - B))"
```
⟨*proof*⟩

**lemma** *psp2:*
```
"[| F ∈ A leadsTo A'; F ∈ B co B' |]
==> F ∈ (B' ∩ A) leadsTo ((B ∩ A') ∪ (B' - B))"
```
⟨*proof*⟩

**lemma** *psp_unless:*
```
"[| F ∈ A leadsTo A';  F ∈ B unless B' |]
 ==> F ∈ (A ∩ B) leadsTo ((A' ∩ B) ∪ B')"
```

⟨*proof*⟩

## 3.5 Proving the induction rules

**lemma** *leadsTo_wf_induct_lemma:*
```
"[| wf r;
    ∀m. F ∈ (A ∩ f-‘{m}) leadsTo
              ((A ∩ f-‘(r^-1 ‘‘ {m})) ∪ B) |]
 ==> F ∈ (A ∩ f-‘{m}) leadsTo B"
```
⟨*proof*⟩

**lemma** *leadsTo_wf_induct:*
```
"[| wf r;
    ∀m. F ∈ (A ∩ f-‘{m}) leadsTo
              ((A ∩ f-‘(r^-1 ‘‘ {m})) ∪ B) |]
 ==> F ∈ A leadsTo B"
```

⟨*proof*⟩

**lemma** *bounded_induct:*
```
"[| wf r;
    ∀m ∈ I. F ∈ (A ∩ f-‘{m}) leadsTo
                ((A ∩ f-‘(r^-1 ‘‘ {m})) ∪ B) |]
 ==> F ∈ A leadsTo ((A - (f-‘I)) ∪ B)"
```
⟨*proof*⟩

**lemma** *lessThan_induct:*
```
"[| !!m::nat. F ∈ (A ∩ f-‘{m}) leadsTo ((A ∩ f-‘{..<m}) ∪ B) |]
 ==> F ∈ A leadsTo B"
```
⟨*proof*⟩

**lemma** *lessThan_bounded_induct:*
```
"!!l::nat. [| ∀m ∈ greaterThan l.
       F ∈ (A ∩ f-‘{m}) leadsTo ((A ∩ f-‘(lessThan m)) ∪ B) |]
 ==> F ∈ A leadsTo ((A ∩ (f-‘(atMost l))) ∪ B)"
```
⟨*proof*⟩

**lemma** *greaterThan_bounded_induct:*
```
"(!!l::nat. ∀m ∈ lessThan l.
            F ∈ (A ∩ f-‘{m}) leadsTo ((A ∩ f-‘(greaterThan m)) ∪ B))
 ==> F ∈ A leadsTo ((A ∩ (f-‘(atLeast l))) ∪ B)"
```
⟨*proof*⟩

## 3.6 wlt

Misra's property W3

**lemma** *wlt_leadsTo:* `"F ∈ (wlt F B) leadsTo B"`
⟨*proof*⟩

**lemma** *leadsTo_subset:* `"F ∈ A leadsTo B ==> A ⊆ wlt F B"`
⟨*proof*⟩

Misra's property W2

**lemma** *leadsTo_eq_subset_wlt:* `"F ∈ A leadsTo B = (A ⊆ wlt F B)"`
⟨*proof*⟩

Misra's property W4

**lemma** *wlt_increasing:* `"B ⊆ wlt F B"`
⟨*proof*⟩

Used in the Trans case below

**lemma** *lemma1:*
```
"[| B ⊆ A2;
    F ∈ (A1 - B) co (A1 ∪ B);
    F ∈ (A2 - C) co (A2 ∪ C) |]
 ==> F ∈ (A1 ∪ A2 - C) co (A1 ∪ A2 ∪ C)"
```

⟨*proof*⟩

Lemma (1,2,3) of Misra's draft book, Chapter 4, "Progress"

**lemma** *leadsTo_123:*
```
"F ∈ A leadsTo A'
 ==> ∃B. A ⊆ B & F ∈ B leadsTo A' & F ∈ (B-A') co (B ∪ A')"
```
⟨*proof*⟩

Misra's property W5

**lemma** *wlt_constrains_wlt: "F ∈ (wlt F B - B) co (wlt F B)"*
⟨*proof*⟩

## 3.7 Completion: Binary and General Finite versions

**lemma** *completion_lemma :*
```
"[| W = wlt F (B' ∪ C);
  F ∈ A leadsTo (A' ∪ C);  F ∈ A' co (A' ∪ C);
  F ∈ B leadsTo (B' ∪ C);  F ∈ B' co (B' ∪ C) |]
==> F ∈ (A ∩ B) leadsTo ((A' ∩ B') ∪ C)"
```
⟨*proof*⟩

**lemmas** *completion = completion_lemma [OF refl]*

**lemma** *finite_completion_lemma:*
```
"finite I ==> (∀i ∈ I. F ∈ (A i) leadsTo (A' i ∪ C)) -->
              (∀i ∈ I. F ∈ (A' i) co (A' i ∪ C)) -->
              F ∈ (⋂i ∈ I. A i) leadsTo ((⋂i ∈ I. A' i) ∪ C)"
```
⟨*proof*⟩

**lemma** *finite_completion:*
```
"[| finite I;
    !!i. i ∈ I ==> F ∈ (A i) leadsTo (A' i ∪ C);
    !!i. i ∈ I ==> F ∈ (A' i) co (A' i ∪ C) |]
 ==> F ∈ (⋂i ∈ I. A i) leadsTo ((⋂i ∈ I. A' i) ∪ C)"
```
⟨*proof*⟩

**lemma** *stable_completion:*
```
"[| F ∈ A leadsTo A';  F ∈ stable A';
    F ∈ B leadsTo B';  F ∈ stable B' |]
==> F ∈ (A ∩ B) leadsTo (A' ∩ B')"
```
⟨*proof*⟩

**lemma** *finite_stable_completion:*
```
"[| finite I;
    !!i. i ∈ I ==> F ∈ (A i) leadsTo (A' i);
    !!i. i ∈ I ==> F ∈ stable (A' i) |]
 ==> F ∈ (⋂i ∈ I. A i) leadsTo (⋂i ∈ I. A' i)"
```
⟨*proof*⟩

**end**

# 4 Weak Safety

**theory** *Constrains* **imports** *UNITY* **begin**

**inductive_set**
  *traces :: "['a set, ('a * 'a)set set] => ('a * 'a list) set"*
  **for** *init :: "'a set"* **and** *acts :: "('a * 'a)set set"*
  **where**

    *Init:  "s ∈ init ==> (s,[]) ∈ traces init acts"*

  *| Acts:  "[| act: acts;  (s,evs) ∈ traces init acts;  (s,s'): act |]*
  *==> (s', s#evs) ∈ traces init acts"*

**inductive_set**
  *reachable :: "'a program => 'a set"*
  **for** *F :: "'a program"*
  **where**
    *Init:  "s ∈ Init F ==> s ∈ reachable F"*

  *| Acts:  "[| act: Acts F;  s ∈ reachable F;  (s,s'): act |]*
  *==> s' ∈ reachable F"*

**constdefs**
  *Constrains :: "['a set, 'a set] => 'a program set"*  (**infixl** *"Co" 60*)
    *"A Co B == {F. F ∈ (reachable F ∩ A)  co  B}"*

  *Unless  :: "['a set, 'a set] => 'a program set"*  (**infixl** *"Unless" 60*)
    *"A Unless B == (A-B) Co (A ∪ B)"*

  *Stable  :: "'a set => 'a program set"*
    *"Stable A == A Co A"*

  *Always :: "'a set => 'a program set"*
    *"Always A == {F. Init F ⊆ A} ∩ Stable A"*

  *Increasing :: "['a => 'b::{order}] => 'a program set"*
    *"Increasing f == ⋂z. Stable {s. z ≤ f s}"*

## 4.1 traces and reachable

**lemma** *reachable_equiv_traces:*
  *"reachable F = {s. ∃evs. (s,evs) ∈ traces (Init F) (Acts F)}"*
⟨*proof*⟩

**lemma** *Init_subset_reachable: "Init F ⊆ reachable F"*
⟨*proof*⟩

**lemma** *stable_reachable [intro!,simp]:*

*"Acts G ⊆ Acts F ==> G ∈ stable (reachable F)"*
⟨*proof*⟩

**lemma** *invariant_reachable: "F ∈ invariant (reachable F)"*
⟨*proof*⟩

**lemma** *invariant_includes_reachable: "F ∈ invariant A ==> reachable F ⊆ A"*
⟨*proof*⟩

## 4.2 Co

**lemmas** *constrains_reachable_Int =*
*subset_refl [THEN stable_reachable [unfolded stable_def],*
*THEN constrains_Int, standard]*

**lemma** *Constrains_eq_constrains:*
*"A Co B = {F. F ∈ (reachable F ∩ A) co (reachable F ∩ B)}"*
⟨*proof*⟩

**lemma** *constrains_imp_Constrains: "F ∈ A co A' ==> F ∈ A Co A'"*
⟨*proof*⟩

**lemma** *stable_imp_Stable: "F ∈ stable A ==> F ∈ Stable A"*
⟨*proof*⟩

**lemma** *ConstrainsI:*
*"(!!act s s'. [| act: Acts F; (s,s') ∈ act; s ∈ A |] ==> s': A')*
*==> F ∈ A Co A'"*
⟨*proof*⟩

**lemma** *Constrains_empty [iff]: "F ∈ {} Co B"*
⟨*proof*⟩

**lemma** *Constrains_UNIV [iff]: "F ∈ A Co UNIV"*
⟨*proof*⟩

**lemma** *Constrains_weaken_R:*
*"[| F ∈ A Co A'; A'<=B' |] ==> F ∈ A Co B'"*
⟨*proof*⟩

**lemma** *Constrains_weaken_L:*
*"[| F ∈ A Co A'; B ⊆ A |] ==> F ∈ B Co A'"*
⟨*proof*⟩

**lemma** *Constrains_weaken:*
*"[| F ∈ A Co A'; B ⊆ A; A'<=B' |] ==> F ∈ B Co B'"*
⟨*proof*⟩

**lemma** *Constrains_Un:*

```
    "[| F ∈ A Co A'; F ∈ B Co B' |] ==> F ∈ (A ∪ B) Co (A' ∪ B')"
```
⟨*proof*⟩

**lemma** *Constrains_UN:*
```
  assumes Co: "!!i. i ∈ I ==> F ∈ (A i) Co (A' i)"
  shows "F ∈ (⋃i ∈ I. A i) Co (⋃i ∈ I. A' i)"
```
⟨*proof*⟩

**lemma** *Constrains_Int:*
```
    "[| F ∈ A Co A'; F ∈ B Co B' |] ==> F ∈ (A ∩ B) Co (A' ∩ B')"
```
⟨*proof*⟩

**lemma** *Constrains_INT:*
```
  assumes Co: "!!i. i ∈ I ==> F ∈ (A i) Co (A' i)"
  shows "F ∈ (⋂i ∈ I. A i) Co (⋂i ∈ I. A' i)"
```
⟨*proof*⟩

**lemma** *Constrains_imp_subset:* `"F ∈ A Co A' ==> reachable F ∩ A ⊆ A'"`
⟨*proof*⟩

**lemma** *Constrains_trans:* `"[| F ∈ A Co B; F ∈ B Co C |] ==> F ∈ A Co C"`
⟨*proof*⟩

**lemma** *Constrains_cancel:*
```
     "[| F ∈ A Co (A' ∪ B); F ∈ B Co B' |] ==> F ∈ A Co (A' ∪ B')"
```
⟨*proof*⟩

## 4.3 Stable

**lemma** *Stable_eq:* `"[| F ∈ Stable A; A = B |] ==> F ∈ Stable B"`
⟨*proof*⟩

**lemma** *Stable_eq_stable:* `"(F ∈ Stable A) = (F ∈ stable (reachable F ∩ A))"`
⟨*proof*⟩

**lemma** *StableI:* `"F ∈ A Co A ==> F ∈ Stable A"`
⟨*proof*⟩

**lemma** *StableD:* `"F ∈ Stable A ==> F ∈ A Co A"`
⟨*proof*⟩

**lemma** *Stable_Un:*
```
    "[| F ∈ Stable A; F ∈ Stable A' |] ==> F ∈ Stable (A ∪ A')"
```
⟨*proof*⟩

**lemma** *Stable_Int:*
```
    "[| F ∈ Stable A; F ∈ Stable A' |] ==> F ∈ Stable (A ∩ A')"
```
⟨*proof*⟩

**lemma** *Stable_Constrains_Un:*
```
    "[| F ∈ Stable C; F ∈ A Co (C ∪ A') |]
     ==> F ∈ (C ∪ A) Co (C ∪ A')"
```

⟨*proof*⟩

**lemma** *Stable_Constrains_Int:*
```
"[| F ∈ Stable C; F ∈ (C ∩ A) Co A' |]
 ==> F ∈ (C ∩ A) Co (C ∩ A')"
```
⟨*proof*⟩

**lemma** *Stable_UN:*
```
"(!!i. i ∈ I ==> F ∈ Stable (A i)) ==> F ∈ Stable (⋃i ∈ I. A i)"
```
⟨*proof*⟩

**lemma** *Stable_INT:*
```
"(!!i. i ∈ I ==> F ∈ Stable (A i)) ==> F ∈ Stable (⋂i ∈ I. A i)"
```
⟨*proof*⟩

**lemma** *Stable_reachable:* `"F ∈ Stable (reachable F)"`
⟨*proof*⟩

## 4.4 Increasing

**lemma** *IncreasingD:*
```
"F ∈ Increasing f ==> F ∈ Stable {s. x ≤ f s}"
```
⟨*proof*⟩

**lemma** *mono_Increasing_o:*
```
"mono g ==> Increasing f ⊆ Increasing (g o f)"
```
⟨*proof*⟩

**lemma** *strict_IncreasingD:*
```
"!!z::nat. F ∈ Increasing f ==> F ∈ Stable {s. z < f s}"
```
⟨*proof*⟩

**lemma** *increasing_imp_Increasing:*
```
"F ∈ increasing f ==> F ∈ Increasing f"
```
⟨*proof*⟩

**lemmas** *Increasing_constant =*
*increasing_constant [THEN increasing_imp_Increasing, standard, iff]*

## 4.5 The Elimination Theorem

**lemma** *Elimination:*
```
"[| ∀m. F ∈ {s. s x = m} Co (B m) |]
 ==> F ∈ {s. s x ∈ M} Co (⋃m ∈ M. B m)"
```
⟨*proof*⟩

**lemma** *Elimination_sing:*
```
"(∀m. F ∈ {m} Co (B m)) ==> F ∈ M Co (⋃m ∈ M. B m)"
```
⟨*proof*⟩

## 4.6 Specialized laws for handling Always

**lemma** *AlwaysI:* `"[| Init F ⊆ A;  F ∈ Stable A |] ==> F ∈ Always A"`

⟨*proof*⟩

**lemma** *AlwaysD: "F ∈ Always A ==> Init F ⊆ A & F ∈ Stable A"*
⟨*proof*⟩

**lemmas** *AlwaysE = AlwaysD [THEN conjE, standard]*
**lemmas** *Always_imp_Stable = AlwaysD [THEN conjunct2, standard]*

**lemma** *Always_includes_reachable: "F ∈ Always A ==> reachable F ⊆ A"*
⟨*proof*⟩

**lemma** *invariant_imp_Always:*
*"F ∈ invariant A ==> F ∈ Always A"*
⟨*proof*⟩

**lemmas** *Always_reachable =*
*invariant_reachable [THEN invariant_imp_Always, standard]*

**lemma** *Always_eq_invariant_reachable:*
*"Always A = {F. F ∈ invariant (reachable F ∩ A)}"*
⟨*proof*⟩

**lemma** *Always_eq_includes_reachable: "Always A = {F. reachable F ⊆ A}"*
⟨*proof*⟩

**lemma** *Always_UNIV_eq [simp]: "Always UNIV = UNIV"*
⟨*proof*⟩

**lemma** *UNIV_AlwaysI: "UNIV ⊆ A ==> F ∈ Always A"*
⟨*proof*⟩

**lemma** *Always_eq_UN_invariant: "Always A = (⋃ I ∈ Pow A. invariant I)"*
⟨*proof*⟩

**lemma** *Always_weaken: "[| F ∈ Always A; A ⊆ B |] ==> F ∈ Always B"*
⟨*proof*⟩

## 4.7 "Co" rules involving Always

**lemma** *Always_Constrains_pre:*
*"F ∈ Always INV ==> (F ∈ (INV ∩ A) Co A') = (F ∈ A Co A')"*
⟨*proof*⟩

**lemma** *Always_Constrains_post:*
*"F ∈ Always INV ==> (F ∈ A Co (INV ∩ A')) = (F ∈ A Co A')"*
⟨*proof*⟩

**lemmas** *Always_ConstrainsI = Always_Constrains_pre [THEN iffD1, standard]*

**lemmas** *Always_ConstrainsD = Always_Constrains_post [THEN iffD2, standard]*

**lemma** *Always_Constrains_weaken:*
*"[| F ∈ Always C; F ∈ A Co A'; C ∩ B ⊆ A; C ∩ A' ⊆ B' |] ==> F ∈ B Co B'"*
⟨*proof*⟩

**lemma** *Always_Int_distrib: "Always (A ∩ B) = Always A ∩ Always B"*
⟨*proof*⟩

**lemma** *Always_INT_distrib: "Always (INTER I A) = (⋂i ∈ I. Always (A i))"*
⟨*proof*⟩

**lemma** *Always_Int_I:*
*"[| F ∈ Always A; F ∈ Always B |] ==> F ∈ Always (A ∩ B)"*
⟨*proof*⟩

**lemma** *Always_Compl_Un_eq:*
*"F ∈ Always A ==> (F ∈ Always (-A ∪ B)) = (F ∈ Always B)"*
⟨*proof*⟩

**lemmas** *Always_thin = thin_rl [of "F ∈ Always A", standard]*

## 4.8 Totalize

**lemma** *reachable_imp_reachable_tot:*
*"s ∈ reachable F ==> s ∈ reachable (totalize F)"*
⟨*proof*⟩

**lemma** *reachable_tot_imp_reachable:*
*"s ∈ reachable (totalize F) ==> s ∈ reachable F"*
⟨*proof*⟩

**lemma** *reachable_tot_eq [simp]: "reachable (totalize F) = reachable F"*
⟨*proof*⟩

**lemma** *totalize_Constrains_iff [simp]: "(totalize F ∈ A Co B) = (F ∈ A Co B)"*
⟨*proof*⟩

**lemma** *totalize_Stable_iff [simp]: "(totalize F ∈ Stable A) = (F ∈ Stable A)"*
⟨*proof*⟩

**lemma** *totalize_Always_iff [simp]: "(totalize F ∈ Always A) = (F ∈ Always A)"*
⟨*proof*⟩

**end**

# 5 Weak Progress

**theory** *SubstAx* **imports** *WFair Constrains* **begin**

**constdefs**
```
  Ensures :: "['a set, 'a set] => 'a program set"    (infixl "Ensures" 60)
   "A Ensures B == {F. F ∈ (reachable F ∩ A) ensures B}"

  LeadsTo :: "['a set, 'a set] => 'a program set"    (infixl "LeadsTo" 60)
   "A LeadsTo B == {F. F ∈ (reachable F ∩ A) leadsTo B}"
```

**syntax** *(xsymbols)*
```
  "op LeadsTo" :: "['a set, 'a set] => 'a program set" (infixl " ⟼w " 60)
```

Resembles the previous definition of LeadsTo

**lemma** *LeadsTo_eq_leadsTo:*
```
    "A LeadsTo B = {F. F ∈ (reachable F ∩ A) leadsTo (reachable F ∩ B)}"
```
⟨*proof*⟩

## 5.1 Specialized laws for handling invariants

**lemma** *Always_LeadsTo_pre:*
```
    "F ∈ Always INV ==> (F ∈ (INV ∩ A) LeadsTo A') = (F ∈ A LeadsTo A')"
```
⟨*proof*⟩

**lemma** *Always_LeadsTo_post:*
```
    "F ∈ Always INV ==> (F ∈ A LeadsTo (INV ∩ A')) = (F ∈ A LeadsTo A')"
```
⟨*proof*⟩

**lemmas** *Always_LeadsToI = Always_LeadsTo_pre [THEN iffD1, standard]*

**lemmas** *Always_LeadsToD = Always_LeadsTo_post [THEN iffD2, standard]*

## 5.2 Introduction rules: Basis, Trans, Union

**lemma** *leadsTo_imp_LeadsTo: "F ∈ A leadsTo B ==> F ∈ A LeadsTo B"*
⟨*proof*⟩

**lemma** *LeadsTo_Trans:*
```
    "[| F ∈ A LeadsTo B;  F ∈ B LeadsTo C |] ==> F ∈ A LeadsTo C"
```
⟨*proof*⟩

**lemma** *LeadsTo_Union:*
```
    "(!!A. A ∈ S ==> F ∈ A LeadsTo B) ==> F ∈ (Union S) LeadsTo B"
```
⟨*proof*⟩

## 5.3 Derived rules

**lemma** *LeadsTo_UNIV [simp]: "F ∈ A LeadsTo UNIV"*
⟨*proof*⟩

Useful with cancellation, disjunction

**lemma** *LeadsTo_Un_duplicate:*
*"F ∈ A LeadsTo (A' ∪ A') ==> F ∈ A LeadsTo A'"*
⟨*proof*⟩

**lemma** *LeadsTo_Un_duplicate2:*
*"F ∈ A LeadsTo (A' ∪ C ∪ C) ==> F ∈ A LeadsTo (A' ∪ C)"*
⟨*proof*⟩

**lemma** *LeadsTo_UN:*
*"(!!i. i ∈ I ==> F ∈ (A i) LeadsTo B) ==> F ∈ (⋃i ∈ I. A i) LeadsTo B"*
⟨*proof*⟩

Binary union introduction rule

**lemma** *LeadsTo_Un:*
*"[| F ∈ A LeadsTo C; F ∈ B LeadsTo C |] ==> F ∈ (A ∪ B) LeadsTo C"*
⟨*proof*⟩

Lets us look at the starting state

**lemma** *single_LeadsTo_I:*
*"(!!s. s ∈ A ==> F ∈ {s} LeadsTo B) ==> F ∈ A LeadsTo B"*
⟨*proof*⟩

**lemma** *subset_imp_LeadsTo: "A ⊆ B ==> F ∈ A LeadsTo B"*
⟨*proof*⟩

**lemmas** *empty_LeadsTo = empty_subsetI [THEN subset_imp_LeadsTo, standard, simp]*

**lemma** *LeadsTo_weaken_R [rule_format]:*
*"[| F ∈ A LeadsTo A'; A' ⊆ B' |] ==> F ∈ A LeadsTo B'"*
⟨*proof*⟩

**lemma** *LeadsTo_weaken_L [rule_format]:*
*"[| F ∈ A LeadsTo A'; B ⊆ A |]*
*==> F ∈ B LeadsTo A'"*
⟨*proof*⟩

**lemma** *LeadsTo_weaken:*
*"[| F ∈ A LeadsTo A';*
*B ⊆ A; A' ⊆ B' |]*
*==> F ∈ B LeadsTo B'"*
⟨*proof*⟩

**lemma** *Always_LeadsTo_weaken:*
*"[| F ∈ Always C; F ∈ A LeadsTo A';*
*C ∩ B ⊆ A; C ∩ A' ⊆ B' |]*
*==> F ∈ B LeadsTo B'"*

⟨*proof*⟩

**lemma** *LeadsTo_Un_post: "F ∈ A LeadsTo B ==> F ∈ (A ∪ B) LeadsTo B"*
⟨*proof*⟩

**lemma** *LeadsTo_Trans_Un:*
*"[| F ∈ A LeadsTo B; F ∈ B LeadsTo C |]*
*==> F ∈ (A ∪ B) LeadsTo C"*
⟨*proof*⟩

**lemma** *LeadsTo_Un_distrib:*
*"(F ∈ (A ∪ B) LeadsTo C) = (F ∈ A LeadsTo C & F ∈ B LeadsTo C)"*
⟨*proof*⟩

**lemma** *LeadsTo_UN_distrib:*
*"(F ∈ (⋃i ∈ I. A i) LeadsTo B) = (∀i ∈ I. F ∈ (A i) LeadsTo B)"*
⟨*proof*⟩

**lemma** *LeadsTo_Union_distrib:*
*"(F ∈ (Union S) LeadsTo B) = (∀A ∈ S. F ∈ A LeadsTo B)"*
⟨*proof*⟩

**lemma** *LeadsTo_Basis: "F ∈ A Ensures B ==> F ∈ A LeadsTo B"*
⟨*proof*⟩

**lemma** *EnsuresI:*
*"[| F ∈ (A-B) Co (A ∪ B); F ∈ transient (A-B) |]*
*==> F ∈ A Ensures B"*
⟨*proof*⟩

**lemma** *Always_LeadsTo_Basis:*
*"[| F ∈ Always INV;*
*F ∈ (INV ∩ (A-A')) Co (A ∪ A');*
*F ∈ transient (INV ∩ (A-A')) |]*
*==> F ∈ A LeadsTo A'"*
⟨*proof*⟩

Set difference: maybe combine with *leadsTo_weaken_L*?? This is the most useful form of the "disjunction" rule

**lemma** *LeadsTo_Diff:*
*"[| F ∈ (A-B) LeadsTo C; F ∈ (A ∩ B) LeadsTo C |]*
*==> F ∈ A LeadsTo C"*
⟨*proof*⟩

**lemma** *LeadsTo_UN_UN:*

```
"(!! i. i ∈ I ==> F ∈ (A i) LeadsTo (A' i))
 ==> F ∈ (⋃i ∈ I. A i) LeadsTo (⋃i ∈ I. A' i)"
```
⟨*proof*⟩

Version with no index set

**lemma** *LeadsTo_UN_UN_noindex:*
```
"(!!i. F ∈ (A i) LeadsTo (A' i)) ==> F ∈ (⋃i. A i) LeadsTo (⋃i. A'
i)"
```
⟨*proof*⟩

Version with no index set

**lemma** *all_LeadsTo_UN_UN:*
```
"∀i. F ∈ (A i) LeadsTo (A' i)
 ==> F ∈ (⋃i. A i) LeadsTo (⋃i. A' i)"
```
⟨*proof*⟩

Binary union version

**lemma** *LeadsTo_Un_Un:*
```
"[| F ∈ A LeadsTo A'; F ∈ B LeadsTo B' |]
      ==> F ∈ (A ∪ B) LeadsTo (A' ∪ B')"
```
⟨*proof*⟩

**lemma** *LeadsTo_cancel2:*
```
"[| F ∈ A LeadsTo (A' ∪ B); F ∈ B LeadsTo B' |]
 ==> F ∈ A LeadsTo (A' ∪ B')"
```
⟨*proof*⟩

**lemma** *LeadsTo_cancel_Diff2:*
```
"[| F ∈ A LeadsTo (A' ∪ B); F ∈ (B-A') LeadsTo B' |]
 ==> F ∈ A LeadsTo (A' ∪ B')"
```
⟨*proof*⟩

**lemma** *LeadsTo_cancel1:*
```
"[| F ∈ A LeadsTo (B ∪ A'); F ∈ B LeadsTo B' |]
 ==> F ∈ A LeadsTo (B' ∪ A')"
```
⟨*proof*⟩

**lemma** *LeadsTo_cancel_Diff1:*
```
"[| F ∈ A LeadsTo (B ∪ A'); F ∈ (B-A') LeadsTo B' |]
 ==> F ∈ A LeadsTo (B' ∪ A')"
```
⟨*proof*⟩

The impossibility law

The set "A" may be non-empty, but it contains no reachable states

**lemma** *LeadsTo_empty: "[|F ∈ A LeadsTo {}; all_total F|] ==> F ∈ Always (-A)"*
⟨*proof*⟩

## 5.4 PSP: Progress-Safety-Progress

Special case of PSP: Misra's "stable conjunction"

**lemma** *PSP_Stable:*
```
"[| F ∈ A LeadsTo A';  F ∈ Stable B |]
 ==> F ∈ (A ∩ B) LeadsTo (A' ∩ B)"
```
⟨*proof*⟩

**lemma** *PSP_Stable2:*
```
"[| F ∈ A LeadsTo A'; F ∈ Stable B |]
 ==> F ∈ (B ∩ A) LeadsTo (B ∩ A')"
```
⟨*proof*⟩

**lemma** *PSP:*
```
"[| F ∈ A LeadsTo A'; F ∈ B Co B' |]
 ==> F ∈ (A ∩ B') LeadsTo ((A' ∩ B) ∪ (B' - B))"
```
⟨*proof*⟩

**lemma** *PSP2:*
```
"[| F ∈ A LeadsTo A'; F ∈ B Co B' |]
 ==> F ∈ (B' ∩ A) LeadsTo ((B ∩ A') ∪ (B' - B))"
```
⟨*proof*⟩

**lemma** *PSP_Unless:*
```
"[| F ∈ A LeadsTo A'; F ∈ B Unless B' |]
 ==> F ∈ (A ∩ B) LeadsTo ((A' ∩ B) ∪ B')"
```
⟨*proof*⟩

**lemma** *Stable_transient_Always_LeadsTo:*
```
"[| F ∈ Stable A;  F ∈ transient C;
    F ∈ Always (-A ∪ B ∪ C) |] ==> F ∈ A LeadsTo B"
```
⟨*proof*⟩

## 5.5 Induction rules

**lemma** *LeadsTo_wf_induct:*
```
"[| wf r;
    ∀m. F ∈ (A ∩ f-`{m}) LeadsTo
               ((A ∩ f-`(r^-1 `` {m})) ∪ B) |]
 ==> F ∈ A LeadsTo B"
```
⟨*proof*⟩

**lemma** *Bounded_induct:*
```
"[| wf r;
    ∀m ∈ I. F ∈ (A ∩ f-`{m}) LeadsTo
                  ((A ∩ f-`(r^-1 `` {m})) ∪ B) |]
 ==> F ∈ A LeadsTo ((A - (f-`I)) ∪ B)"
```
⟨*proof*⟩

**lemma** *LessThan_induct:*
```
"(!!m::nat. F ∈ (A ∩ f-`{m}) LeadsTo ((A ∩ f-`(lessThan m)) ∪ B))
 ==> F ∈ A LeadsTo B"
```
⟨*proof*⟩

Integer version. Could generalize from 0 to any lower bound

**lemma** *integ_0_le_induct:*

```
"[| F ∈ Always {s. (0::int) ≤ f s};
    !! z. F ∈ (A ∩ {s. f s = z}) LeadsTo
              ((A ∩ {s. f s < z}) ∪ B) |]
 ==> F ∈ A LeadsTo B"
```

⟨*proof*⟩

**lemma** *LessThan_bounded_induct:*

```
"!!l::nat. ∀m ∈ greaterThan l.
              F ∈ (A ∩ f-`{m}) LeadsTo ((A ∩ f-`(lessThan m)) ∪ B)
      ==> F ∈ A LeadsTo ((A ∩ (f-`(atMost l))) ∪ B)"
```

⟨*proof*⟩

**lemma** *GreaterThan_bounded_induct:*

```
"!!l::nat. ∀m ∈ lessThan l.
            F ∈ (A ∩ f-`{m}) LeadsTo ((A ∩ f-`(greaterThan m)) ∪ B)
 ==> F ∈ A LeadsTo ((A ∩ (f-`(atLeast l))) ∪ B)"
```

⟨*proof*⟩

## 5.6 Completion: Binary and General Finite versions

**lemma** *Completion:*

```
"[| F ∈ A LeadsTo (A' ∪ C);  F ∈ A' Co (A' ∪ C);
    F ∈ B LeadsTo (B' ∪ C);  F ∈ B' Co (B' ∪ C) |]
 ==> F ∈ (A ∩ B) LeadsTo ((A' ∩ B') ∪ C)"
```

⟨*proof*⟩

**lemma** *Finite_completion_lemma:*

```
"finite I
 ==> (∀i ∈ I. F ∈ (A i) LeadsTo (A' i ∪ C)) -->
     (∀i ∈ I. F ∈ (A' i) Co (A' i ∪ C)) -->
     F ∈ (⋂i ∈ I. A i) LeadsTo ((⋂i ∈ I. A' i) ∪ C)"
```

⟨*proof*⟩

**lemma** *Finite_completion:*

```
"[| finite I;
    !!i. i ∈ I ==> F ∈ (A i) LeadsTo (A' i ∪ C);
    !!i. i ∈ I ==> F ∈ (A' i) Co (A' i ∪ C) |]
 ==> F ∈ (⋂i ∈ I. A i) LeadsTo ((⋂i ∈ I. A' i) ∪ C)"
```

⟨*proof*⟩

**lemma** *Stable_completion:*

```
"[| F ∈ A LeadsTo A';  F ∈ Stable A';
    F ∈ B LeadsTo B';  F ∈ Stable B' |]
 ==> F ∈ (A ∩ B) LeadsTo (A' ∩ B')"
```

⟨*proof*⟩

**lemma** *Finite_stable_completion:*

```
"[| finite I;
    !!i. i ∈ I ==> F ∈ (A i) LeadsTo (A' i);
    !!i. i ∈ I ==> F ∈ Stable (A' i) |]
 ==> F ∈ (⋂i ∈ I. A i) LeadsTo (⋂i ∈ I. A' i)"
```

⟨*proof*⟩

**end**

# 6 The Detects Relation

**theory** *Detects* **imports** *FP SubstAx* **begin**

**consts**
```
  op_Detects  :: "['a set, 'a set] => 'a program set"  (infixl "Detects" 60)
  op_Equality :: "['a set, 'a set] => 'a set"          (infixl "<==>" 60)
```

**defs**
```
  Detects_def:  "A Detects B == (Always (-A ∪ B)) ∩ (B LeadsTo A)"
  Equality_def: "A <==> B == (-A ∪ B) ∩ (A ∪ -B)"
```

**lemma** *Always_at_FP:*
```
     "[|F ∈ A LeadsTo B; all_total F|] ==> F ∈ Always (-((FP F) ∩ A ∩ -B))"
```
⟨*proof*⟩

**lemma** *Detects_Trans:*
```
     "[| F ∈ A Detects B; F ∈ B Detects C |] ==> F ∈ A Detects C"
```
⟨*proof*⟩

**lemma** *Detects_refl:* `"F ∈ A Detects A"`
⟨*proof*⟩

**lemma** *Detects_eq_Un:* `"(A<==>B) = (A ∩ B) ∪ (-A ∩ -B)"`
⟨*proof*⟩

**lemma** *Detects_antisym:*
```
     "[| F ∈ A Detects B;  F ∈ B Detects A|] ==> F ∈ Always (A <==> B)"
```
⟨*proof*⟩

**lemma** *Detects_Always:*
```
     "[|F ∈ A Detects B; all_total F|] ==> F ∈ Always (-(FP F) ∪ (A <==>
B))"
```
⟨*proof*⟩

**lemma** *Detects_Imp_LeadstoEQ:*
```
     "F ∈ A Detects B ==> F ∈ UNIV LeadsTo (A <==> B)"
```
⟨*proof*⟩

**end**

# 7 Unions of Programs

**theory** *Union* **imports** *SubstAx FP* **begin**

**constdefs**

```
  ok :: "['a program, 'a program] => bool"      (infixl "ok" 65)
    "F ok G == Acts F ⊆ AllowedActs G &
               Acts G ⊆ AllowedActs F"


  OK  :: "['a set, 'a => 'b program] => bool"
    "OK I F == (∀i ∈ I. ∀j ∈ I-{i}. Acts (F i) ⊆ AllowedActs (F j))"

  JOIN  :: "['a set, 'a => 'b program] => 'b program"
    "JOIN I F == mk_program (⋂i ∈ I. Init (F i), ⋃i ∈ I. Acts (F i),
                             ⋂i ∈ I. AllowedActs (F i))"

  Join :: "['a program, 'a program] => 'a program"      (infixl "Join" 65)
    "F Join G == mk_program (Init F ∩ Init G, Acts F ∪ Acts G,
                             AllowedActs F ∩ AllowedActs G)"

  SKIP :: "'a program"
    "SKIP == mk_program (UNIV, {}, UNIV)"


  safety_prop :: "'a program set => bool"
    "safety_prop X == SKIP: X & (∀G. Acts G ⊆ UNION X Acts --> G ∈ X)"
```

**syntax**

```
  "@JOIN1"     :: "[pttrns, 'b set] => 'b set"         ("(3JN _./ _)" 10)
  "@JOIN"      :: "[pttrn, 'a set, 'b set] => 'b set"  ("(3JN _:_./ _)" 10)
```

**translations**

```
  "JN x : A. B"   == "JOIN A (%x. B)"
  "JN x y. B"   == "JN x. JN y. B"
  "JN x. B"     == "JOIN UNIV (%x. B)"
```

**syntax** *(xsymbols)*

```
  SKIP     :: "'a program"                                ("⊥")
  Join     :: "['a program, 'a program] => 'a program"    (infixl "⊔" 65)
  "@JOIN1" :: "[pttrns, 'b set] => 'b set"                ("(3⨆ _./ _)" 10)
  "@JOIN"  :: "[pttrn, 'a set, 'b set] => 'b set"         ("(3⨆ _∈_./ _)" 10)
```

## 7.1 SKIP

**lemma** *Init_SKIP [simp]: "Init SKIP = UNIV"*

⟨*proof*⟩

**lemma** *Acts_SKIP [simp]: "Acts SKIP = {Id}"*
⟨*proof*⟩

**lemma** *AllowedActs_SKIP [simp]: "AllowedActs SKIP = UNIV"*
⟨*proof*⟩

**lemma** *reachable_SKIP [simp]: "reachable SKIP = UNIV"*
⟨*proof*⟩

## 7.2 SKIP and safety properties

**lemma** *SKIP_in_constrains_iff [iff]: "(SKIP ∈ A co B) = (A ⊆ B)"*
⟨*proof*⟩

**lemma** *SKIP_in_Constrains_iff [iff]: "(SKIP ∈ A Co B) = (A ⊆ B)"*
⟨*proof*⟩

**lemma** *SKIP_in_stable [iff]: "SKIP ∈ stable A"*
⟨*proof*⟩

**declare** *SKIP_in_stable [THEN stable_imp_Stable, iff]*

## 7.3 Join

**lemma** *Init_Join [simp]: "Init (F⊔G) = Init F ∩ Init G"*
⟨*proof*⟩

**lemma** *Acts_Join [simp]: "Acts (F⊔G) = Acts F ∪ Acts G"*
⟨*proof*⟩

**lemma** *AllowedActs_Join [simp]:*
*"AllowedActs (F⊔G) = AllowedActs F ∩ AllowedActs G"*
⟨*proof*⟩

## 7.4 JN

**lemma** *JN_empty [simp]: "(⨆i∈{}. F i) = SKIP"*
⟨*proof*⟩

**lemma** *JN_insert [simp]: "(⨆i ∈ insert a I. F i) = (F a)⊔(⨆i ∈ I. F i)"*
⟨*proof*⟩

**lemma** *Init_JN [simp]: "Init (⨆i ∈ I. F i) = (⋂i ∈ I. Init (F i))"*
⟨*proof*⟩

**lemma** *Acts_JN [simp]: "Acts (⨆i ∈ I. F i) = insert Id (⋃i ∈ I. Acts (F i))"*
⟨*proof*⟩

**lemma** *AllowedActs_JN [simp]:*
*"AllowedActs (⨆i ∈ I. F i) = (⋂i ∈ I. AllowedActs (F i))"*
⟨*proof*⟩

**lemma** *JN_cong [cong]:*
*"[| I=J; !!i. i ∈ J ==> F i = G i |] ==> (⨆i ∈ I. F i) = (⨆i ∈ J. G i)"*
⟨*proof*⟩

## 7.5 Algebraic laws

**lemma** *Join_commute: "F⊔G = G⊔F"*
⟨*proof*⟩

**lemma** *Join_assoc: "(F⊔G)⊔H = F⊔(G⊔H)"*
⟨*proof*⟩

**lemma** *Join_left_commute: "A⊔(B⊔C) = B⊔(A⊔C)"*
⟨*proof*⟩

**lemma** *Join_SKIP_left [simp]: "SKIP⊔F = F"*
⟨*proof*⟩

**lemma** *Join_SKIP_right [simp]: "F⊔SKIP = F"*
⟨*proof*⟩

**lemma** *Join_absorb [simp]: "F⊔F = F"*
⟨*proof*⟩

**lemma** *Join_left_absorb: "F⊔(F⊔G) = F⊔G"*
⟨*proof*⟩

**lemmas** *Join_ac = Join_assoc Join_left_absorb Join_commute Join_left_commute*

## 7.6 Laws Governing ⨆

**lemma** *JN_absorb: "k ∈ I ==> F k⊔(⨆i ∈ I. F i) = (⨆i ∈ I. F i)"*
⟨*proof*⟩

**lemma** *JN_Un: "(⨆i ∈ I ∪ J. F i) = ((⨆i ∈ I. F i)⊔(⨆i ∈ J. F i))"*
⟨*proof*⟩

**lemma** *JN_constant: "(⨆i ∈ I. c) = (if I={} then SKIP else c)"*
⟨*proof*⟩

**lemma** *JN_Join_distrib:*
*"(⨆i ∈ I. F i⊔G i) = (⨆i ∈ I. F i) ⊔ (⨆i ∈ I. G i)"*
⟨*proof*⟩

**lemma** *JN_Join_miniscope:*
*"i ∈ I ==> (⨆i ∈ I. F i⊔G) = ((⨆i ∈ I. F i)⊔G)"*
⟨*proof*⟩

**lemma** *JN_Join_diff: "i ∈ I ==> F i⊔JOIN (I - {i}) F = JOIN I F"*

⟨*proof*⟩

## 7.7 Safety: co, stable, FP

**lemma** *JN_constrains:*
```
"i ∈ I ==> (⋃i ∈ I. F i) ∈ A co B = (∀i ∈ I. F i ∈ A co B)"
```
⟨*proof*⟩

**lemma** *Join_constrains [simp]:*
```
"(F⊔G ∈ A co B) = (F ∈ A co B & G ∈ A co B)"
```
⟨*proof*⟩

**lemma** *Join_unless [simp]:*
```
"(F⊔G ∈ A unless B) = (F ∈ A unless B & G ∈ A unless B)"
```
⟨*proof*⟩

**lemma** *Join_constrains_weaken:*
```
"[| F ∈ A co A';  G ∈ B co B' |]
 ==> F⊔G ∈ (A ∩ B) co (A' ∪ B')"
```
⟨*proof*⟩

**lemma** *JN_constrains_weaken:*
```
"[| ∀i ∈ I. F i ∈ A i co A' i;  i ∈ I |]
 ==> (⋃i ∈ I. F i) ∈ (⋂i ∈ I. A i) co (⋃i ∈ I. A' i)"
```
⟨*proof*⟩

**lemma** *JN_stable:* `"(⋃i ∈ I. F i) ∈ stable A = (∀i ∈ I. F i ∈ stable A)"`
⟨*proof*⟩

**lemma** *invariant_JN_I:*
```
"[| !!i. i ∈ I ==> F i ∈ invariant A;  i ∈ I |]
  ==> (⋃i ∈ I. F i) ∈ invariant A"
```
⟨*proof*⟩

**lemma** *Join_stable [simp]:*
```
"(F⊔G ∈ stable A) =
 (F ∈ stable A & G ∈ stable A)"
```
⟨*proof*⟩

**lemma** *Join_increasing [simp]:*
```
"(F⊔G ∈ increasing f) =
 (F ∈ increasing f & G ∈ increasing f)"
```
⟨*proof*⟩

**lemma** *invariant_JoinI:*
```
"[| F ∈ invariant A; G ∈ invariant A |]
 ==> F⊔G ∈ invariant A"
```
⟨*proof*⟩

**lemma** *FP_JN:* `"FP (⋃i ∈ I. F i) = (⋂i ∈ I. FP (F i))"`

⟨*proof*⟩

## 7.8 Progress: transient, ensures

**lemma** *JN_transient:*
```
"i ∈ I ==>
(⨆i ∈ I. F i) ∈ transient A = (∃i ∈ I. F i ∈ transient A)"
```
⟨*proof*⟩

**lemma** *Join_transient [simp]:*
```
"F⊔G ∈ transient A =
 (F ∈ transient A | G ∈ transient A)"
```
⟨*proof*⟩

**lemma** *Join_transient_I1:* `"F ∈ transient A ==> F⊔G ∈ transient A"`
⟨*proof*⟩

**lemma** *Join_transient_I2:* `"G ∈ transient A ==> F⊔G ∈ transient A"`
⟨*proof*⟩

**lemma** *JN_ensures:*
```
"i ∈ I ==>
 (⨆i ∈ I. F i) ∈ A ensures B =
 ((∀i ∈ I. F i ∈ (A-B) co (A ∪ B)) & (∃i ∈ I. F i ∈ A ensures B))"
```
⟨*proof*⟩

**lemma** *Join_ensures:*
```
"F⊔G ∈ A ensures B =
 (F ∈ (A-B) co (A ∪ B) & G ∈ (A-B) co (A ∪ B) &
  (F ∈ transient (A-B) | G ∈ transient (A-B)))"
```
⟨*proof*⟩

**lemma** *stable_Join_constrains:*
```
"[| F ∈ stable A;  G ∈ A co A' |]
 ==> F⊔G ∈ A co A'"
```
⟨*proof*⟩

**lemma** *stable_Join_Always1:*
`"[| F ∈ stable A;  G ∈ invariant A |] ==> F⊔G ∈ Always A"`
⟨*proof*⟩

**lemma** *stable_Join_Always2:*
`"[| F ∈ invariant A;  G ∈ stable A |] ==> F⊔G ∈ Always A"`
⟨*proof*⟩

**lemma** *stable_Join_ensures1:*
`"[| F ∈ stable A;  G ∈ A ensures B |] ==> F⊔G ∈ A ensures B"`
⟨*proof*⟩

**lemma** *stable_Join_ensures2:*

```
"[| F ∈ A ensures B;  G ∈ stable A |] ==> F⊔G ∈ A ensures B"
```
⟨*proof*⟩

## 7.9 the ok and OK relations

**lemma** *ok_SKIP1 [iff]: "SKIP ok F"*
⟨*proof*⟩

**lemma** *ok_SKIP2 [iff]: "F ok SKIP"*
⟨*proof*⟩

**lemma** *ok_Join_commute:*
```
"(F ok G & (F⊔G) ok H) = (G ok H & F ok (G⊔H))"
```
⟨*proof*⟩

**lemma** *ok_commute: "(F ok G) = (G ok F)"*
⟨*proof*⟩

**lemmas** *ok_sym = ok_commute [THEN iffD1, standard]*

**lemma** *ok_iff_OK:*
```
"OK {(0::int,F),(1,G),(2,H)} snd = (F ok G & (F⊔G) ok H)"
```
⟨*proof*⟩

**lemma** *ok_Join_iff1 [iff]: "F ok (G⊔H) = (F ok G & F ok H)"*
⟨*proof*⟩

**lemma** *ok_Join_iff2 [iff]: "(G⊔H) ok F = (G ok F & H ok F)"*
⟨*proof*⟩

**lemma** *ok_Join_commute_I: "[| F ok G; (F⊔G) ok H |] ==> F ok (G⊔H)"*
⟨*proof*⟩

**lemma** *ok_JN_iff1 [iff]: "F ok (JOIN I G) = (∀i ∈ I. F ok G i)"*
⟨*proof*⟩

**lemma** *ok_JN_iff2 [iff]: "(JOIN I G) ok F =  (∀i ∈ I. G i ok F)"*
⟨*proof*⟩

**lemma** *OK_iff_ok: "OK I F = (∀i ∈ I. ∀j ∈ I-{i}. (F i) ok (F j))"*
⟨*proof*⟩

**lemma** *OK_imp_ok: "[| OK I F; i ∈ I; j ∈ I; i ≠ j|] ==> (F i) ok (F j)"*
⟨*proof*⟩

## 7.10 Allowed

**lemma** *Allowed_SKIP [simp]: "Allowed SKIP = UNIV"*
⟨*proof*⟩

**lemma** *Allowed_Join [simp]: "Allowed (F⊔G) = Allowed F ∩ Allowed G"*
⟨*proof*⟩

**lemma** *Allowed_JN [simp]: "Allowed (JOIN I F) = (⋂i ∈ I. Allowed (F i))"*
⟨*proof*⟩

**lemma** *ok_iff_Allowed: "F ok G = (F ∈ Allowed G & G ∈ Allowed F)"*
⟨*proof*⟩

**lemma** *OK_iff_Allowed: "OK I F = (∀i ∈ I. ∀j ∈ I-{i}. F i ∈ Allowed(F j))"*
⟨*proof*⟩

## 7.11 *safety_prop*, for reasoning about given instances of "ok"

**lemma** *safety_prop_Acts_iff:*
*"safety_prop X ==> (Acts G ⊆ insert Id (UNION X Acts)) = (G ∈ X)"*
⟨*proof*⟩

**lemma** *safety_prop_AllowedActs_iff_Allowed:*
*"safety_prop X ==> (UNION X Acts ⊆ AllowedActs F) = (X ⊆ Allowed F)"*
⟨*proof*⟩

**lemma** *Allowed_eq:*
*"safety_prop X ==> Allowed (mk_program (init, acts, UNION X Acts)) = X"*
⟨*proof*⟩

**lemma** *safety_prop_constrains [iff]: "safety_prop (A co B) = (A ⊆ B)"*
⟨*proof*⟩

**lemma** *safety_prop_stable [iff]: "safety_prop (stable A)"*
⟨*proof*⟩

**lemma** *safety_prop_Int [simp]:*
*"[| safety_prop X; safety_prop Y |] ==> safety_prop (X ∩ Y)"*
⟨*proof*⟩

**lemma** *safety_prop_INTER1 [simp]:*
*"(!!i. safety_prop (X i)) ==> safety_prop (⋂i. X i)"*
⟨*proof*⟩

**lemma** *safety_prop_INTER [simp]:*
*"(!!i. i ∈ I ==> safety_prop (X i)) ==> safety_prop (⋂i ∈ I. X i)"*
⟨*proof*⟩

**lemma** *def_prg_Allowed:*
*"[| F == mk_program (init, acts, UNION X Acts) ; safety_prop X |]*
*==> Allowed F = X"*
⟨*proof*⟩

**lemma** *Allowed_totalize [simp]: "Allowed (totalize F) = Allowed F"*
⟨*proof*⟩

**lemma** *def_total_prg_Allowed:*
*"[| F == mk_total_program (init, acts, UNION X Acts) ; safety_prop X |]*

*==> Allowed F = X"*

⟨*proof*⟩

**lemma** *def_UNION_ok_iff:*
```
"[| F == mk_program(init,acts,UNION X Acts); safety_prop X |]
 ==> F ok G = (G ∈ X & acts ⊆ AllowedActs G)"
```
⟨*proof*⟩

The union of two total programs is total.

**lemma** *totalize_Join:* `"totalize F⊔totalize G = totalize (F⊔G)"`
⟨*proof*⟩

**lemma** *all_total_Join:* `"[|all_total F; all_total G|] ==> all_total (F⊔G)"`
⟨*proof*⟩

**lemma** *totalize_JN:* `"(⨆i ∈ I. totalize (F i)) = totalize(⨆i ∈ I. F i)"`
⟨*proof*⟩

**lemma** *all_total_JN:* `"(!!i. i∈I ==> all_total (F i)) ==> all_total(⨆i∈I. F i)"`
⟨*proof*⟩

**end**

# 8 Composition: Basic Primitives

**theory** *Comp* **imports** *Union* **begin**

**instance** *program :: (type) ord* ⟨*proof*⟩

**defs**
```
component_def:          "F ≤ H == ∃G. F⊔G = H"
strict_component_def:   "(F < (H::'a program)) == (F ≤ H & F ≠ H)"
```

**constdefs**
```
component_of :: "'a program =>'a program=> bool"
                                    (infixl "component'_of" 50)
"F component_of H == ∃G. F ok G & F⊔G = H"

strict_component_of :: "'a program⇒'a program=> bool"
                                    (infixl "strict'_component'_of" 50)
"F strict_component_of H == F component_of H & F≠H"

preserves :: "('a=>'b) => 'a program set"
  "preserves v == ⋂z. stable {s. v s = z}"

localize  :: "('a=>'b) => 'a program => 'a program"
"localize v F == mk_program(Init F, Acts F,
                            AllowedActs F ∩ (⋃G ∈ preserves v. Acts G))"

funPair      :: "['a => 'b, 'a => 'c, 'a] => 'b * 'c"
"funPair f g == %x. (f x, g x)"
```

## 8.1 The component relation

**lemma** *componentI: "H ≤ F | H ≤ G ==> H ≤ (F⊔G)"*
⟨*proof*⟩

**lemma** *component_eq_subset:*
*"(F ≤ G) =*
*(Init G ⊆ Init F & Acts F ⊆ Acts G & AllowedActs G ⊆ AllowedActs F)"*
⟨*proof*⟩

**lemma** *component_SKIP [iff]: "SKIP ≤ F"*
⟨*proof*⟩

**lemma** *component_refl [iff]: "F ≤ (F :: 'a program)"*
⟨*proof*⟩

**lemma** *SKIP_minimal: "F ≤ SKIP ==> F = SKIP"*
⟨*proof*⟩

**lemma** *component_Join1: "F ≤ (F⊔G)"*
⟨*proof*⟩

**lemma** *component_Join2: "G ≤ (F⊔G)"*
⟨*proof*⟩

**lemma** *Join_absorb1: "F ≤ G ==> F⊔G = G"*
⟨*proof*⟩

**lemma** *Join_absorb2: "G ≤ F ==> F⊔G = F"*
⟨*proof*⟩

**lemma** *JN_component_iff: "((JOIN I F) ≤ H) = (∀i ∈ I. F i ≤ H)"*
⟨*proof*⟩

**lemma** *component_JN: "i ∈ I ==> (F i) ≤ (⨆i ∈ I. (F i))"*
⟨*proof*⟩

**lemma** *component_trans: "[| F ≤ G; G ≤ H |] ==> F ≤ (H :: 'a program)"*
⟨*proof*⟩

**lemma** *component_antisym: "[| F ≤ G; G ≤ F |] ==> F = (G :: 'a program)"*
⟨*proof*⟩

**lemma** *Join_component_iff: "((F⊔G) ≤ H) = (F ≤ H & G ≤ H)"*
⟨*proof*⟩

**lemma** *component_constrains: "[| F ≤ G; G ∈ A co B |] ==> F ∈ A co B"*
⟨*proof*⟩

**lemma** *component_stable: "[| F ≤ G; G ∈ stable A |] ==> F ∈ stable A"*
⟨*proof*⟩

**lemmas** *program_less_le = strict_component_def [THEN meta_eq_to_obj_eq]*

## 8.2 The preserves property

**lemma** *preservesI: "(!!z. F ∈ stable {s. v s = z}) ==> F ∈ preserves v"*
⟨*proof*⟩

**lemma** *preserves_imp_eq:*
*"[| F ∈ preserves v; act ∈ Acts F; (s,s') ∈ act |] ==> v s = v s'"*
⟨*proof*⟩

**lemma** *Join_preserves [iff]:*
*"(F⊔G ∈ preserves v) = (F ∈ preserves v & G ∈ preserves v)"*
⟨*proof*⟩

**lemma** *JN_preserves [iff]:*
*"(JOIN I F ∈ preserves v) = (∀i ∈ I. F i ∈ preserves v)"*
⟨*proof*⟩

**lemma** *SKIP_preserves [iff]: "SKIP ∈ preserves v"*
⟨*proof*⟩

**lemma** *funPair_apply [simp]: "(funPair f g) x = (f x, g x)"*
⟨*proof*⟩

**lemma** *preserves_funPair: "preserves (funPair v w) = preserves v ∩ preserves w"*
⟨*proof*⟩

**declare** *preserves_funPair [THEN eqset_imp_iff, iff]*

**lemma** *funPair_o_distrib: "(funPair f g) o h = funPair (f o h) (g o h)"*
⟨*proof*⟩

**lemma** *fst_o_funPair [simp]: "fst o (funPair f g) = f"*
⟨*proof*⟩

**lemma** *snd_o_funPair [simp]: "snd o (funPair f g) = g"*
⟨*proof*⟩

**lemma** *subset_preserves_o: "preserves v ⊆ preserves (w o v)"*
⟨*proof*⟩

**lemma** *preserves_subset_stable: "preserves v ⊆ stable {s. P (v s)}"*
⟨*proof*⟩

**lemma** *preserves_subset_increasing: "preserves v ⊆ increasing v"*
⟨*proof*⟩

**lemma** *preserves_id_subset_stable: "preserves id ⊆ stable A"*
⟨*proof*⟩

**lemma** *safety_prop_preserves [iff]: "safety_prop (preserves v)"*
⟨*proof*⟩

**lemma** *stable_localTo_stable2:*
*"[| F ∈ stable {s. P (v s) (w s)};*
*G ∈ preserves v; G ∈ preserves w |]*
*==> F⊔G ∈ stable {s. P (v s) (w s)}"*
⟨*proof*⟩

**lemma** *Increasing_preserves_Stable:*
*"[| F ∈ stable {s. v s ≤ w s}; G ∈ preserves v; F⊔G ∈ Increasing w*
*|]*
*==> F⊔G ∈ Stable {s. v s ≤ w s}"*
⟨*proof*⟩

**lemma** *component_of_imp_component: "F component_of H ==> F ≤ H"*
⟨*proof*⟩

**lemma** *component_of_refl [simp]: "F component_of F"*
⟨*proof*⟩

**lemma** *component_of_SKIP [simp]: "SKIP component_of F"*
⟨*proof*⟩

**lemma** *component_of_trans:*
*"[| F component_of G; G component_of H |] ==> F component_of H"*
⟨*proof*⟩

**lemmas** *strict_component_of_eq =*
*strict_component_of_def [THEN meta_eq_to_obj_eq, standard]*

**lemma** *localize_Init_eq [simp]: "Init (localize v F) = Init F"*
⟨*proof*⟩

**lemma** *localize_Acts_eq [simp]: "Acts (localize v F) = Acts F"*
⟨*proof*⟩

**lemma** *localize_AllowedActs_eq [simp]:*
*"AllowedActs (localize v F) = AllowedActs F ∩ (⋃G ∈ preserves v. Acts*
*G)"*
⟨*proof*⟩

**end**

# 9 Guarantees Specifications

**theory** *Guar* **imports** *Comp* **begin**

**instance** *program :: (type) order*
⟨*proof*⟩

Existential and Universal properties. I formalize the two-program case, proving equivalence with Chandy and Sanders's n-ary definitions

**constdefs**

```
  ex_prop  :: "'a program set => bool"
   "ex_prop X == ∀F G. F ok G -->F ∈ X | G ∈ X --> (F⊔G) ∈ X"

  strict_ex_prop  :: "'a program set => bool"
   "strict_ex_prop X == ∀F G.  F ok G --> (F ∈ X | G ∈ X) = (F⊔G ∈ X)"

  uv_prop  :: "'a program set => bool"
   "uv_prop X == SKIP ∈ X & (∀F G. F ok G --> F ∈ X & G ∈ X --> (F⊔G) ∈
X)"

  strict_uv_prop  :: "'a program set => bool"
   "strict_uv_prop X ==
      SKIP ∈ X & (∀F G. F ok G --> (F ∈ X & G ∈ X) = (F⊔G ∈ X))"
```

Guarantees properties

**constdefs**

```
  guar :: "['a program set, 'a program set] => 'a program set"
          (infixl "guarantees" 55)
   "X guarantees Y == {F. ∀G. F ok G --> F⊔G ∈ X --> F⊔G ∈ Y}"



   wg :: "['a program, 'a program set] =>  'a program set"
  "wg F Y == Union({X. F ∈ X guarantees Y})"


   wx :: "('a program) set => ('a program)set"
   "wx X == Union({Y. Y ⊆ X & ex_prop Y})"


  welldef :: "'a program set"
  "welldef == {F. Init F ≠ {}}"

  refines :: "['a program, 'a program, 'a program set] => bool"
                        ("(3_ refines _ wrt _)" [10,10,10] 10)
  "G refines F wrt X ==
     ∀H. (F ok H & G ok H & F⊔H ∈ welldef ∩ X) -->
         (G⊔H ∈ welldef ∩ X)"

  iso_refines :: "['a program, 'a program, 'a program set] => bool"
                              ("(3_ iso'_refines _ wrt _)" [10,10,10] 10)
  "G iso_refines F wrt X ==
```

```
  F ∈ welldef ∩ X --> G ∈ welldef ∩ X"
```

**lemma** *OK_insert_iff:*

```
     "(OK (insert i I) F) =
      (if i ∈ I then OK I F else OK I F & (F i ok JOIN I F))"
```

⟨*proof*⟩

## 9.1 Existential Properties

**lemma** *ex1 [rule_format]:*

```
 "[| ex_prop X; finite GG |] ==>
     GG ∩ X ≠ {}--> OK GG (%G. G) --> (⨆G ∈ GG. G) ∈ X"
```

⟨*proof*⟩

**lemma** *ex2:*

```
     "∀GG. finite GG & GG ∩ X ≠ {} --> OK GG (%G. G) -->(⨆G ∈ GG. G):X

      ==> ex_prop X"
```

⟨*proof*⟩

**lemma** *ex_prop_finite:*

```
     "ex_prop X =
      (∀GG. finite GG & GG ∩ X ≠ {} & OK GG (%G. G)--> (⨆G ∈ GG. G) ∈
X)"
```

⟨*proof*⟩

**lemma** *ex_prop_equiv:*

```
     "ex_prop X = (∀G. G ∈ X = (∀H. (G component_of H) --> H ∈ X))"
```

⟨*proof*⟩

## 9.2 Universal Properties

**lemma** *uv1 [rule_format]:*

```
     "[| uv_prop X; finite GG |]
      ==> GG ⊆ X & OK GG (%G. G) --> (⨆G ∈ GG. G) ∈ X"
```

⟨*proof*⟩

**lemma** *uv2:*

```
     "∀GG. finite GG & GG ⊆ X & OK GG (%G. G) --> (⨆G ∈ GG. G) ∈ X
      ==> uv_prop X"
```

⟨*proof*⟩

**lemma** *uv_prop_finite:*

```
     "uv_prop X =
      (∀GG. finite GG & GG ⊆ X & OK GG (%G. G) --> (⨆G ∈ GG. G): X)"
```

⟨*proof*⟩

## 9.3 Guarantees

**lemma** *guaranteesI:*
```
"(!!G. [| F ok G; F⊔G ∈ X |] ==> F⊔G ∈ Y) ==> F ∈ X guarantees Y"
```
⟨*proof*⟩

**lemma** *guaranteesD:*
```
"[| F ∈ X guarantees Y;  F ok G;  F⊔G ∈ X |] ==> F⊔G ∈ Y"
```
⟨*proof*⟩

**lemma** *component_guaranteesD:*
```
"[| F ∈ X guarantees Y;  F⊔G = H;  H ∈ X;  F ok G |] ==> H ∈ Y"
```
⟨*proof*⟩

**lemma** *guarantees_weaken:*
```
"[| F ∈ X guarantees X'; Y ⊆ X; X' ⊆ Y' |] ==> F ∈ Y guarantees Y'"
```
⟨*proof*⟩

**lemma** *subset_imp_guarantees_UNIV:* `"X ⊆ Y ==> X guarantees Y = UNIV"`
⟨*proof*⟩

**lemma** *subset_imp_guarantees:* `"X ⊆ Y ==> F ∈ X guarantees Y"`
⟨*proof*⟩

**lemma** *ex_prop_imp:* `"ex_prop Y ==> (Y = UNIV guarantees Y)"`
⟨*proof*⟩

**lemma** *guarantees_imp:* `"(Y = UNIV guarantees Y) ==> ex_prop(Y)"`
⟨*proof*⟩

**lemma** *ex_prop_equiv2:* `"(ex_prop Y) = (Y = UNIV guarantees Y)"`
⟨*proof*⟩

## 9.4 Distributive Laws. Re-Orient to Perform Miniscoping

**lemma** *guarantees_UN_left:*
```
"(⋃i ∈ I. X i) guarantees Y = (⋂i ∈ I. X i guarantees Y)"
```
⟨*proof*⟩

**lemma** *guarantees_Un_left:*
```
"(X ∪ Y) guarantees Z = (X guarantees Z) ∩ (Y guarantees Z)"
```
⟨*proof*⟩

**lemma** *guarantees_INT_right:*
```
"X guarantees (⋂i ∈ I. Y i) = (⋂i ∈ I. X guarantees Y i)"
```
⟨*proof*⟩

**lemma** *guarantees_Int_right:*
```
"Z guarantees (X ∩ Y) = (Z guarantees X) ∩ (Z guarantees Y)"
```
⟨*proof*⟩

**lemma** *guarantees_Int_right_I:*
```
"[| F ∈ Z guarantees X;  F ∈ Z guarantees Y |]
 ==> F ∈ Z guarantees (X ∩ Y)"
```
⟨*proof*⟩

**lemma** *guarantees_INT_right_iff:*
```
"(F ∈ X guarantees (INTER I Y)) = (∀i∈I. F ∈ X guarantees (Y i))"
```
⟨*proof*⟩

**lemma** *shunting:* `"(X guarantees Y) = (UNIV guarantees (-X ∪ Y))"`
⟨*proof*⟩

**lemma** *contrapositive:* `"(X guarantees Y) = -Y guarantees -X"`
⟨*proof*⟩

**lemma** *combining1:*
```
"[| F ∈ V guarantees X;  F ∈ (X ∩ Y) guarantees Z |]
 ==> F ∈ (V ∩ Y) guarantees Z"
```
⟨*proof*⟩

**lemma** *combining2:*
```
"[| F ∈ V guarantees (X ∪ Y);  F ∈ Y guarantees Z |]
 ==> F ∈ V guarantees (X ∪ Z)"
```
⟨*proof*⟩

**lemma** *all_guarantees:*
```
"∀i∈I. F ∈ X guarantees (Y i) ==> F ∈ X guarantees (⋂i ∈ I. Y i)"
```
⟨*proof*⟩

**lemma** *ex_guarantees:*
```
"∃i∈I. F ∈ X guarantees (Y i) ==> F ∈ X guarantees (⋃i ∈ I. Y i)"
```
⟨*proof*⟩

## 9.5 Guarantees: Additional Laws (by lcp)

**lemma** *guarantees_Join_Int:*
```
"[| F ∈ U guarantees V;  G ∈ X guarantees Y; F ok G |]
 ==> F⊔G ∈ (U ∩ X) guarantees (V ∩ Y)"
```
⟨*proof*⟩

**lemma** *guarantees_Join_Un:*
```
"[| F ∈ U guarantees V;  G ∈ X guarantees Y; F ok G |]
 ==> F⊔G ∈ (U ∪ X) guarantees (V ∪ Y)"
```
⟨*proof*⟩

**lemma** *guarantees_JN_INT:*
```
"[| ∀i∈I. F i ∈ X i guarantees Y i;  OK I F |]
 ==> (JOIN I F) ∈ (INTER I X) guarantees (INTER I Y)"
```

⟨*proof*⟩

**lemma** *guarantees_JN_UN:*
```
"[| ∀i∈I. F i ∈ X i guarantees Y i;  OK I F |]
 ==> (JOIN I F) ∈ (UNION I X) guarantees (UNION I Y)"
```
⟨*proof*⟩

## 9.6 Guarantees Laws for Breaking Down the Program (by lcp)

**lemma** *guarantees_Join_I1:*
```
"[| F ∈ X guarantees Y;  F ok G |] ==> F⊔G ∈ X guarantees Y"
```
⟨*proof*⟩

**lemma** *guarantees_Join_I2:*
```
"[| G ∈ X guarantees Y;  F ok G |] ==> F⊔G ∈ X guarantees Y"
```
⟨*proof*⟩

**lemma** *guarantees_JN_I:*
```
"[| i ∈ I;  F i ∈ X guarantees Y;  OK I F |]
 ==> (⨆i ∈ I. (F i)) ∈ X guarantees Y"
```
⟨*proof*⟩

**lemma** *Join_welldef_D1:* `"F⊔G ∈ welldef ==> F ∈ welldef"`
⟨*proof*⟩

**lemma** *Join_welldef_D2:* `"F⊔G ∈ welldef ==> G ∈ welldef"`
⟨*proof*⟩

**lemma** *refines_refl:* `"F refines F wrt X"`
⟨*proof*⟩

**lemma** *refines_trans:*
```
"[| H refines G wrt X;  G refines F wrt X |] ==> H refines F wrt X"
```
⟨*proof*⟩

**lemma** *strict_ex_refine_lemma:*
```
"strict_ex_prop X
 ==> (∀H. F ok H & G ok H & F⊔H ∈ X --> G⊔H ∈ X)
        = (F ∈ X --> G ∈ X)"
```
⟨*proof*⟩

**lemma** *strict_ex_refine_lemma_v:*
```
"strict_ex_prop X
 ==> (∀H. F ok H & G ok H & F⊔H ∈ welldef & F⊔H ∈ X --> G⊔H ∈ X) =

     (F ∈ welldef ∩ X --> G ∈ X)"
```

⟨*proof*⟩

**lemma** *ex_refinement_thm:*
```
"[| strict_ex_prop X;
    ∀H. F ok H & G ok H & F⊔H ∈ welldef ∩ X --> G⊔H ∈ welldef |]
 ==> (G refines F wrt X) = (G iso_refines F wrt X)"
```
⟨*proof*⟩

**lemma** *strict_uv_refine_lemma:*
```
"strict_uv_prop X ==>
 (∀H. F ok H & G ok H & F⊔H ∈ X --> G⊔H ∈ X) = (F ∈ X --> G ∈ X)"
```
⟨*proof*⟩

**lemma** *strict_uv_refine_lemma_v:*
```
"strict_uv_prop X
 ==> (∀H. F ok H & G ok H & F⊔H ∈ welldef & F⊔H ∈ X --> G⊔H ∈ X) =

     (F ∈ welldef ∩ X --> G ∈ X)"
```
⟨*proof*⟩

**lemma** *uv_refinement_thm:*
```
"[| strict_uv_prop X;
    ∀H. F ok H & G ok H & F⊔H ∈ welldef ∩ X -->
        G⊔H ∈ welldef |]
 ==> (G refines F wrt X) = (G iso_refines F wrt X)"
```
⟨*proof*⟩

**lemma** *guarantees_equiv:*
```
"(F ∈ X guarantees Y) = (∀H. H ∈ X ⟶ (F component_of H ⟶ H ∈ Y))"
```
⟨*proof*⟩

**lemma** *wg_weakest:* `"!!X. F∈ (X guarantees Y) ==> X ⊆ (wg F Y)"`
⟨*proof*⟩

**lemma** *wg_guarantees:* `"F∈ ((wg F Y) guarantees Y)"`
⟨*proof*⟩

**lemma** *wg_equiv:* `"(H ∈ wg F X) = (F component_of H --> H ∈ X)"`
⟨*proof*⟩

**lemma** *component_of_wg:* `"F component_of H ==> (H ∈ wg F X) = (H ∈ X)"`
⟨*proof*⟩

**lemma** *wg_finite:*
```
"∀FF. finite FF & FF ∩ X ≠ {} --> OK FF (%F. F)
      --> (∀F∈FF. ((⨆F ∈ FF. F): wg F X) = ((⨆F ∈ FF. F):X))"
```
⟨*proof*⟩

**lemma** *wg_ex_prop:* `"ex_prop X ==> (F ∈ X) = (∀H. H ∈ wg F X)"`
⟨*proof*⟩

**lemma** *wx_subset: "(wx X)<=X"*
⟨*proof*⟩

**lemma** *wx_ex_prop: "ex_prop (wx X)"*
⟨*proof*⟩

**lemma** *wx_weakest: "∀Z. Z<= X --> ex_prop Z --> Z ⊆ wx X"*
⟨*proof*⟩

**lemma** *wx'_ex_prop: "ex_prop({F. ∀G. F ok G --> F⊔G ∈ X})"*
⟨*proof*⟩

Equivalence with the other definition of wx

**lemma** *wx_equiv: "wx X = {F. ∀G. F ok G --> (F⊔G) ∈ X}"*
⟨*proof*⟩

Propositions 7 to 11 are about this second definition of wx. They are the same as the ones proved for the first definition of wx, by equivalence

**lemma** *guarantees_wx_eq: "(X guarantees Y) = wx(-X ∪ Y)"*
⟨*proof*⟩

**lemma** *stable_guarantees_Always:*
*"Init F ⊆ A ==> F ∈ (stable A) guarantees (Always A)"*
⟨*proof*⟩

**lemma** *constrains_guarantees_leadsTo:*
*"F ∈ transient A ==> F ∈ (A co A ∪ B) guarantees (A leadsTo (B-A))"*
⟨*proof*⟩

**end**

# 10 Extending State Sets

**theory** *Extend* **imports** *Guar* **begin**

**constdefs**

```
Restrict :: "[ 'a set, ('a*'b) set] => ('a*'b) set"
  "Restrict A r == r ∩ (A <*> UNIV)"

good_map :: "['a*'b => 'c] => bool"
  "good_map h == surj h & (∀x y. fst (inv h (h (x,y))) = x)"


extend_set :: "['a*'b => 'c, 'a set] => 'c set"
  "extend_set h A == h ` (A <*> UNIV)"
```

```
  project_set :: "['a*'b => 'c, 'c set] => 'a set"
    "project_set h C == {x. ∃y. h(x,y) ∈ C}"

  extend_act :: "['a*'b => 'c, ('a*'a) set] => ('c*'c) set"
    "extend_act h == %act. ⋃(s,s') ∈ act. ⋃y. {(h(s,y), h(s',y))}"

  project_act :: "['a*'b => 'c, ('c*'c) set] => ('a*'a) set"
    "project_act h act == {(x,x'). ∃y y'. (h(x,y), h(x',y')) ∈ act}"

  extend :: "['a*'b => 'c, 'a program] => 'c program"
    "extend h F == mk_program (extend_set h (Init F),
                               extend_act h `` Acts F,
                               project_act h -`` AllowedActs F)"


  project :: "['a*'b => 'c, 'c set, 'c program] => 'a program"
    "project h C F ==
       mk_program (project_set h (Init F),
                   project_act h `` Restrict C `` Acts F,
                   {act. Restrict (project_set h C) act :
                         project_act h `` Restrict C `` AllowedActs F})"

locale Extend =
  fixes f     :: "'c => 'a"
    and g     :: "'c => 'b"
    and h     :: "'a*'b => 'c"
    and slice :: "['c set, 'b] => 'a set"
  assumes
    good_h:  "good_map h"
  defines f_def: "f z == fst (inv h z)"
      and g_def: "g z == snd (inv h z)"
      and slice_def: "slice Z y == {x. h(x,y) ∈ Z}"
```

## 10.1 Restrict

**lemma** `Restrict_iff [iff]: "((x,y): Restrict A r) = ((x,y): r & x ∈ A)"`
⟨*proof*⟩

**lemma** `Restrict_UNIV [simp]: "Restrict UNIV = id"`
⟨*proof*⟩

**lemma** `Restrict_empty [simp]: "Restrict {} r = {}"`
⟨*proof*⟩

**lemma** `Restrict_Int [simp]: "Restrict A (Restrict B r) = Restrict (A ∩ B) r"`
⟨*proof*⟩

**lemma** `Restrict_triv: "Domain r ⊆ A ==> Restrict A r = r"`
⟨*proof*⟩

**lemma** `Restrict_subset: "Restrict A r ⊆ r"`
⟨*proof*⟩

**lemma** *Restrict_eq_mono:*
```
"[| A ⊆ B;  Restrict B r = Restrict B s |]
 ==> Restrict A r = Restrict A s"
```
⟨*proof*⟩

**lemma** *Restrict_imageI:*
```
"[| s ∈ RR;  Restrict A r = Restrict A s |]
 ==> Restrict A r ∈ Restrict A ‘ RR"
```
⟨*proof*⟩

**lemma** *Domain_Restrict [simp]: "Domain (Restrict A r) = A ∩ Domain r"*
⟨*proof*⟩

**lemma** *Image_Restrict [simp]: "(Restrict A r) `` B = r `` (A ∩ B)"*
⟨*proof*⟩

**lemma** *good_mapI:*
**assumes** *surj_h: "surj h"*
**and** *prem: "!! x x' y y'. h(x,y) = h(x',y') ==> x=x'"*
**shows** *"good_map h"*
⟨*proof*⟩

**lemma** *good_map_is_surj: "good_map h ==> surj h"*
⟨*proof*⟩

**lemma** *fst_inv_equalityI:*
**assumes** *surj_h: "surj h"*
**and** *prem: "!! x y. g (h(x,y)) = x"*
**shows** *"fst (inv h z) = g z"*
⟨*proof*⟩

## 10.2 Trivial properties of f, g, h

**lemma** (**in** *Extend*) *f_h_eq [simp]: "f(h(x,y)) = x"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *h_inject1 [dest]: "h(x,y) = h(x',y') ==> x=x'"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *h_f_g_equiv: "h(f z, g z) == z"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *h_f_g_eq: "h(f z, g z) = z"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *split_extended_all:*
```
"(!!z. PROP P z) == (!!u y. PROP P (h (u, y)))"
```
⟨*proof*⟩

## 10.3 extend_set: basic properties

**lemma** *project_set_iff [iff]:*
*"(x ∈ project_set h C) = (∃y. h(x,y) ∈ C)"*
⟨*proof*⟩

**lemma** *extend_set_mono: "A ⊆ B ==> extend_set h A ⊆ extend_set h B"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *mem_extend_set_iff [iff]: "z ∈ extend_set h A = (f z ∈ A)"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_set_strict_mono [iff]:*
*"(extend_set h A ⊆ extend_set h B) = (A ⊆ B)"*
⟨*proof*⟩

**lemma** *extend_set_empty [simp]: "extend_set h {} = {}"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_set_eq_Collect: "extend_set h {s. P s} = {s. P(f s)}"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_set_sing: "extend_set h {x} = {s. f s = x}"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_set_inverse [simp]:*
*"project_set h (extend_set h C) = C"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_set_project_set:*
*"C ⊆ extend_set h (project_set h C)"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *inj_extend_set: "inj (extend_set h)"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_set_UNIV_eq [simp]: "extend_set h UNIV = UNIV"*
⟨*proof*⟩

## 10.4 project_set: basic properties

**lemma** (**in** *Extend*) *project_set_eq: "project_set h C = f ‘ C"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_set_I: "!!z. z ∈ C ==> f z ∈ project_set h C"*
⟨*proof*⟩

## 10.5 More laws

**lemma** (**in** *Extend*) *project_set_extend_set_Int:*
*"project_set h ((extend_set h A) ∩ B) = A ∩ (project_set h B)"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_set_extend_set_Un:*
*"project_set h ((extend_set h A) ∪ B) = A ∪ (project_set h B)"*
⟨*proof*⟩

**lemma** *project_set_Int_subset:*
*"project_set h (A ∩ B) ⊆ (project_set h A) ∩ (project_set h B)"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_set_Un_distrib:*
*"extend_set h (A ∪ B) = extend_set h A ∪ extend_set h B"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_set_Int_distrib:*
*"extend_set h (A ∩ B) = extend_set h A ∩ extend_set h B"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_set_INT_distrib:*
*"extend_set h (INTER A B) = (⋂x ∈ A. extend_set h (B x))"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_set_Diff_distrib:*
*"extend_set h (A - B) = extend_set h A - extend_set h B"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_set_Union:*
*"extend_set h (Union A) = (⋃X ∈ A. extend_set h X)"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_set_subset_Compl_eq:*
*"(extend_set h A ⊆ - extend_set h B) = (A ⊆ - B)"*
⟨*proof*⟩

## 10.6 *extend_act*

**lemma** (**in** *Extend*) *mem_extend_act_iff [iff]:*
*"((h(s,y), h(s',y)) ∈ extend_act h act) = ((s, s') ∈ act)"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_act_D:*
*"(z, z') ∈ extend_act h act ==> (f z, f z') ∈ act"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_act_inverse [simp]:*
*"project_act h (extend_act h act) = act"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_act_extend_act_restrict [simp]:*
*"project_act h (Restrict C (extend_act h act)) =*
*Restrict (project_set h C) act"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *subset_extend_act_D:*
```
"act' ⊆ extend_act h act ==> project_act h act' ⊆ act"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *inj_extend_act: "inj (extend_act h)"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_act_Image [simp]:*
```
"extend_act h act `` (extend_set h A) = extend_set h (act `` A)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_act_strict_mono [iff]:*
```
"(extend_act h act' ⊆ extend_act h act) = (act'<=act)"
```
⟨*proof*⟩

**declare** (**in** *Extend*) *inj_extend_act [THEN inj_eq, iff]*

**lemma** *Domain_extend_act:*
```
"Domain (extend_act h act) = extend_set h (Domain act)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_act_Id [simp]:*
```
"extend_act h Id = Id"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_act_I:*
```
"!!z z'. (z, z') ∈ act ==> (f z, f z') ∈ project_act h act"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_act_Id [simp]: "project_act h Id = Id"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *Domain_project_act:*
```
"Domain (project_act h act) = project_set h (Domain act)"
```
⟨*proof*⟩

## 10.7 extend

Basic properties

**lemma** *Init_extend [simp]:*
```
"Init (extend h F) = extend_set h (Init F)"
```
⟨*proof*⟩

**lemma** *Init_project [simp]:*
```
"Init (project h C F) = project_set h (Init F)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *Acts_extend [simp]:*
```
"Acts (extend h F) = (extend_act h ` Acts F)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *AllowedActs_extend [simp]:*

```
     "AllowedActs (extend h F) = project_act h -‘ AllowedActs F"
```
⟨*proof*⟩

**lemma** *Acts_project [simp]:*
```
     "Acts(project h C F) = insert Id (project_act h ‘ Restrict C ‘ Acts F)"
```
⟨*proof*⟩

**lemma** **(in** *Extend)* *AllowedActs_project [simp]:*
```
     "AllowedActs(project h C F) =
        {act. Restrict (project_set h C) act
               ∈ project_act h ‘ Restrict C ‘ AllowedActs F}"
```
⟨*proof*⟩

**lemma** **(in** *Extend)* *Allowed_extend:*
```
     "Allowed (extend h F) = project h UNIV -‘ Allowed F"
```
⟨*proof*⟩

**lemma** **(in** *Extend)* *extend_SKIP [simp]: "extend h SKIP = SKIP"*
⟨*proof*⟩

**lemma** *project_set_UNIV [simp]: "project_set h UNIV = UNIV"*
⟨*proof*⟩

**lemma** *project_set_Union:*
```
     "project_set h (Union A) = (⋃X ∈ A. project_set h X)"
```
⟨*proof*⟩

**lemma** **(in** *Extend)* *project_act_Restrict_subset:*
```
     "project_act h (Restrict C act) ⊆
      Restrict (project_set h C) (project_act h act)"
```
⟨*proof*⟩

**lemma** **(in** *Extend)* *project_act_Restrict_Id_eq:*
```
     "project_act h (Restrict C Id) = Restrict (project_set h C) Id"
```
⟨*proof*⟩

**lemma** **(in** *Extend)* *project_extend_eq:*
```
     "project h C (extend h F) =
      mk_program (Init F, Restrict (project_set h C) ‘ Acts F,
                  {act. Restrict (project_set h C) act
                          ∈ project_act h ‘ Restrict C ‘
                                    (project_act h -‘ AllowedActs F)})"
```
⟨*proof*⟩

**lemma** **(in** *Extend)* *extend_inverse [simp]:*
```
     "project h UNIV (extend h F) = F"
```
⟨*proof*⟩

**lemma** **(in** *Extend)* *inj_extend: "inj (extend h)"*
⟨*proof*⟩

**lemma** **(in** *Extend)* *extend_Join [simp]:*

```
    "extend h (F⊔G) = extend h F⊔extend h G"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_JN [simp]:*
```
    "extend h (JOIN I F) = (⨆i ∈ I. extend h (F i))"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_mono:* `"F ≤ G ==> extend h F ≤ extend h G"`
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_mono:* `"F ≤ G ==> project h C F ≤ project h C G"`
⟨*proof*⟩

**lemma** (**in** *Extend*) *all_total_extend:* `"all_total F ==> all_total (extend h F)"`
⟨*proof*⟩

## 10.8 Safety: co, stable

**lemma** (**in** *Extend*) *extend_constrains:*
```
    "(extend h F ∈ (extend_set h A) co (extend_set h B)) =
     (F ∈ A co B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_stable:*
```
    "(extend h F ∈ stable (extend_set h A)) = (F ∈ stable A)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_invariant:*
```
    "(extend h F ∈ invariant (extend_set h A)) = (F ∈ invariant A)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_constrains_project_set:*
```
    "extend h F ∈ A co B ==> F ∈ (project_set h A) co (project_set h B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_stable_project_set:*
```
    "extend h F ∈ stable A ==> F ∈ stable (project_set h A)"
```
⟨*proof*⟩

## 10.9 Weak safety primitives: Co, Stable

**lemma** (**in** *Extend*) *reachable_extend_f:*
```
    "p ∈ reachable (extend h F) ==> f p ∈ reachable F"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *h_reachable_extend:*
```
    "h(s,y) ∈ reachable (extend h F) ==> s ∈ reachable F"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *reachable_extend_eq:*
```
"reachable (extend h F) = extend_set h (reachable F)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_Constrains:*
```
"(extend h F ∈ (extend_set h A) Co (extend_set h B)) =
 (F ∈ A Co B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_Stable:*
```
"(extend h F ∈ Stable (extend_set h A)) = (F ∈ Stable A)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_Always:*
```
"(extend h F ∈ Always (extend_set h A)) = (F ∈ Always A)"
```
⟨*proof*⟩

**lemma** *project_act_mono:*
```
"D ⊆ C ==>
 project_act h (Restrict D act) ⊆ project_act h (Restrict C act)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_constrains_mono:*
```
"[| D ⊆ C; project h C F ∈ A co B |] ==> project h D F ∈ A co B"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_stable_mono:*
```
"[| D ⊆ C;  project h C F ∈ stable A |] ==> project h D F ∈ stable A"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_constrains:*
```
"(project h C F ∈ A co B)  =
 (F ∈ (C ∩ extend_set h A) co (extend_set h B) & A ⊆ B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_stable:*
```
"(project h UNIV F ∈ stable A) = (F ∈ stable (extend_set h A))"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_stable_I:*
```
"F ∈ stable (extend_set h A) ==> project h C F ∈ stable A"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *Int_extend_set_lemma:*
```
"A ∩ extend_set h ((project_set h A) ∩ B) = A ∩ extend_set h B"
```
⟨*proof*⟩

**lemma** *project_constrains_project_set:*
```
"G ∈ C co B ==> project h C G ∈ project_set h C co project_set h B"
```
⟨*proof*⟩

**lemma** *project_stable_project_set:*
```
"G ∈ stable C ==> project h C G ∈ stable (project_set h C)"
```
⟨*proof*⟩

## 10.10 Progress: transient, ensures

**lemma** (**in** *Extend*) *extend_transient:*
```
"(extend h F ∈ transient (extend_set h A)) = (F ∈ transient A)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_ensures:*
```
"(extend h F ∈ (extend_set h A) ensures (extend_set h B)) =
 (F ∈ A ensures B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *leadsTo_imp_extend_leadsTo:*
```
"F ∈ A leadsTo B
 ==> extend h F ∈ (extend_set h A) leadsTo (extend_set h B)"
```
⟨*proof*⟩

## 10.11 Proving the converse takes some doing!

**lemma** (**in** *Extend*) *slice_iff [iff]*: `"(x ∈ slice C y) = (h(x,y) ∈ C)"`
⟨*proof*⟩

**lemma** (**in** *Extend*) *slice_Union*: `"slice (Union S) y = (⋃x ∈ S. slice x y)"`
⟨*proof*⟩

**lemma** (**in** *Extend*) *slice_extend_set*: `"slice (extend_set h A) y = A"`
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_set_is_UN_slice:*
```
"project_set h A = (⋃y. slice A y)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_transient_slice:*
```
"extend h F ∈ transient A ==> F ∈ transient (slice A y)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_constrains_slice:*
```
"extend h F ∈ A co B ==> F ∈ (slice A y) co (slice B y)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_ensures_slice:*
```
"extend h F ∈ A ensures B ==> F ∈ (slice A y) ensures (project_set h
B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *leadsTo_slice_project_set:*

```
"∀y. F ∈ (slice B y) leadsTo CU ==> F ∈ (project_set h B) leadsTo CU"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_leadsTo_slice [rule_format]:*
```
"extend h F ∈ AU leadsTo BU
 ==> ∀y. F ∈ (slice AU y) leadsTo (project_set h BU)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_leadsTo:*
```
"(extend h F ∈ (extend_set h A) leadsTo (extend_set h B)) =
 (F ∈ A leadsTo B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_LeadsTo:*
```
"(extend h F ∈ (extend_set h A) LeadsTo (extend_set h B)) =
 (F ∈ A LeadsTo B)"
```
⟨*proof*⟩

## 10.12 preserves

**lemma** (**in** *Extend*) *project_preserves_I:*
```
"G ∈ preserves (v o f) ==> project h C G ∈ preserves v"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_preserves_id_I:*
```
"G ∈ preserves f ==> project h C G ∈ preserves id"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_preserves:*
```
"(extend h G ∈ preserves (v o f)) = (G ∈ preserves v)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *inj_extend_preserves: "inj h ==> (extend h G ∈ preserves g)"*
⟨*proof*⟩

## 10.13 Guarantees

**lemma** (**in** *Extend*) *project_extend_Join:*
```
"project h UNIV ((extend h F)⊔G) = F⊔(project h UNIV G)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_Join_eq_extend_D:*
```
"(extend h F)⊔G = extend h H ==> H = F⊔(project h UNIV G)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *ok_extend_imp_ok_project:*
```
"extend h F ok G ==> F ok project h UNIV G"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *ok_extend_iff: "(extend h F ok extend h G) = (F ok G)"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *OK_extend_iff: "OK I (%i. extend h (F i)) = (OK I F)"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *guarantees_imp_extend_guarantees:*
*"F ∈ X guarantees Y ==>*
*extend h F ∈ (extend h ‘ X) guarantees (extend h ‘ Y)"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_guarantees_imp_guarantees:*
*"extend h F ∈ (extend h ‘ X) guarantees (extend h ‘ Y)*
*==> F ∈ X guarantees Y"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_guarantees_eq:*
*"(extend h F ∈ (extend h ‘ X) guarantees (extend h ‘ Y)) =*
*(F ∈ X guarantees Y)"*
⟨*proof*⟩

**end**

# 11 Renaming of State Sets

**theory** *Rename* **imports** *Extend* **begin**

**constdefs**

*rename :: "['a => 'b, 'a program] => 'b program"*
*"rename h == extend (%(x,u::unit). h x)"*

**declare** *image_inv_f_f [simp] image_surj_f_inv_f [simp]*

**declare** *Extend.intro [simp,intro]*

**lemma** *good_map_bij [simp,intro]: "bij h ==> good_map (%(x,u). h x)"*
⟨*proof*⟩

**lemma** *fst_o_inv_eq_inv: "bij h ==> fst (inv (%(x,u). h x) s) = inv h s"*
⟨*proof*⟩

**lemma** *mem_rename_set_iff: "bij h ==> z ∈ h‘A = (inv h z ∈ A)"*
⟨*proof*⟩

**lemma** *extend_set_eq_image [simp]: "extend_set (%(x,u). h x) A = h‘A"*
⟨*proof*⟩

**lemma** *Init_rename [simp]: "Init (rename h F) = h‘(Init F)"*
⟨*proof*⟩

## 11.1 inverse properties

**lemma** *extend_set_inv:*
*"bij h*
*==> extend_set (%(x,u::'c). inv h x) = project_set (%(x,u::'c). h x)"*
⟨*proof*⟩

**lemma** *bij_extend_act_eq_project_act: "bij h*
*==> extend_act (%(x,u::'c). h x) = project_act (%(x,u::'c). inv h x)"*
⟨*proof*⟩

**lemma** *bij_extend_act: "bij h ==> bij (extend_act (%(x,u::'c). h x))"*
⟨*proof*⟩

**lemma** *bij_project_act: "bij h ==> bij (project_act (%(x,u::'c). h x))"*
⟨*proof*⟩

**lemma** *bij_inv_project_act_eq: "bij h ==> inv (project_act (%(x,u::'c). inv h x)) =*
*project_act (%(x,u::'c). h x)"*
⟨*proof*⟩

**lemma** *extend_inv: "bij h*
*==> extend (%(x,u::'c). inv h x) = project (%(x,u::'c). h x) UNIV"*
⟨*proof*⟩

**lemma** *rename_inv_rename [simp]: "bij h ==> rename (inv h) (rename h F) = F"*
⟨*proof*⟩

**lemma** *rename_rename_inv [simp]: "bij h ==> rename h (rename (inv h) F) = F"*
⟨*proof*⟩

**lemma** *rename_inv_eq: "bij h ==> rename (inv h) = inv (rename h)"*
⟨*proof*⟩

**lemma** *bij_extend: "bij h ==> bij (extend (%(x,u::'c). h x))"*
⟨*proof*⟩

**lemma** *bij_project: "bij h ==> bij (project (%(x,u::'c). h x) UNIV)"*
⟨*proof*⟩

**lemma** *inv_project_eq:*
*"bij h*
*==> inv (project (%(x,u::'c). h x) UNIV) = extend (%(x,u::'c). h x)"*
⟨*proof*⟩

**lemma** *Allowed_rename [simp]:*
*"bij h ==> Allowed (rename h F) = rename h ` Allowed F"*

⟨*proof*⟩

**lemma** *bij_rename: "bij h ==> bij (rename h)"*
⟨*proof*⟩
**lemmas** *surj_rename = bij_rename [THEN bij_is_surj, standard]*

**lemma** *inj_rename_imp_inj: "inj (rename h) ==> inj h"*
⟨*proof*⟩

**lemma** *surj_rename_imp_surj: "surj (rename h) ==> surj h"*
⟨*proof*⟩

**lemma** *bij_rename_imp_bij: "bij (rename h) ==> bij h"*
⟨*proof*⟩

**lemma** *bij_rename_iff [simp]: "bij (rename h) = bij h"*
⟨*proof*⟩

## 11.2 the lattice operations

**lemma** *rename_SKIP [simp]: "bij h ==> rename h SKIP = SKIP"*
⟨*proof*⟩

**lemma** *rename_Join [simp]:*
*"bij h ==> rename h (F Join G) = rename h F Join rename h G"*
⟨*proof*⟩

**lemma** *rename_JN [simp]:*
*"bij h ==> rename h (JOIN I F) = (⨆i ∈ I. rename h (F i))"*
⟨*proof*⟩

## 11.3 Strong Safety: co, stable

**lemma** *rename_constrains:*
*"bij h ==> (rename h F ∈ (h‘A) co (h‘B)) = (F ∈ A co B)"*
⟨*proof*⟩

**lemma** *rename_stable:*
*"bij h ==> (rename h F ∈ stable (h‘A)) = (F ∈ stable A)"*
⟨*proof*⟩

**lemma** *rename_invariant:*
*"bij h ==> (rename h F ∈ invariant (h‘A)) = (F ∈ invariant A)"*
⟨*proof*⟩

**lemma** *rename_increasing:*
*"bij h ==> (rename h F ∈ increasing func) = (F ∈ increasing (func o h))"*
⟨*proof*⟩

## 11.4 Weak Safety: Co, Stable

**lemma** *reachable_rename_eq:*
*"bij h ==> reachable (rename h F) = h ‘ (reachable F)"*

⟨*proof*⟩

**lemma** *rename_Constrains:*
```
"bij h ==> (rename h F ∈ (h`A) Co (h`B)) = (F ∈ A Co B)"
```
⟨*proof*⟩

**lemma** *rename_Stable:*
```
"bij h ==> (rename h F ∈ Stable (h`A)) = (F ∈ Stable A)"
```
⟨*proof*⟩

**lemma** *rename_Always:*
```
"bij h ==> (rename h F ∈ Always (h`A)) = (F ∈ Always A)"
```
⟨*proof*⟩

**lemma** *rename_Increasing:*
```
"bij h ==> (rename h F ∈ Increasing func) = (F ∈ Increasing (func o h))"
```
⟨*proof*⟩

## 11.5 Progress: transient, ensures

**lemma** *rename_transient:*
```
"bij h ==> (rename h F ∈ transient (h`A)) = (F ∈ transient A)"
```
⟨*proof*⟩

**lemma** *rename_ensures:*
```
"bij h ==> (rename h F ∈ (h`A) ensures (h`B)) = (F ∈ A ensures B)"
```
⟨*proof*⟩

**lemma** *rename_leadsTo:*
```
"bij h ==> (rename h F ∈ (h`A) leadsTo (h`B)) = (F ∈ A leadsTo B)"
```
⟨*proof*⟩

**lemma** *rename_LeadsTo:*
```
"bij h ==> (rename h F ∈ (h`A) LeadsTo (h`B)) = (F ∈ A LeadsTo B)"
```
⟨*proof*⟩

**lemma** *rename_rename_guarantees_eq:*
```
"bij h ==> (rename h F ∈ (rename h ` X) guarantees
                          (rename h ` Y)) =
           (F ∈ X guarantees Y)"
```
⟨*proof*⟩

**lemma** *rename_guarantees_eq_rename_inv:*
```
"bij h ==> (rename h F ∈ X guarantees Y) =
           (F ∈ (rename (inv h) ` X) guarantees
                (rename (inv h) ` Y))"
```
⟨*proof*⟩

**lemma** *rename_preserves:*
```
"bij h ==> (rename h G ∈ preserves v) = (G ∈ preserves (v o h))"
```
⟨*proof*⟩

**lemma** *ok_rename_iff [simp]:*
```
"bij h ==> (rename h F ok rename h G) = (F ok
```

```
G)"
```
⟨*proof*⟩

**lemma** `OK_rename_iff [simp]: "bij h ==> OK I (%i. rename h (F i)) = (OK I F)"`
⟨*proof*⟩

## 11.6 "image" versions of the rules, for lifting "guarantees" properties

**lemmas** `bij_eq_rename = surj_rename [THEN surj_f_inv_f, symmetric]`

**lemma** `rename_image_constrains:`
```
    "bij h ==> rename h ` (A co B) = (h ` A) co (h`B)"
```
⟨*proof*⟩

**lemma** `rename_image_stable: "bij h ==> rename h ` stable A = stable (h ` A)"`
⟨*proof*⟩

**lemma** `rename_image_increasing:`
```
    "bij h ==> rename h ` increasing func = increasing (func o inv h)"
```
⟨*proof*⟩

**lemma** `rename_image_invariant:`
```
    "bij h ==> rename h ` invariant A = invariant (h ` A)"
```
⟨*proof*⟩

**lemma** `rename_image_Constrains:`
```
    "bij h ==> rename h ` (A Co B) = (h ` A) Co (h`B)"
```
⟨*proof*⟩

**lemma** `rename_image_preserves:`
```
    "bij h ==> rename h ` preserves v = preserves (v o inv h)"
```
⟨*proof*⟩

**lemma** `rename_image_Stable:`
```
    "bij h ==> rename h ` Stable A = Stable (h ` A)"
```
⟨*proof*⟩

**lemma** `rename_image_Increasing:`
```
    "bij h ==> rename h ` Increasing func = Increasing (func o inv h)"
```
⟨*proof*⟩

**lemma** `rename_image_Always: "bij h ==> rename h ` Always A = Always (h ` A)"`
⟨*proof*⟩

**lemma** `rename_image_leadsTo:`
```
    "bij h ==> rename h ` (A leadsTo B) = (h ` A) leadsTo (h`B)"
```
⟨*proof*⟩

**lemma** `rename_image_LeadsTo:`
```
    "bij h ==> rename h ` (A LeadsTo B) = (h ` A) LeadsTo (h`B)"
```
⟨*proof*⟩

**end**

# 12 Replication of Components

**theory** *Lift_prog* **imports** *Rename* **begin**

**constdefs**

```
  insert_map :: "[nat, 'b, nat=>'b] => (nat=>'b)"
    "insert_map i z f k == if k<i then f k
                           else if k=i then z
                           else f(k - 1)"

  delete_map :: "[nat, nat=>'b] => (nat=>'b)"
    "delete_map i g k == if k<i then g k else g (Suc k)"

  lift_map :: "[nat, 'b * ((nat=>'b) * 'c)] => (nat=>'b) * 'c"
    "lift_map i == %(s,(f,uu)). (insert_map i s f, uu)"

  drop_map :: "[nat, (nat=>'b) * 'c] => 'b * ((nat=>'b) * 'c)"
    "drop_map i == %(g, uu). (g i, (delete_map i g, uu))"

  lift_set :: "[nat, ('b * ((nat=>'b) * 'c)) set] => ((nat=>'b) * 'c) set"
    "lift_set i A == lift_map i ` A"

  lift :: "[nat, ('b * ((nat=>'b) * 'c)) program] => ((nat=>'b) * 'c) program"
    "lift i == rename (lift_map i)"


  sub :: "['a, 'a=>'b] => 'b"
    "sub == %i f. f i"
```

**declare** *insert_map_def [simp] delete_map_def [simp]*

**lemma** *insert_map_inverse: "delete_map i (insert_map i x f) = f"*
⟨*proof*⟩

**lemma** *insert_map_delete_map_eq: "(insert_map i x (delete_map i g)) = g(i:=x)"*
⟨*proof*⟩

## 12.1 Injectiveness proof

**lemma** *insert_map_inject1: "(insert_map i x f) = (insert_map i y g) ==> x=y"*
⟨*proof*⟩

**lemma** *insert_map_inject2: "(insert_map i x f) = (insert_map i y g) ==> f=g"*
⟨*proof*⟩

**lemma** *insert_map_inject':*
*"(insert_map i x f) = (insert_map i y g) ==> x=y & f=g"*
⟨*proof*⟩

**lemmas** *insert_map_inject = insert_map_inject' [THEN conjE, elim!]*

**lemma** *lift_map_eq_iff [iff]:*
```
"(lift_map i (s,(f,uu)) = lift_map i' (s',(f',uu')))
 = (uu = uu' & insert_map i s f = insert_map i' s' f')"
```
⟨*proof*⟩

**lemma** *drop_map_lift_map_eq [simp]: "!!s. drop_map i (lift_map i s) = s"*
⟨*proof*⟩

**lemma** *inj_lift_map: "inj (lift_map i)"*
⟨*proof*⟩

## 12.2 Surjectiveness proof

**lemma** *lift_map_drop_map_eq [simp]: "!!s. lift_map i (drop_map i s) = s"*
⟨*proof*⟩

**lemma** *drop_map_inject [dest!]: "(drop_map i s) = (drop_map i s') ==> s=s'"*
⟨*proof*⟩

**lemma** *surj_lift_map: "surj (lift_map i)"*
⟨*proof*⟩

**lemma** *bij_lift_map [iff]: "bij (lift_map i)"*
⟨*proof*⟩

**lemma** *inv_lift_map_eq [simp]: "inv (lift_map i) = drop_map i"*
⟨*proof*⟩

**lemma** *inv_drop_map_eq [simp]: "inv (drop_map i) = lift_map i"*
⟨*proof*⟩

**lemma** *bij_drop_map [iff]: "bij (drop_map i)"*
⟨*proof*⟩

**lemma** *sub_apply [simp]: "sub i f = f i"*
⟨*proof*⟩

**lemma** *all_total_lift: "all_total F ==> all_total (lift i F)"*
⟨*proof*⟩

**lemma** *insert_map_upd_same: "(insert_map i t f)(i := s) = insert_map i s f"*
⟨*proof*⟩

**lemma** *insert_map_upd:*
```
"(insert_map j t f)(i := s) =
 (if i=j then insert_map i s f
  else if i<j then insert_map j t (f(i:=s))
  else insert_map j t (f(i - Suc 0 := s)))"
```

⟨*proof*⟩

**lemma** *insert_map_eq_diff:*
```
"[| insert_map i s f = insert_map j t g;  i≠j |]
 ==> ∃g'. insert_map i s' f = insert_map j t g'"
```
⟨*proof*⟩

**lemma** *lift_map_eq_diff:*
```
"[| lift_map i (s,(f,uu)) = lift_map j (t,(g,vv));  i≠j |]
 ==> ∃g'. lift_map i (s',(f,uu)) = lift_map j (t,(g',vv))"
```
⟨*proof*⟩

## 12.3 The Operator *lift_set*

**lemma** *lift_set_empty [simp]: "lift_set i {} = {}"*
⟨*proof*⟩

**lemma** *lift_set_iff: "(lift_map i x ∈ lift_set i A) = (x ∈ A)"*
⟨*proof*⟩

**lemma** *lift_set_iff2 [iff]:*
```
"((f,uu) ∈ lift_set i A) = ((f i, (delete_map i f, uu)) ∈ A)"
```
⟨*proof*⟩

**lemma** *lift_set_mono: "A ⊆ B ==> lift_set i A ⊆ lift_set i B"*
⟨*proof*⟩

**lemma** *lift_set_Un_distrib: "lift_set i (A ∪ B) = lift_set i A ∪ lift_set i B"*
⟨*proof*⟩

**lemma** *lift_set_Diff_distrib: "lift_set i (A-B) = lift_set i A - lift_set i B"*
⟨*proof*⟩

## 12.4 The Lattice Operations

**lemma** *bij_lift [iff]: "bij (lift i)"*
⟨*proof*⟩

**lemma** *lift_SKIP [simp]: "lift i SKIP = SKIP"*
⟨*proof*⟩

**lemma** *lift_Join [simp]: "lift i (F Join G) = lift i F Join lift i G"*
⟨*proof*⟩

**lemma** *lift_JN [simp]: "lift j (JOIN I F) = (⨆i ∈ I. lift j (F i))"*
⟨*proof*⟩

## 12.5 Safety: constrains, stable, invariant

**lemma** *lift_constrains:*

```
    "(lift i F ∈ (lift_set i A) co (lift_set i B)) = (F ∈ A co B)"
```
⟨*proof*⟩

**lemma** *lift_stable:*
```
    "(lift i F ∈ stable (lift_set i A)) = (F ∈ stable A)"
```
⟨*proof*⟩

**lemma** *lift_invariant:*
```
    "(lift i F ∈ invariant (lift_set i A)) = (F ∈ invariant A)"
```
⟨*proof*⟩

**lemma** *lift_Constrains:*
```
    "(lift i F ∈ (lift_set i A) Co (lift_set i B)) = (F ∈ A Co B)"
```
⟨*proof*⟩

**lemma** *lift_Stable:*
```
    "(lift i F ∈ Stable (lift_set i A)) = (F ∈ Stable A)"
```
⟨*proof*⟩

**lemma** *lift_Always:*
```
    "(lift i F ∈ Always (lift_set i A)) = (F ∈ Always A)"
```
⟨*proof*⟩

## 12.6 Progress: transient, ensures

**lemma** *lift_transient:*
```
    "(lift i F ∈ transient (lift_set i A)) = (F ∈ transient A)"
```
⟨*proof*⟩

**lemma** *lift_ensures:*
```
    "(lift i F ∈ (lift_set i A) ensures (lift_set i B)) =
     (F ∈ A ensures B)"
```
⟨*proof*⟩

**lemma** *lift_leadsTo:*
```
    "(lift i F ∈ (lift_set i A) leadsTo (lift_set i B)) =
     (F ∈ A leadsTo B)"
```
⟨*proof*⟩

**lemma** *lift_LeadsTo:*
```
    "(lift i F ∈ (lift_set i A) LeadsTo (lift_set i B)) =
     (F ∈ A LeadsTo B)"
```
⟨*proof*⟩

**lemma** *lift_lift_guarantees_eq:*
```
    "(lift i F ∈ (lift i ‘ X) guarantees (lift i ‘ Y)) =
     (F ∈ X guarantees Y)"
```
⟨*proof*⟩

**lemma** *lift_guarantees_eq_lift_inv:*
```
    "(lift i F ∈ X guarantees Y) =
```

```
     (F ∈ (rename (drop_map i) ‘ X) guarantees (rename (drop_map i) ‘ Y))"
```
⟨*proof*⟩

**lemma** *lift_preserves_snd_I: "F ∈ preserves snd ==> lift i F ∈ preserves snd"*
⟨*proof*⟩

**lemma** *delete_map_eqE':*
```
     "(delete_map i g) = (delete_map i g') ==> ∃x. g = g'(i:=x)"
```
⟨*proof*⟩

**lemmas** *delete_map_eqE = delete_map_eqE' [THEN exE, elim!]*

**lemma** *delete_map_neq_apply:*
```
     "[| delete_map j g = delete_map j g';  i≠j |] ==> g i = g' i"
```
⟨*proof*⟩

**lemma** *vimage_o_fst_eq [simp]: "(f o fst) -‘ A = (f-‘A) <*> UNIV"*
⟨*proof*⟩

**lemma** *vimage_sub_eq_lift_set [simp]:*
```
     "(sub i -‘A) <*> UNIV = lift_set i (A <*> UNIV)"
```
⟨*proof*⟩

**lemma** *mem_lift_act_iff [iff]:*
```
     "((s,s') ∈ extend_act (%(x,u::unit). lift_map i x) act) =
      ((drop_map i s, drop_map i s') ∈ act)"
```
⟨*proof*⟩

**lemma** *preserves_snd_lift_stable:*
```
     "[| F ∈ preserves snd;  i≠j |]
      ==> lift j F ∈ stable (lift_set i (A <*> UNIV))"
```
⟨*proof*⟩

**lemma** *constrains_imp_lift_constrains:*
```
    "[| F i ∈ (A <*> UNIV) co (B <*> UNIV);
        F j ∈ preserves snd |]
     ==> lift j (F j) ∈ (lift_set i (A <*> UNIV)) co (lift_set i (B <*> UNIV))"
```
⟨*proof*⟩

**lemma** *lift_map_image_Times:*
```
     "lift_map i ‘ (A <*> UNIV) =
      (⋃s ∈ A. ⋃f. {insert_map i s f}) <*> UNIV"
```
⟨*proof*⟩

**lemma** *lift_preserves_eq:*
```
     "(lift i F ∈ preserves v) = (F ∈ preserves (v o lift_map i))"
```
⟨*proof*⟩

**lemma** *lift_preserves_sub:*
```
"F ∈ preserves snd
 ==> lift i F ∈ preserves (v o sub j o fst) =
     (if i=j then F ∈ preserves (v o fst) else True)"
```
⟨*proof*⟩

## 12.7 Lemmas to Handle Function Composition (o) More Consistently

**lemma** *o_equiv_assoc:* `"f o g = h ==> f' o f o g = f' o h"`
⟨*proof*⟩

**lemma** *o_equiv_apply:* `"f o g = h ==> ∀x. f(g x) = h x"`
⟨*proof*⟩

**lemma** *fst_o_lift_map:* `"sub i o fst o lift_map i = fst"`
⟨*proof*⟩

**lemma** *snd_o_lift_map:* `"snd o lift_map i = snd o snd"`
⟨*proof*⟩

## 12.8 More lemmas about extend and project

They could be moved to theory Extend or Project

**lemma** *extend_act_extend_act:*
```
"extend_act h' (extend_act h act) =
 extend_act (%(x,(y,y')). h'(h(x,y),y')) act"
```
⟨*proof*⟩

**lemma** *project_act_project_act:*
```
"project_act h (project_act h' act) =
 project_act (%(x,(y,y')). h'(h(x,y),y')) act"
```
⟨*proof*⟩

**lemma** *project_act_extend_act:*
```
"project_act h (extend_act h' act) =
   {(x,x'). ∃s s' y y' z. (s,s') ∈ act &
            h(x,y) = h'(s,z) & h(x',y') = h'(s',z)}"
```
⟨*proof*⟩

## 12.9 OK and "lift"

**lemma** *act_in_UNION_preserves_fst:*
```
"act ⊆ {(x,x'). fst x = fst x'} ==> act ∈ UNION (preserves fst) Acts"
```
⟨*proof*⟩

**lemma** *UNION_OK_lift_I:*
```
"[| ∀i ∈ I. F i ∈ preserves snd;
    ∀i ∈ I. UNION (preserves fst) Acts ⊆ AllowedActs (F i) |]
 ==> OK I (%i. lift i (F i))"
```
⟨*proof*⟩

**lemma** *OK_lift_I:*
```
"[| ∀i ∈ I. F i ∈ preserves snd;
    ∀i ∈ I. preserves fst ⊆ Allowed (F i) |]
 ==> OK I (%i. lift i (F i))"
```
⟨*proof*⟩

**lemma** *Allowed_lift [simp]: "Allowed (lift i F) = lift i ‘ (Allowed F)"*
⟨*proof*⟩

**lemma** *lift_image_preserves:*
```
"lift i ‘ preserves v = preserves (v o drop_map i)"
```
⟨*proof*⟩

**end**

**theory** *PPROD* **imports** *Lift_prog* **begin**

**constdefs**

```
PLam  :: "[nat set, nat => ('b * ((nat=>'b) * 'c)) program]
          => ((nat=>'b) * 'c) program"
  "PLam I F == ⨆i ∈ I. lift i (F i)"
```

**syntax**
```
"@PLam" :: "[pttrn, nat set, 'b set] => (nat => 'b) set"
            ("(3plam _:_./ _)" 10)
```

**translations**
```
"plam x : A. B"   == "PLam A (%x. B)"
```

**lemma** *Init_PLam [simp]: "Init (PLam I F) = (⋂i ∈ I. lift_set i (Init (F i)))"*
⟨*proof*⟩

**lemma** *PLam_empty [simp]: "PLam {} F = SKIP"*
⟨*proof*⟩

**lemma** *PLam_SKIP [simp]: "(plam i : I. SKIP) = SKIP"*
⟨*proof*⟩

**lemma** *PLam_insert: "PLam (insert i I) F = (lift i (F i)) Join (PLam I F)"*
⟨*proof*⟩

**lemma** *PLam_component_iff: "((PLam I F) ≤ H) = (∀i ∈ I. lift i (F i) ≤ H)"*
⟨*proof*⟩

**lemma** *component_PLam: "i ∈ I ==> lift i (F i) ≤ (PLam I F)"*

⟨*proof*⟩

**lemma** *PLam_constrains:*
```
"[| i ∈ I;  ∀j. F j ∈ preserves snd |]
 ==> (PLam I F ∈ (lift_set i (A <*> UNIV)) co
                  (lift_set i (B <*> UNIV)))  =
     (F i ∈ (A <*> UNIV) co (B <*> UNIV))"
```
⟨*proof*⟩

**lemma** *PLam_stable:*
```
"[| i ∈ I;  ∀j. F j ∈ preserves snd |]
 ==> (PLam I F ∈ stable (lift_set i (A <*> UNIV))) =
     (F i ∈ stable (A <*> UNIV))"
```
⟨*proof*⟩

**lemma** *PLam_transient:*
```
"i ∈ I ==>
PLam I F ∈ transient A = (∃i ∈ I. lift i (F i) ∈ transient A)"
```
⟨*proof*⟩

This holds because the `F j` cannot change `lift_set i`

**lemma** *PLam_ensures:*
```
"[| i ∈ I;  F i ∈ (A <*> UNIV) ensures (B <*> UNIV);
    ∀j. F j ∈ preserves snd |]
 ==> PLam I F ∈ lift_set i (A <*> UNIV) ensures lift_set i (B <*> UNIV)"
```
⟨*proof*⟩

**lemma** *PLam_leadsTo_Basis:*
```
"[| i ∈ I;
    F i ∈ ((A <*> UNIV) - (B <*> UNIV)) co
          ((A <*> UNIV) ∪ (B <*> UNIV));
    F i ∈ transient ((A <*> UNIV) - (B <*> UNIV));
    ∀j. F j ∈ preserves snd |]
 ==> PLam I F ∈ lift_set i (A <*> UNIV) leadsTo lift_set i (B <*> UNIV)"
```
⟨*proof*⟩

**lemma** *invariant_imp_PLam_invariant:*
```
"[| F i ∈ invariant (A <*> UNIV);  i ∈ I;
    ∀j. F j ∈ preserves snd |]
 ==> PLam I F ∈ invariant (lift_set i (A <*> UNIV))"
```
⟨*proof*⟩

**lemma** *PLam_preserves_fst [simp]:*
```
"∀j. F j ∈ preserves snd
 ==> (PLam I F ∈ preserves (v o sub j o fst)) =
     (if j ∈ I then F j ∈ preserves (v o fst) else True)"
```

⟨*proof*⟩

**lemma** *PLam_preserves_snd [simp,intro]:*
```
"∀j. F j ∈ preserves snd ==> PLam I F ∈ preserves snd"
```
⟨*proof*⟩

This rule looks unsatisfactory because it refers to *lift*. One must use *lift_guarantees_eq_lift_inv* to rewrite the first subgoal and something like *lift_preserves_sub* to rewrite the third. However there's no obvious alternative for the third premise.

**lemma** *guarantees_PLam_I:*
```
"[| lift i (F i): X guarantees Y;  i ∈ I;
    OK I (%i. lift i (F i)) |]
 ==> (PLam I F) ∈ X guarantees Y"
```
⟨*proof*⟩

**lemma** *Allowed_PLam [simp]:*
```
"Allowed (PLam I F) = (⋂i ∈ I. lift i ‘ Allowed(F i))"
```
⟨*proof*⟩

**lemma** *PLam_preserves [simp]:*
```
"(PLam I F) ∈ preserves v = (∀i ∈ I. F i ∈ preserves (v o lift_map
i))"
```
⟨*proof*⟩

**end**

# 13 The Prefix Ordering on Lists

**theory** *ListOrder* **imports** *Main* **begin**

**inductive_set**
```
  genPrefix :: "('a * 'a)set => ('a list * 'a list)set"
  for r :: "('a * 'a)set"
 where
   Nil:     "([],[]) : genPrefix(r)"

 | prepend: "[| (xs,ys) : genPrefix(r);  (x,y) : r |] ==>
             (x#xs, y#ys) : genPrefix(r)"

 | append:  "(xs,ys) : genPrefix(r) ==> (xs, ys@zs) : genPrefix(r)"
```

**instance** *list :: (type)ord* ⟨*proof*⟩

**defs**

```
  prefix_def:        "xs <= zs  ==  (xs,zs) : genPrefix Id"

  strict_prefix_def: "xs < zs  ==  xs <= zs & xs ~= (zs::'a list)"
```

**constdefs**

```
  Le :: "(nat*nat) set"
    "Le == {(x,y). x <= y}"

  Ge :: "(nat*nat) set"
    "Ge == {(x,y). y <= x}"
```

**abbreviation**

```
  pfixLe :: "[nat list, nat list] => bool"  (infixl "pfixLe" 50)  where
  "xs pfixLe ys == (xs,ys) : genPrefix Le"
```

**abbreviation**

```
  pfixGe :: "[nat list, nat list] => bool"  (infixl "pfixGe" 50)  where
  "xs pfixGe ys == (xs,ys) : genPrefix Ge"
```

## 13.1 preliminary lemmas

**lemma** *Nil_genPrefix [iff]: "([], xs) : genPrefix r"*
⟨*proof*⟩

**lemma** *genPrefix_length_le: "(xs,ys) : genPrefix r ==> length xs <= length ys"*
⟨*proof*⟩

**lemma** *cdlemma:*

```
     "[| (xs', ys'): genPrefix r |]
      ==> (ALL x xs. xs' = x#xs --> (EX y ys. ys' = y#ys & (x,y) : r & (xs, ys) : genPrefix r))"
```

⟨*proof*⟩

**lemma** *cons_genPrefixE [elim!]:*

```
     "[| (x#xs, zs): genPrefix r;
         !!y ys. [| zs = y#ys;  (x,y) : r;  (xs, ys) : genPrefix r |] ==> P

      |] ==> P"
```

⟨*proof*⟩

**lemma** *Cons_genPrefix_Cons [iff]:*

```
     "((x#xs,y#ys) : genPrefix r) = ((x,y) : r & (xs,ys) : genPrefix r)"
```

⟨*proof*⟩

## 13.2 genPrefix is a partial order

**lemma** *refl_genPrefix: "reflexive r ==> reflexive (genPrefix r)"*

⟨*proof*⟩

**lemma** *genPrefix_refl [simp]: "reflexive r ==> (l,l) : genPrefix r"*
⟨*proof*⟩

**lemma** *genPrefix_mono: "r<=s ==> genPrefix r <= genPrefix s"*
⟨*proof*⟩

**lemma** *append_genPrefix [rule_format]:*
*"ALL zs. (xs @ ys, zs) : genPrefix r --> (xs, zs) : genPrefix r"*
⟨*proof*⟩

**lemma** *genPrefix_trans_O [rule_format]:*
*"(x, y) : genPrefix r*
*==> ALL z. (y,z) : genPrefix s --> (x, z) : genPrefix (s O r)"*
⟨*proof*⟩

**lemma** *genPrefix_trans [rule_format]:*
*"[| (x,y) : genPrefix r; (y,z) : genPrefix r; trans r |]*
*==> (x,z) : genPrefix r"*
⟨*proof*⟩

**lemma** *prefix_genPrefix_trans [rule_format]:*
*"[| x<=y; (y,z) : genPrefix r |] ==> (x, z) : genPrefix r"*
⟨*proof*⟩

**lemma** *genPrefix_prefix_trans [rule_format]:*
*"[| (x,y) : genPrefix r; y<=z |] ==> (x,z) : genPrefix r"*
⟨*proof*⟩

**lemma** *trans_genPrefix: "trans r ==> trans (genPrefix r)"*
⟨*proof*⟩

**lemma** *genPrefix_antisym [rule_format]:*
*"[| (xs,ys) : genPrefix r; antisym r |]*
*==> (ys,xs) : genPrefix r --> xs = ys"*
⟨*proof*⟩

**lemma** *antisym_genPrefix: "antisym r ==> antisym (genPrefix r)"*
⟨*proof*⟩

## 13.3 recursion equations

**lemma** *genPrefix_Nil [simp]: "((xs, []) : genPrefix r) = (xs = [])"*
⟨*proof*⟩

**lemma** *same_genPrefix_genPrefix [simp]:*
*"reflexive r ==> ((xs@ys, xs@zs) : genPrefix r) = ((ys,zs) : genPrefix*

```
r)"
```
⟨*proof*⟩

**lemma** *genPrefix_Cons:*
```
     "((xs, y#ys) : genPrefix r) =
      (xs=[] | (EX z zs. xs=z#zs & (z,y) : r & (zs,ys) : genPrefix r))"
```
⟨*proof*⟩

**lemma** *genPrefix_take_append:*
```
     "[| reflexive r;  (xs,ys) : genPrefix r |]
      ==>  (xs@zs, take (length xs) ys @ zs) : genPrefix r"
```
⟨*proof*⟩

**lemma** *genPrefix_append_both:*
```
     "[| reflexive r;  (xs,ys) : genPrefix r;  length xs = length ys |]
      ==>  (xs@zs, ys @ zs) : genPrefix r"
```
⟨*proof*⟩

**lemma** *append_cons_eq:* `"xs @ y # ys = (xs @ [y]) @ ys"`
⟨*proof*⟩

**lemma** *aolemma:*
```
     "[| (xs,ys) : genPrefix r;  reflexive r |]
      ==> length xs < length ys --> (xs @ [ys ! length xs], ys) : genPrefix
r"
```
⟨*proof*⟩

**lemma** *append_one_genPrefix:*
```
     "[| (xs,ys) : genPrefix r;  length xs < length ys;  reflexive r |]
      ==> (xs @ [ys ! length xs], ys) : genPrefix r"
```
⟨*proof*⟩

**lemma** *genPrefix_imp_nth [rule_format]:*
```
     "ALL i ys. i < length xs
                --> (xs, ys) : genPrefix r --> (xs ! i, ys ! i) : r"
```
⟨*proof*⟩

**lemma** *nth_imp_genPrefix [rule_format]:*
```
     "ALL ys. length xs <= length ys
      --> (ALL i. i < length xs --> (xs ! i, ys ! i) : r)
      --> (xs, ys) : genPrefix r"
```
⟨*proof*⟩

**lemma** *genPrefix_iff_nth:*
```
     "((xs,ys) : genPrefix r) =
      (length xs <= length ys & (ALL i. i < length xs --> (xs!i, ys!i) : r))"
```
⟨*proof*⟩

## 13.4 The type of lists is partially ordered

**declare** *reflexive_Id [iff]*
*antisym_Id [iff]*
*trans_Id [iff]*

**lemma** *prefix_refl [iff]: "xs <= (xs::'a list)"*
⟨*proof*⟩

**lemma** *prefix_trans: "!!xs::'a list. [| xs <= ys; ys <= zs |] ==> xs <= zs"*
⟨*proof*⟩

**lemma** *prefix_antisym: "!!xs::'a list. [| xs <= ys; ys <= xs |] ==> xs = ys"*
⟨*proof*⟩

**lemma** *prefix_less_le: "!!xs::'a list. (xs < zs) = (xs <= zs & xs ~= zs)"*
⟨*proof*⟩

**instance** *list :: (type) order*
⟨*proof*⟩

**lemma** *set_mono: "xs <= ys ==> set xs <= set ys"*
⟨*proof*⟩

**lemma** *Nil_prefix [iff]: "[] <= xs"*
⟨*proof*⟩

**lemma** *prefix_Nil [simp]: "(xs <= []) = (xs = [])"*
⟨*proof*⟩

**lemma** *Cons_prefix_Cons [simp]: "(x#xs <= y#ys) = (x=y & xs<=ys)"*
⟨*proof*⟩

**lemma** *same_prefix_prefix [simp]: "(xs@ys <= xs@zs) = (ys <= zs)"*
⟨*proof*⟩

**lemma** *append_prefix [iff]: "(xs@ys <= xs) = (ys <= [])"*
⟨*proof*⟩

**lemma** *prefix_appendI [simp]: "xs <= ys ==> xs <= ys@zs"*
⟨*proof*⟩

**lemma** *prefix_Cons:*
*"(xs <= y#ys) = (xs=[] | (? zs. xs=y#zs & zs <= ys))"*
⟨*proof*⟩

**lemma** *append_one_prefix:*
*"[| xs <= ys; length xs < length ys |] ==> xs @ [ys ! length xs] <= ys"*
⟨*proof*⟩

**lemma** *prefix_length_le: "xs <= ys ==> length xs <= length ys"*
⟨*proof*⟩

**lemma** *splemma: "xs<=ys ==> xs~=ys --> length xs < length ys"*
⟨*proof*⟩

**lemma** *strict_prefix_length_less: "xs < ys ==> length xs < length ys"*
⟨*proof*⟩

**lemma** *mono_length: "mono length"*
⟨*proof*⟩

**lemma** *prefix_iff: "(xs <= zs) = (EX ys. zs = xs@ys)"*
⟨*proof*⟩

**lemma** *prefix_snoc [simp]: "(xs <= ys@[y]) = (xs = ys@[y] | xs <= ys)"*
⟨*proof*⟩

**lemma** *prefix_append_iff:*
*"(xs <= ys@zs) = (xs <= ys | (? us. xs = ys@us & us <= zs))"*
⟨*proof*⟩

**lemma** *common_prefix_linear [rule_format]:*
*"!!zs::'a list. xs <= zs --> ys <= zs --> xs <= ys | ys <= xs"*
⟨*proof*⟩

## 13.5 pfixLe, pfixGe: properties inherited from the translations

**lemma** *reflexive_Le [iff]: "reflexive Le"*
⟨*proof*⟩

**lemma** *antisym_Le [iff]: "antisym Le"*
⟨*proof*⟩

**lemma** *trans_Le [iff]: "trans Le"*
⟨*proof*⟩

**lemma** *pfixLe_refl [iff]: "x pfixLe x"*
⟨*proof*⟩

**lemma** *pfixLe_trans: "[| x pfixLe y; y pfixLe z |] ==> x pfixLe z"*
⟨*proof*⟩

**lemma** *pfixLe_antisym: "[| x pfixLe y; y pfixLe x |] ==> x = y"*
⟨*proof*⟩

**lemma** *prefix_imp_pfixLe: "xs<=ys ==> xs pfixLe ys"*
⟨*proof*⟩

**lemma** *reflexive_Ge [iff]: "reflexive Ge"*
⟨*proof*⟩

**lemma** *antisym_Ge [iff]: "antisym Ge"*
⟨*proof*⟩

**lemma** *trans_Ge [iff]: "trans Ge"*
⟨*proof*⟩

**lemma** *pfixGe_refl [iff]: "x pfixGe x"*
⟨*proof*⟩

**lemma** *pfixGe_trans: "[| x pfixGe y; y pfixGe z |] ==> x pfixGe z"*
⟨*proof*⟩

**lemma** *pfixGe_antisym: "[| x pfixGe y; y pfixGe x |] ==> x = y"*
⟨*proof*⟩

**lemma** *prefix_imp_pfixGe: "xs<=ys ==> xs pfixGe ys"*
⟨*proof*⟩

**end**

# 14 Multisets

**theory** *Multiset*
**imports** *List*
**begin**

## 14.1 The type of multisets

**typedef** *'a multiset = "{f::'a => nat. finite {x . f x > 0}}"*
⟨*proof*⟩

**lemmas** *multiset_typedef [simp] =*
*Abs_multiset_inverse Rep_multiset_inverse Rep_multiset*
**and** *[simp] = Rep_multiset_inject [symmetric]*

**definition**
*Mempty :: "'a multiset" ("{#}")* **where**
*"{#} = Abs_multiset (λa. 0)"*

**definition**
*single :: "'a => 'a multiset"* **where**
*"single a = Abs_multiset (λb. if b = a then 1 else 0)"*

**declare**
*Mempty_def[code func del] single_def[code func del]*

**definition**
*count :: "'a multiset => 'a => nat"* **where**
*"count = Rep_multiset"*

**definition**

```
  MCollect :: "'a multiset => ('a => bool) => 'a multiset" where
  "MCollect M P = Abs_multiset (λx. if P x then Rep_multiset M x else 0)"

abbreviation
  Melem :: "'a => 'a multiset => bool"  ("(_/ :# _)" [50, 51] 50) where
  "a :# M == 0 < count M a"

notation (xsymbols)
  Melem (infix "∈#" 50)

syntax
  "_MCollect" :: "pttrn => 'a multiset => bool => 'a multiset"    ("(1{# _
:# _./ _#})")
translations
  "{#x :# M. P#}" == "CONST MCollect M (λx. P)"

definition
  set_of :: "'a multiset => 'a set" where
  "set_of M = {x. x :# M}"

instantiation multiset :: (type) "{plus, minus, zero, size}"
begin

definition
  union_def[code func del]:
  "M + N = Abs_multiset (λa. Rep_multiset M a + Rep_multiset N a)"

definition
  diff_def: "M - N = Abs_multiset (λa. Rep_multiset M a - Rep_multiset N a)"

definition
  Zero_multiset_def [simp]: "0 = {#}"

definition
  size_def[code func del]: "size M = setsum (count M) (set_of M)"

instance ⟨proof⟩

end

definition
  multiset_inter :: "'a multiset ⇒ 'a multiset ⇒ 'a multiset"  (infixl "#∩"
70) where
  "multiset_inter A B = A - (A - B)"
```

Multiset Enumeration

```
syntax
  "_multiset" :: "args => 'a multiset"    ("{#(_)#}")
translations
  "{#x, xs#}" == "{#x#} + {#xs#}"
  "{#x#}" == "CONST single x"
```

Preservation of the representing set *multiset*.

```
lemma const0_in_multiset: "(λa. 0) ∈ multiset"
```

⟨*proof*⟩

**lemma** *only1_in_multiset: "(λb. if b = a then 1 else 0) ∈ multiset"*
⟨*proof*⟩

**lemma** *union_preserves_multiset:*
*"M ∈ multiset ==> N ∈ multiset ==> (λa. M a + N a) ∈ multiset"*
⟨*proof*⟩

**lemma** *diff_preserves_multiset:*
*"M ∈ multiset ==> (λa. M a - N a) ∈ multiset"*
⟨*proof*⟩

**lemma** *MCollect_preserves_multiset:*
*"M ∈ multiset ==> (λx. if P x then M x else 0) ∈ multiset"*
⟨*proof*⟩

**lemmas** *in_multiset = const0_in_multiset only1_in_multiset*
*union_preserves_multiset diff_preserves_multiset MCollect_preserves_multiset*

## 14.2 Algebraic properties

### 14.2.1 Union

**lemma** *union_empty [simp]: "M + {#} = M ∧ {#} + M = M"*
⟨*proof*⟩

**lemma** *union_commute: "M + N = N + (M::'a multiset)"*
⟨*proof*⟩

**lemma** *union_assoc: "(M + N) + K = M + (N + (K::'a multiset))"*
⟨*proof*⟩

**lemma** *union_lcomm: "M + (N + K) = N + (M + (K::'a multiset))"*
⟨*proof*⟩

**lemmas** *union_ac = union_assoc union_commute union_lcomm*

**instance** *multiset :: (type) comm_monoid_add*
⟨*proof*⟩

### 14.2.2 Difference

**lemma** *diff_empty [simp]: "M - {#} = M ∧ {#} - M = {#}"*
⟨*proof*⟩

**lemma** *diff_union_inverse2 [simp]: "M + {#a#} - {#a#} = M"*
⟨*proof*⟩

**lemma** *diff_cancel: "A - A = {#}"*
⟨*proof*⟩

### 14.2.3 Count of elements

**lemma** *count_empty [simp]: "count {#} a = 0"*

⟨*proof*⟩

**lemma** *count_single [simp]: "count {#b#} a = (if b = a then 1 else 0)"*
⟨*proof*⟩

**lemma** *count_union [simp]: "count (M + N) a = count M a + count N a"*
⟨*proof*⟩

**lemma** *count_diff [simp]: "count (M - N) a = count M a - count N a"*
⟨*proof*⟩

**lemma** *count_MCollect [simp]:*
*"count {# x:#M. P x #} a = (if P a then count M a else 0)"*
⟨*proof*⟩

### 14.2.4 Set of elements

**lemma** *set_of_empty [simp]: "set_of {#} = {}"*
⟨*proof*⟩

**lemma** *set_of_single [simp]: "set_of {#b#} = {b}"*
⟨*proof*⟩

**lemma** *set_of_union [simp]: "set_of (M + N) = set_of M ∪ set_of N"*
⟨*proof*⟩

**lemma** *set_of_eq_empty_iff [simp]: "(set_of M = {}) = (M = {#})"*
⟨*proof*⟩

**lemma** *mem_set_of_iff [simp]: "(x ∈ set_of M) = (x :# M)"*
⟨*proof*⟩

**lemma** *set_of_MCollect [simp]: "set_of {# x:#M. P x #} = set_of M ∩ {x. P x}"*
⟨*proof*⟩

### 14.2.5 Size

**lemma** *size_empty [simp,code func]: "size {#} = 0"*
⟨*proof*⟩

**lemma** *size_single [simp,code func]: "size {#b#} = 1"*
⟨*proof*⟩

**lemma** *finite_set_of [iff]: "finite (set_of M)"*
⟨*proof*⟩

**lemma** *setsum_count_Int:*
*"finite A ==> setsum (count N) (A ∩ set_of N) = setsum (count N) A"*
⟨*proof*⟩

**lemma** *size_union[simp,code func]: "size (M + N::'a multiset) = size M + size N"*
⟨*proof*⟩

**lemma** *size_eq_0_iff_empty [iff]: "(size M = 0) = (M = {#})"*
⟨*proof*⟩

**lemma** *nonempty_has_size: "(S ≠ {#}) = (0 < size S)"*
⟨*proof*⟩

**lemma** *size_eq_Suc_imp_elem: "size M = Suc n ==> ∃a. a :# M"*
⟨*proof*⟩

### 14.2.6 Equality of multisets

**lemma** *multiset_eq_conv_count_eq: "(M = N) = (∀a. count M a = count N a)"*
⟨*proof*⟩

**lemma** *single_not_empty [simp]: "{#a#} ≠ {#} ∧ {#} ≠ {#a#}"*
⟨*proof*⟩

**lemma** *single_eq_single [simp]: "({#a#} = {#b#}) = (a = b)"*
⟨*proof*⟩

**lemma** *union_eq_empty [iff]: "(M + N = {#}) = (M = {#} ∧ N = {#})"*
⟨*proof*⟩

**lemma** *empty_eq_union [iff]: "({#} = M + N) = (M = {#} ∧ N = {#})"*
⟨*proof*⟩

**lemma** *union_right_cancel [simp]: "(M + K = N + K) = (M = (N::'a multiset))"*
⟨*proof*⟩

**lemma** *union_left_cancel [simp]: "(K + M = K + N) = (M = (N::'a multiset))"*
⟨*proof*⟩

**lemma** *union_is_single:*
*"(M + N = {#a#}) = (M = {#a#} ∧ N={#} ∨ M = {#} ∧ N = {#a#})"*
⟨*proof*⟩

**lemma** *single_is_union:*
*"({#a#} = M + N) ⟷ ({#a#} = M ∧ N = {#} ∨ M = {#} ∧ {#a#} = N)"*
⟨*proof*⟩

**lemma** *add_eq_conv_diff:*
*"(M + {#a#} = N + {#b#}) =*
*(M = N ∧ a = b ∨ M = N - {#a#} + {#b#} ∧ N = M - {#b#} + {#a#})"*
⟨*proof*⟩

**declare** *Rep_multiset_inject [symmetric, simp del]*

**instance** *multiset :: (type) cancel_ab_semigroup_add*
⟨*proof*⟩

**lemma** *insert_DiffM:*
*"x ∈# M ⟹ {#x#} + (M - {#x#}) = M"*
⟨*proof*⟩

**lemma** *insert_DiffM2[simp]:*
*"x ∈# M ⟹ M - {#x#} + {#x#} = M"*
⟨*proof*⟩

**lemma** *multi_union_self_other_eq:*
*"(A::'a multiset) + X = A + Y ⟹ X = Y"*
⟨*proof*⟩

**lemma** *multi_self_add_other_not_self[simp]: "(A = A + {#x#}) = False"*
⟨*proof*⟩

**lemma** *insert_noteq_member:*
**assumes** *BC: "B + {#b#} = C + {#c#}"*
**and** *bnotc: "b ≠ c"*
**shows** *"c ∈# B"*
⟨*proof*⟩

**lemma** *add_eq_conv_ex:*
*"(M + {#a#} = N + {#b#}) =*
*(M = N ∧ a = b ∨ (∃K. M = K + {#b#} ∧ N = K + {#a#}))"*
⟨*proof*⟩

**lemma** *empty_multiset_count:*
*"(∀x. count A x = 0) = (A = {#})"*
⟨*proof*⟩

### 14.2.7 Intersection

**lemma** *multiset_inter_count:*
*"count (A #∩ B) x = min (count A x) (count B x)"*
⟨*proof*⟩

**lemma** *multiset_inter_commute: "A #∩ B = B #∩ A"*
⟨*proof*⟩

**lemma** *multiset_inter_assoc: "A #∩ (B #∩ C) = A #∩ B #∩ C"*
⟨*proof*⟩

**lemma** *multiset_inter_left_commute: "A #∩ (B #∩ C) = B #∩ (A #∩ C)"*
⟨*proof*⟩

**lemmas** *multiset_inter_ac =*
*multiset_inter_commute*
*multiset_inter_assoc*
*multiset_inter_left_commute*

**lemma** *multiset_inter_single: "a ≠ b ⟹ {#a#} #∩ {#b#} = {#}"*
⟨*proof*⟩

**lemma** *multiset_union_diff_commute: "B #∩ C = {#} ⟹ A + B - C = A - C + B"*
⟨*proof*⟩

### 14.2.8 Comprehension (filter)

**lemma** *MCollect_empty[simp, code func]: "MCollect {#} P = {#}"*
⟨*proof*⟩

**lemma** *MCollect_single[simp, code func]:*
*"MCollect {#x#} P = (if P x then {#x#} else {#})"*
⟨*proof*⟩

**lemma** *MCollect_union[simp, code func]:*
*"MCollect (M+N) f = MCollect M f + MCollect N f"*
⟨*proof*⟩

## 14.3 Induction and case splits

**lemma** *setsum_decr:*
*"finite F ==> (0::nat) < f a ==>*
*setsum (f (a := f a - 1)) F = (if a∈F then setsum f F - 1 else setsum f F)"*
⟨*proof*⟩

**lemma** *rep_multiset_induct_aux:*
**assumes** *1: "P (λa. (0::nat))"*
**and** *2: "!!f b. f ∈ multiset ==> P f ==> P (f (b := f b + 1))"*
**shows** *"∀f. f ∈ multiset --> setsum f {x. f x ≠ 0} = n --> P f"*
⟨*proof*⟩

**theorem** *rep_multiset_induct:*
*"f ∈ multiset ==> P (λa. 0) ==>*
*(!!f b. f ∈ multiset ==> P f ==> P (f (b := f b + 1))) ==> P f"*
⟨*proof*⟩

**theorem** *multiset_induct [case_names empty add, induct type: multiset]:*
**assumes** *empty: "P {#}"*
**and** *add: "!!M x. P M ==> P (M + {#x#})"*
**shows** *"P M"*
⟨*proof*⟩

**lemma** *multi_nonempty_split: "M ≠ {#} ⟹ ∃A a. M = A + {#a#}"*
⟨*proof*⟩

**lemma** *multiset_cases [cases type, case_names empty add]:*
**assumes** *em: "M = {#} ⟹ P"*
**assumes** *add: "⋀N x. M = N + {#x#} ⟹ P"*
**shows** *"P"*
⟨*proof*⟩

**lemma** *multi_member_split: "x ∈# M ⟹ ∃A. M = A + {#x#}"*
⟨*proof*⟩

**lemma** *multiset_partition: "M = {# x:#M. P x #} + {# x:#M. ¬ P x #}"*
⟨*proof*⟩

**declare** *multiset_typedef [simp del]*

**lemma** *multi_drop_mem_not_eq: "c ∈# B ⟹ B - {#c#} ≠ B"*
⟨*proof*⟩

## 14.4 Orderings

### 14.4.1 Well-foundedness

**definition**
*mult1 :: "('a × 'a) set => ('a multiset × 'a multiset) set"* **where**
*"mult1 r =*
*{(N, M). ∃a M0 K. M = M0 + {#a#} ∧ N = M0 + K ∧*
*(∀b. b :# K --> (b, a) ∈ r)}"*

**definition**
*mult :: "('a × 'a) set => ('a multiset × 'a multiset) set"* **where**
*"mult r = (mult1 r)⁺"*

**lemma** *not_less_empty [iff]: "(M, {#}) ∉ mult1 r"*
⟨*proof*⟩

**lemma** *less_add: "(N, M0 + {#a#}) ∈ mult1 r ==>*
*(∃M. (M, M0) ∈ mult1 r ∧ N = M + {#a#}) ∨*
*(∃K. (∀b. b :# K --> (b, a) ∈ r) ∧ N = M0 + K)"*
**(is** *"_ ⟹ ?case1 (mult1 r) ∨ ?case2")*
⟨*proof*⟩

**lemma** *all_accessible: "wf r ==> ∀M. M ∈ acc (mult1 r)"*
⟨*proof*⟩

**theorem** *wf_mult1: "wf r ==> wf (mult1 r)"*
⟨*proof*⟩

**theorem** *wf_mult: "wf r ==> wf (mult r)"*
⟨*proof*⟩

### 14.4.2 Closure-free presentation

**lemma** *diff_union_single_conv: "a :# J ==> I + J - {#a#} = I + (J - {#a#})"*
⟨*proof*⟩

One direction.

**lemma** *mult_implies_one_step:*
*"trans r ==> (M, N) ∈ mult r ==>*
*∃I J K. N = I + J ∧ M = I + K ∧ J ≠ {#} ∧*
*(∀k ∈ set_of K. ∃j ∈ set_of J. (k, j) ∈ r)"*
⟨*proof*⟩

**lemma** *elem_imp_eq_diff_union: "a :# M ==> M = M - {#a#} + {#a#}"*
⟨*proof*⟩

**lemma** *size_eq_Suc_imp_eq_union: "size M = Suc n ==> ∃a N. M = N + {#a#}"*
⟨*proof*⟩

**lemma** *one_step_implies_mult_aux:*
*"trans r ==>*

```
    ∀I J K. (size J = n ∧ J ≠ {#} ∧ (∀k ∈ set_of K. ∃j ∈ set_of J. (k,
j) ∈ r))
      --> (I + K, I + J) ∈ mult r"
```
⟨*proof*⟩

**lemma** *one_step_implies_mult:*
```
  "trans r ==> J ≠ {#} ==> ∀k ∈ set_of K. ∃j ∈ set_of J. (k, j) ∈ r
    ==> (I + K, I + J) ∈ mult r"
```
⟨*proof*⟩

### 14.4.3 Partial-order properties

**instantiation** *multiset :: (order) order*
**begin**

**definition**
```
  less_multiset_def: "M' < M ⟷ (M', M) ∈ mult {(x', x). x' < x}"
```

**definition**
```
  le_multiset_def: "M' <= M ⟷ M' = M ∨ M' < (M::'a multiset)"
```

**lemma** *trans_base_order: "trans {(x', x). x' < (x::'a::order)}"*
⟨*proof*⟩

Irreflexivity.

**lemma** *mult_irrefl_aux:*
```
  "finite A ==> (∀x ∈ A. ∃y ∈ A. x < (y::'a::order)) ⟹ A = {}"
```
⟨*proof*⟩

**lemma** *mult_less_not_refl: "¬ M < (M::'a::order multiset)"*
⟨*proof*⟩

**lemma** *mult_less_irrefl [elim!]: "M < (M::'a::order multiset) ==> R"*
⟨*proof*⟩

Transitivity.

**theorem** *mult_less_trans: "K < M ==> M < N ==> K < (N::'a::order multiset)"*
⟨*proof*⟩

Asymmetry.

**theorem** *mult_less_not_sym: "M < N ==> ¬ N < (M::'a::order multiset)"*
⟨*proof*⟩

**theorem** *mult_less_asym:*
```
  "M < N ==> (¬ P ==> N < (M::'a::order multiset)) ==> P"
```
⟨*proof*⟩

**theorem** *mult_le_refl [iff]: "M <= (M::'a::order multiset)"*
⟨*proof*⟩

Anti-symmetry.

**theorem** *mult_le_antisym:*
```
  "M <= N ==> N <= M ==> M = (N::'a::order multiset)"
```

⟨*proof*⟩

Transitivity.

**theorem** *mult_le_trans:*
*"K <= M ==> M <= N ==> K <= (N::'a::order multiset)"*
⟨*proof*⟩

**theorem** *mult_less_le: "(M < N) = (M <= N ∧ M ≠ (N::'a::order multiset))"*
⟨*proof*⟩

**instance**
⟨*proof*⟩

**end**

### 14.4.4 Monotonicity of multiset union

**lemma** *mult1_union:*
*"(B, D) ∈ mult1 r ==> trans r ==> (C + B, C + D) ∈ mult1 r"*
⟨*proof*⟩

**lemma** *union_less_mono2: "B < D ==> C + B < C + (D::'a::order multiset)"*
⟨*proof*⟩

**lemma** *union_less_mono1: "B < D ==> B + C < D + (C::'a::order multiset)"*
⟨*proof*⟩

**lemma** *union_less_mono:*
*"A < C ==> B < D ==> A + B < C + (D::'a::order multiset)"*
⟨*proof*⟩

**lemma** *union_le_mono:*
*"A <= C ==> B <= D ==> A + B <= C + (D::'a::order multiset)"*
⟨*proof*⟩

**lemma** *empty_leI [iff]: "{#} <= (M::'a::order multiset)"*
⟨*proof*⟩

**lemma** *union_upper1: "A <= A + (B::'a::order multiset)"*
⟨*proof*⟩

**lemma** *union_upper2: "B <= A + (B::'a::order multiset)"*
⟨*proof*⟩

**instance** *multiset :: (order) pordered_ab_semigroup_add*
⟨*proof*⟩

## 14.5 Link with lists

**primrec** *multiset_of :: "'a list ⇒ 'a multiset"* **where**
*"multiset_of [] = {#}" |*
*"multiset_of (a # x) = multiset_of x + {# a #}"*

**lemma** *multiset_of_zero_iff[simp]: "(multiset_of x = {#}) = (x = [])"*

⟨*proof*⟩

**lemma** *multiset_of_zero_iff_right[simp]: "({#} = multiset_of x) = (x = [])"*
⟨*proof*⟩

**lemma** *set_of_multiset_of[simp]: "set_of(multiset_of x) = set x"*
⟨*proof*⟩

**lemma** *mem_set_multiset_eq: "x ∈ set xs = (x :# multiset_of xs)"*
⟨*proof*⟩

**lemma** *multiset_of_append [simp]:*
*"multiset_of (xs @ ys) = multiset_of xs + multiset_of ys"*
⟨*proof*⟩

**lemma** *surj_multiset_of: "surj multiset_of"*
⟨*proof*⟩

**lemma** *set_count_greater_0: "set x = {a. count (multiset_of x) a > 0}"*
⟨*proof*⟩

**lemma** *distinct_count_atmost_1:*
*"distinct x = (! a. count (multiset_of x) a = (if a ∈ set x then 1 else 0))"*
⟨*proof*⟩

**lemma** *multiset_of_eq_setD:*
*"multiset_of xs = multiset_of ys ⟹ set xs = set ys"*
⟨*proof*⟩

**lemma** *set_eq_iff_multiset_of_eq_distinct:*
*"distinct x ⟹ distinct y ⟹*
*(set x = set y) = (multiset_of x = multiset_of y)"*
⟨*proof*⟩

**lemma** *set_eq_iff_multiset_of_remdups_eq:*
*"(set x = set y) = (multiset_of (remdups x) = multiset_of (remdups y))"*
⟨*proof*⟩

**lemma** *multiset_of_compl_union [simp]:*
*"multiset_of [x←xs. P x] + multiset_of [x←xs. ¬P x] = multiset_of xs"*
⟨*proof*⟩

**lemma** *count_filter:*
*"count (multiset_of xs) x = length [y ← xs. y = x]"*
⟨*proof*⟩

**lemma** *nth_mem_multiset_of: "i < length ls ⟹ (ls ! i) :# multiset_of ls"*
⟨*proof*⟩

**lemma** *multiset_of_remove1: "multiset_of (remove1 a xs) = multiset_of xs - {#a#}"*
⟨*proof*⟩

**lemma** *multiset_of_eq_length:*
**assumes** *"multiset_of xs = multiset_of ys"*
**shows** *"length xs = length ys"*
⟨*proof*⟩

This lemma shows which properties suffice to show that a function `f` with `f xs = ys` behaves like sort.

**lemma** *properties_for_sort:*
*"multiset_of ys = multiset_of xs ⟹ sorted ys ⟹ sort xs = ys"*
⟨*proof*⟩

## 14.6 Pointwise ordering induced by count

**definition**
*mset_le :: "'a multiset ⇒ 'a multiset ⇒ bool"* (**infix** *"≤#" 50*) **where**
*"(A ≤# B) = (∀a. count A a ≤ count B a)"*

**definition**
*mset_less :: "'a multiset ⇒ 'a multiset ⇒ bool"* (**infix** *"<#" 50*) **where**
*"(A <# B) = (A ≤# B ∧ A ≠ B)"*

**notation** *mset_le* (**infix** *"⊆#" 50*)
**notation** *mset_less* (**infix** *"⊂#" 50*)

**lemma** *mset_le_refl[simp]: "A ≤# A"*
⟨*proof*⟩

**lemma** *mset_le_trans: "A ≤# B ⟹ B ≤# C ⟹ A ≤# C"*
⟨*proof*⟩

**lemma** *mset_le_antisym: "A ≤# B ⟹ B ≤# A ⟹ A = B"*
⟨*proof*⟩

**lemma** *mset_le_exists_conv: "(A ≤# B) = (∃C. B = A + C)"*
⟨*proof*⟩

**lemma** *mset_le_mono_add_right_cancel[simp]: "(A + C ≤# B + C) = (A ≤# B)"*
⟨*proof*⟩

**lemma** *mset_le_mono_add_left_cancel[simp]: "(C + A ≤# C + B) = (A ≤# B)"*
⟨*proof*⟩

**lemma** *mset_le_mono_add: "⟦ A ≤# B; C ≤# D ⟧ ⟹ A + C ≤# B + D"*
⟨*proof*⟩

**lemma** *mset_le_add_left[simp]: "A ≤# A + B"*
⟨*proof*⟩

**lemma** *mset_le_add_right[simp]: "B ≤# A + B"*
⟨*proof*⟩

**lemma** *mset_le_single: "a :# B ⟹ {#a#} ≤# B"*
⟨*proof*⟩

**lemma** *multiset_diff_union_assoc: "C ≤# B ⟹ A + B - C = A + (B - C)"*
⟨*proof*⟩

**lemma** *mset_le_multiset_union_diff_commute:*
**assumes** *"B ≤# A"*
**shows** *"A - B + C = A + C - B"*
⟨*proof*⟩

**lemma** *multiset_of_remdups_le: "multiset_of (remdups xs) ≤# multiset_of xs"*
⟨*proof*⟩

**lemma** *multiset_of_update:*
*"i < length ls ⟹ multiset_of (ls[i := v]) = multiset_of ls - {#ls ! i#} + {#v#}"*
⟨*proof*⟩

**lemma** *multiset_of_swap:*
*"i < length ls ⟹ j < length ls ⟹*
*multiset_of (ls[j := ls ! i, i := ls ! j]) = multiset_of ls"*
⟨*proof*⟩

**interpretation** *mset_order: order ["op ≤#" "op <#"]*
⟨*proof*⟩

**interpretation** *mset_order_cancel_semigroup:*
*pordered_cancel_ab_semigroup_add ["op +" "op ≤#" "op <#"]*
⟨*proof*⟩

**interpretation** *mset_order_semigroup_cancel:*
*pordered_ab_semigroup_add_imp_le ["op +" "op ≤#" "op <#"]*
⟨*proof*⟩

**lemma** *mset_lessD: "A ⊂# B ⟹ x ∈# A ⟹ x ∈# B"*
⟨*proof*⟩

**lemma** *mset_leD: "A ⊆# B ⟹ x ∈# A ⟹ x ∈# B"*
⟨*proof*⟩

**lemma** *mset_less_insertD: "(A + {#x#} ⊂# B) ⟹ (x ∈# B ∧ A ⊂# B)"*
⟨*proof*⟩

**lemma** *mset_le_insertD: "(A + {#x#} ⊆# B) ⟹ (x ∈# B ∧ A ⊆# B)"*
⟨*proof*⟩

**lemma** *mset_less_of_empty[simp]: "A ⊂# {#} = False"*
⟨*proof*⟩

**lemma** *multi_psub_of_add_self[simp]: "A ⊂# A + {#x#}"*
⟨*proof*⟩

**lemma** *multi_psub_self[simp]: "A ⊂# A = False"*
⟨*proof*⟩

**lemma** *mset_less_add_bothsides:*
*"T + {#x#} ⊂# S + {#x#} ⟹ T ⊂# S"*
⟨*proof*⟩

**lemma** *mset_less_empty_nonempty: "({#} ⊂# S) = (S ≠ {#})"*
⟨*proof*⟩

**lemma** *mset_less_size: "A ⊂# B ⟹ size A < size B"*
⟨*proof*⟩

**lemmas** *mset_less_trans = mset_order.less_eq_less.less_trans*

**lemma** *mset_less_diff_self: "c ∈# B ⟹ B - {#c#} ⊂# B"*
⟨*proof*⟩

## 14.7 Strong induction and subset induction for multisets

Well-foundedness of proper subset operator:

proper multiset subset

**definition**
*mset_less_rel :: "('a multiset * 'a multiset) set"* **where**
*"mset_less_rel = {(A,B). A ⊂# B}"*

**lemma** *multiset_add_sub_el_shuffle:*
**assumes** *"c ∈# B"* **and** *"b ≠ c"*
**shows** *"B - {#c#} + {#b#} = B + {#b#} - {#c#}"*
⟨*proof*⟩

**lemma** *wf_mset_less_rel: "wf mset_less_rel"*
⟨*proof*⟩

The induction rules:

**lemma** *full_multiset_induct [case_names less]:*
**assumes** *ih: "⋀B. ∀A. A ⊂# B ⟶ P A ⟹ P B"*
**shows** *"P B"*
⟨*proof*⟩

**lemma** *multi_subset_induct [consumes 2, case_names empty add]:*
**assumes** *"F ⊆# A"*
**and** *empty: "P {#}"*
**and** *insert: "⋀a F. a ∈# A ⟹ P F ⟹ P (F + {#a#})"*
**shows** *"P F"*
⟨*proof*⟩

A consequence: Extensionality.

**lemma** *multi_count_eq: "(∀x. count A x = count B x) = (A = B)"*
⟨*proof*⟩

**lemmas** *multi_count_ext = multi_count_eq [THEN iffD1, rule_format]*

## 14.8 The fold combinator

The intended behaviour is *fold_mset f z {#x$_1$, ..., x$_n$#} = f x$_1$ (... (f x$_n$ z)...)* if *f* is associative-commutative.

The graph of *fold_mset*, *z*: the start element, *f*: folding function, *A*: the multiset, *y*: the result.

**inductive**
  *fold_msetG :: "('a ⇒ 'b ⇒ 'b) ⇒ 'b ⇒ 'a multiset ⇒ 'b ⇒ bool"*
  **for** *f :: "'a ⇒ 'b ⇒ 'b"*
  **and** *z :: 'b*
**where**
  *emptyI [intro]: "fold_msetG f z {#} z"*
*| insertI [intro]: "fold_msetG f z A y ⟹ fold_msetG f z (A + {#x#}) (f x y)"*

**inductive_cases** *empty_fold_msetGE [elim!]: "fold_msetG f z {#} x"*
**inductive_cases** *insert_fold_msetGE: "fold_msetG f z (A + {#}) y"*

**definition**
  *fold_mset :: "('a ⇒ 'b ⇒ 'b) ⇒ 'b ⇒ 'a multiset ⇒ 'b"* **where**
  *"fold_mset f z A = (THE x. fold_msetG f z A x)"*

**lemma** *Diff1_fold_msetG:*
  *"fold_msetG f z (A - {#x#}) y ⟹ x ∈# A ⟹ fold_msetG f z A (f x y)"*
⟨*proof*⟩

**lemma** *fold_msetG_nonempty: "∃x. fold_msetG f z A x"*
⟨*proof*⟩

**lemma** *fold_mset_empty[simp]: "fold_mset f z {#} = z"*
⟨*proof*⟩

**locale** *left_commutative =*
**fixes** *f :: "'a => 'b => 'b"*
**assumes** *left_commute: "f x (f y z) = f y (f x z)"*
**begin**

**lemma** *fold_msetG_determ:*
  *"fold_msetG f z A x ⟹ fold_msetG f z A y ⟹ y = x"*
⟨*proof*⟩

**lemma** *fold_mset_insert_aux:*
  *"(fold_msetG f z (A + {#x#}) v) =*
    *(∃y. fold_msetG f z A y ∧ v = f x y)"*
⟨*proof*⟩

**lemma** *fold_mset_equality: "fold_msetG f z A y ⟹ fold_mset f z A = y"*
⟨*proof*⟩

**lemma** *fold_mset_insert:*
  *"fold_mset f z (A + {#x#}) = f x (fold_mset f z A)"*
⟨*proof*⟩

**lemma** *fold_mset_insert_idem:*
*"fold_mset f z (A + {#a#}) = f a (fold_mset f z A)"*
⟨*proof*⟩

**lemma** *fold_mset_commute: "f x (fold_mset f z A) = fold_mset f (f x z) A"*
⟨*proof*⟩

**lemma** *fold_mset_single [simp]: "fold_mset f z {#x#} = f x z"*
⟨*proof*⟩

**lemma** *fold_mset_union [simp]:*
*"fold_mset f z (A+B) = fold_mset f (fold_mset f z A) B"*
⟨*proof*⟩

**lemma** *fold_mset_fusion:*
**includes** *left_commutative g*
**shows** *"(⋀x y. h (g x y) = f x (h y)) ⟹ h (fold_mset g w A) = fold_mset f (h w) A"*
⟨*proof*⟩

**lemma** *fold_mset_rec:*
**assumes** *"a ∈# A"*
**shows** *"fold_mset f z A = f a (fold_mset f z (A - {#a#}))"*
⟨*proof*⟩

**end**

A note on code generation: When defining some function containing a subterm *fold_mset F*, code generation is not automatic. When interpreting locale *left_commutative* with *F*, the would be code thms for *fold_mset* become thms like *fold_mset F z {#} = z* where *F* is not a pattern but contains defined symbols, i.e. is not a code thm. Hence a separate constant with its own code thms needs to be introduced for *F*. See the image operator below.

## 14.9 Image

**definition** *[code func del]: "image_mset f == fold_mset (op + o single o f) {#}"*

**interpretation** *image_left_comm: left_commutative ["op + o single o f"]*
⟨*proof*⟩

**lemma** *image_mset_empty [simp, code func]: "image_mset f {#} = {#}"*
⟨*proof*⟩

**lemma** *image_mset_single [simp, code func]: "image_mset f {#x#} = {#f x#}"*
⟨*proof*⟩

**lemma** *image_mset_insert:*
*"image_mset f (M + {#a#}) = image_mset f M + {#f a#}"*
⟨*proof*⟩

**lemma** *image_mset_union[simp, code func]:*
*"image_mset f (M+N) = image_mset f M + image_mset f N"*

⟨*proof*⟩

**lemma** *size_image_mset [simp]: "size (image_mset f M) = size M"*
⟨*proof*⟩

**lemma** *image_mset_is_empty_iff [simp]: "image_mset f M = {#} ⟷ M = {#}"*
⟨*proof*⟩

**syntax**
*comprehension1_mset :: "'a ⇒ 'b ⇒ 'b multiset ⇒ 'a multiset"*
*("({#_/. _ :# _#})")*
**translations**
*"{#e. x:#M#}" == "CONST image_mset (%x. e) M"*

**syntax**
*comprehension2_mset :: "'a ⇒ 'b ⇒ 'b multiset ⇒ bool ⇒ 'a multiset"*
*("({#_/ | _ :# _./ _#})")*
**translations**
*"{#e | x:#M. P#}" => "{#e. x :# {# x:#M. P#}#}"*

This allows to write not just filters like *{# x :# M. x < c#}* but also images like *{#x + x. x :# M#}* and *"{#x+x|x:#M. x<c#}"*, where the latter is currently displayed as *{#x + x. x :# {# x :# M. x < c#}#}*.

**end**

# 15 The Follows Relation of Charpentier and Sivilotte

**theory** *Follows* **imports** *SubstAx ListOrder Multiset* **begin**

**constdefs**

*Follows :: "['a => 'b::{order}, 'a => 'b::{order}] => 'a program set"*
**(infixl** *"Fols" 65)*
*"f Fols g == Increasing g ∩ Increasing f Int*
*Always {s. f s ≤ g s} Int*
*(⋂k. {s. k ≤ g s} LeadsTo {s. k ≤ f s})"*

**lemma** *mono_Always_o:*
*"mono h ==> Always {s. f s ≤ g s} ⊆ Always {s. h (f s) ≤ h (g s)}"*
⟨*proof*⟩

**lemma** *mono_LeadsTo_o:*
*"mono (h::'a::order => 'b::order)*
*==> (⋂j. {s. j ≤ g s} LeadsTo {s. j ≤ f s}) ⊆*
*(⋂k. {s. k ≤ h (g s)} LeadsTo {s. k ≤ h (f s)})"*
⟨*proof*⟩

**lemma** *Follows_constant [iff]: "F ∈ (%s. c) Fols (%s. c)"*
⟨*proof*⟩

**lemma** *mono_Follows_o: "mono h ==> f Fols g ⊆ (h o f) Fols (h o g)"*
⟨*proof*⟩

**lemma** *mono_Follows_apply:*
*"mono h ==> f Fols g ⊆ (%x. h (f x)) Fols (%x. h (g x))"*
⟨*proof*⟩

**lemma** *Follows_trans:*
*"[| F ∈ f Fols g;  F ∈ g Fols h |] ==> F ∈ f Fols h"*
⟨*proof*⟩

## 15.1 Destruction rules

**lemma** *Follows_Increasing1: "F ∈ f Fols g ==> F ∈ Increasing f"*
⟨*proof*⟩

**lemma** *Follows_Increasing2: "F ∈ f Fols g ==> F ∈ Increasing g"*
⟨*proof*⟩

**lemma** *Follows_Bounded: "F ∈ f Fols g ==> F ∈ Always {s. f s ≤ g s}"*
⟨*proof*⟩

**lemma** *Follows_LeadsTo:*
*"F ∈ f Fols g ==> F ∈ {s. k ≤ g s} LeadsTo {s. k ≤ f s}"*
⟨*proof*⟩

**lemma** *Follows_LeadsTo_pfixLe:*
*"F ∈ f Fols g ==> F ∈ {s. k pfixLe g s} LeadsTo {s. k pfixLe f s}"*
⟨*proof*⟩

**lemma** *Follows_LeadsTo_pfixGe:*
*"F ∈ f Fols g ==> F ∈ {s. k pfixGe g s} LeadsTo {s. k pfixGe f s}"*
⟨*proof*⟩

**lemma** *Always_Follows1:*
*"[| F ∈ Always {s. f s = f' s}; F ∈ f Fols g |] ==> F ∈ f' Fols g"*

⟨*proof*⟩

**lemma** *Always_Follows2:*
*"[| F ∈ Always {s. g s = g' s}; F ∈ f Fols g |] ==> F ∈ f Fols g'"*
⟨*proof*⟩

## 15.2 Union properties (with the subset ordering)

**lemma** *increasing_Un:*
*"[| F ∈ increasing f;  F ∈ increasing g |]*
*==> F ∈ increasing (%s. (f s) ∪ (g s))"*
⟨*proof*⟩

**lemma** *Increasing_Un:*
*"[| F ∈ Increasing f;  F ∈ Increasing g |]*

```
     ==> F ∈ Increasing (%s. (f s) ∪ (g s))"
```
⟨*proof*⟩

**lemma** *Always_Un:*
```
     "[| F ∈ Always {s. f' s ≤ f s}; F ∈ Always {s. g' s ≤ g s} |]
      ==> F ∈ Always {s. f' s ∪ g' s ≤ f s ∪ g s}"
```
⟨*proof*⟩

**lemma** *Follows_Un_lemma:*
```
     "[| F ∈ Increasing f; F ∈ Increasing g;
         F ∈ Increasing g'; F ∈ Always {s. f' s ≤ f s};
         ∀k. F ∈ {s. k ≤ f s} LeadsTo {s. k ≤ f' s} |]
      ==> F ∈ {s. k ≤ f s ∪ g s} LeadsTo {s. k ≤ f' s ∪ g s}"
```
⟨*proof*⟩

**lemma** *Follows_Un:*
```
    "[| F ∈ f' Fols f;  F ∈ g' Fols g |]
     ==> F ∈ (%s. (f' s) ∪ (g' s)) Fols (%s. (f s) ∪ (g s))"
```
⟨*proof*⟩

## 15.3 Multiset union properties (with the multiset ordering)

**lemma** *increasing_union:*
```
    "[| F ∈ increasing f;  F ∈ increasing g |]
     ==> F ∈ increasing (%s. (f s) + (g s :: ('a::order) multiset))"
```
⟨*proof*⟩

**lemma** *Increasing_union:*
```
    "[| F ∈ Increasing f;  F ∈ Increasing g |]
     ==> F ∈ Increasing (%s. (f s) + (g s :: ('a::order) multiset))"
```
⟨*proof*⟩

**lemma** *Always_union:*
```
     "[| F ∈ Always {s. f' s ≤ f s}; F ∈ Always {s. g' s ≤ g s} |]
      ==> F ∈ Always {s. f' s + g' s ≤ f s + (g s :: ('a::order) multiset)}"
```
⟨*proof*⟩

**lemma** *Follows_union_lemma:*
```
     "[| F ∈ Increasing f; F ∈ Increasing g;
         F ∈ Increasing g'; F ∈ Always {s. f' s ≤ f s};
         ∀k::('a::order) multiset.
           F ∈ {s. k ≤ f s} LeadsTo {s. k ≤ f' s} |]
      ==> F ∈ {s. k ≤ f s + g s} LeadsTo {s. k ≤ f' s + g s}"
```
⟨*proof*⟩

**lemma** *Follows_union:*
```
     "!!g g' ::'b => ('a::order) multiset.
        [| F ∈ f' Fols f;  F ∈ g' Fols g |]
        ==> F ∈ (%s. (f' s) + (g' s)) Fols (%s. (f s) + (g s))"
```

⟨*proof*⟩

**lemma** *Follows_setsum:*
```
"!!f ::['c,'b] => ('a::order) multiset.
   [| ∀i ∈ I. F ∈ f' i Fols f i;  finite I |]
   ==> F ∈ (%s. ∑i ∈ I. f' i s) Fols (%s. ∑i ∈ I. f i s)"
```
⟨*proof*⟩

**lemma** *Increasing_imp_Stable_pfixGe:*
```
"F ∈ Increasing func ==> F ∈ Stable {s. h pfixGe (func s)}"
```
⟨*proof*⟩

**lemma** *LeadsTo_le_imp_pfixGe:*
```
"∀z. F ∈ {s. z ≤ f s} LeadsTo {s. z ≤ g s}
 ==> F ∈ {s. z pfixGe f s} LeadsTo {s. z pfixGe g s}"
```
⟨*proof*⟩

**end**

# 16 Predicate Transformers

**theory** *Transformers* **imports** *Comp* **begin**

## 16.1 Defining the Predicate Transformers *wp*, *awp* and *wens*

**constdefs**

*wp :: "[('a*'a) set, 'a set] => 'a set"*
— Dijkstra's weakest-precondition operator (for an individual command)
*"wp act B == - (act^-1 `` (-B))"*

*awp :: "['a program, 'a set] => 'a set"*
— Dijkstra's weakest-precondition operator (for a program)
*"awp F B == (⋂act ∈ Acts F. wp act B)"*

*wens :: "['a program, ('a*'a) set, 'a set] => 'a set"*
— The weakest-ensures transformer
*"wens F act B == gfp(λX. (wp act B ∩ awp F (B ∪ X)) ∪ B)"*

The fundamental theorem for wp

**theorem** *wp_iff:* `"(A <= wp act B) = (act `` A <= B)"`
⟨*proof*⟩

This lemma is a good deal more intuitive than the definition!

**lemma** *in_wp_iff:* `"(a ∈ wp act B) = (∀x. (a,x) ∈ act --> x ∈ B)"`
⟨*proof*⟩

**lemma** *Compl_Domain_subset_wp:* `"- (Domain act) ⊆ wp act B"`
⟨*proof*⟩

**lemma** *wp_empty [simp]: "wp act {} = - (Domain act)"*
⟨*proof*⟩

The identity relation is the skip action

**lemma** *wp_Id [simp]: "wp Id B = B"*
⟨*proof*⟩

**lemma** *wp_totalize_act:*
*"wp (totalize_act act) B = (wp act B ∩ Domain act) ∪ (B - Domain act)"*
⟨*proof*⟩

**lemma** *awp_subset: "(awp F A ⊆ A)"*
⟨*proof*⟩

**lemma** *awp_Int_eq: "awp F (A∩B) = awp F A ∩ awp F B"*
⟨*proof*⟩

The fundamental theorem for awp

**theorem** *awp_iff_constrains: "(A <= awp F B) = (F ∈ A co B)"*
⟨*proof*⟩

**lemma** *awp_iff_stable: "(A ⊆ awp F A) = (F ∈ stable A)"*
⟨*proof*⟩

**lemma** *stable_imp_awp_ident: "F ∈ stable A ==> awp F A = A"*
⟨*proof*⟩

**lemma** *wp_mono: "(A ⊆ B) ==> wp act A ⊆ wp act B"*
⟨*proof*⟩

**lemma** *awp_mono: "(A ⊆ B) ==> awp F A ⊆ awp F B"*
⟨*proof*⟩

**lemma** *wens_unfold:*
*"wens F act B = (wp act B ∩ awp F (B ∪ wens F act B)) ∪ B"*
⟨*proof*⟩

**lemma** *wens_Id [simp]: "wens F Id B = B"*
⟨*proof*⟩

These two theorems justify the claim that *wens* returns the weakest assertion satisfying the ensures property

**lemma** *ensures_imp_wens: "F ∈ A ensures B ==> ∃act ∈ Acts F. A ⊆ wens F act B"*
⟨*proof*⟩

**lemma** *wens_ensures: "act ∈ Acts F ==> F ∈ (wens F act B) ensures B"*
⟨*proof*⟩

These two results constitute assertion (4.13) of the thesis

**lemma** *wens_mono: "(A ⊆ B) ==> wens F act A ⊆ wens F act B"*
⟨*proof*⟩

**lemma** *wens_weakening: "B ⊆ wens F act B"*
⟨*proof*⟩

Assertion (6), or 4.16 in the thesis

**lemma** *subset_wens: "A-B ⊆ wp act B ∩ awp F (B ∪ A) ==> A ⊆ wens F act B"*
⟨*proof*⟩

Assertion 4.17 in the thesis

**lemma** *Diff_wens_constrains: "F ∈ (wens F act A - A) co wens F act A"*
⟨*proof*⟩

Assertion (7): 4.18 in the thesis. NOTE that many of these results hold for an arbitrary action. We often do not require *act ∈ Acts F*

**lemma** *stable_wens: "F ∈ stable A ==> F ∈ stable (wens F act A)"*
⟨*proof*⟩

Assertion 4.20 in the thesis.

**lemma** *wens_Int_eq_lemma:*
*"[|T-B ⊆ awp F T; act ∈ Acts F|]*
*==> T ∩ wens F act B ⊆ wens F act (T∩B)"*
⟨*proof*⟩

Assertion (8): 4.21 in the thesis. Here we indeed require *act ∈ Acts F*

**lemma** *wens_Int_eq:*
*"[|T-B ⊆ awp F T; act ∈ Acts F|]*
*==> T ∩ wens F act B = T ∩ wens F act (T∩B)"*
⟨*proof*⟩

## 16.2 Defining the Weakest Ensures Set

**inductive_set**
*wens_set :: "['a program, 'a set] => 'a set set"*
**for** *F :: "'a program"* **and** *B :: "'a set"*
**where**

*Basis: "B ∈ wens_set F B"*

*| Wens: "[|X ∈ wens_set F B; act ∈ Acts F|] ==> wens F act X ∈ wens_set F B"*

*| Union: "W ≠ {} ==> ∀U ∈ W. U ∈ wens_set F B ==> ⋃W ∈ wens_set F B"*

**lemma** *wens_set_imp_co: "A ∈ wens_set F B ==> F ∈ (A-B) co A"*
⟨*proof*⟩

**lemma** *wens_set_imp_leadsTo: "A ∈ wens_set F B ==> F ∈ A leadsTo B"*
⟨*proof*⟩

**lemma** *leadsTo_imp_wens_set: "F ∈ A leadsTo B ==> ∃C ∈ wens_set F B. A ⊆ C"*
⟨*proof*⟩

Assertion (9): 4.27 in the thesis.

**lemma** *leadsTo_iff_wens_set: "(F ∈ A leadsTo B) = (∃C ∈ wens_set F B. A ⊆ C)"*
⟨*proof*⟩

This is the result that requires the definition of *wens_set* to require *W* to be non-empty in the Unio case, for otherwise we should always have *{} ∈ wens_set F B*.

**lemma** *wens_set_imp_subset: "A ∈ wens_set F B ==> B ⊆ A"*
⟨*proof*⟩

## 16.3 Properties Involving Program Union

Assertion (4.30) of thesis, reoriented

**lemma** *awp_Join_eq: "awp (F⊔G) B = awp F B ∩ awp G B"*
⟨*proof*⟩

**lemma** *wens_subset: "wens F act B - B ⊆ wp act B ∩ awp F (B ∪ wens F act B)"*
⟨*proof*⟩

Assertion (4.31)

**lemma** *subset_wens_Join:*
*"[|A = T ∩ wens F act B; T-B ⊆ awp F T; A-B ⊆ awp G (A ∪ B)|]*
*==> A ⊆ wens (F⊔G) act B"*
⟨*proof*⟩

Assertion (4.32)

**lemma** *wens_Join_subset: "wens (F⊔G) act B ⊆ wens F act B"*
⟨*proof*⟩

Lemma, because the inductive step is just too messy.

**lemma** *wens_Union_inductive_step:*
**assumes** *awpF: "T-B ⊆ awp F T"*
**and** *awpG: "!!X. X ∈ wens_set F B ==> (T∩X) - B ⊆ awp G (T∩X)"*
**shows** *"[|X ∈ wens_set F B; act ∈ Acts F; Y ⊆ X; T∩X = T∩Y|]*
*==> wens (F⊔G) act Y ⊆ wens F act X ∧*
*T ∩ wens F act X = T ∩ wens (F⊔G) act Y"*
⟨*proof*⟩

**theorem** *wens_Union:*
**assumes** *awpF: "T-B ⊆ awp F T"*
**and** *awpG: "!!X. X ∈ wens_set F B ==> (T∩X) - B ⊆ awp G (T∩X)"*
**and** *major: "X ∈ wens_set F B"*
**shows** *"∃Y ∈ wens_set (F⊔G) B. Y ⊆ X & T∩X = T∩Y"*
⟨*proof*⟩

**theorem** *leadsTo_Join:*
**assumes** *leadsTo: "F ∈ A leadsTo B"*
**and** *awpF: "T-B ⊆ awp F T"*
**and** *awpG: "!!X. X ∈ wens_set F B ==> (T∩X) - B ⊆ awp G (T∩X)"*
**shows** *"F⊔G ∈ T∩A leadsTo B"*
⟨*proof*⟩

## 16.4 The Set *wens_set F B* for a Single-Assignment Program

Thesis Section 4.3.3

We start by proving laws about single-assignment programs

**lemma** *awp_single_eq [simp]:*
```
"awp (mk_program (init, {act}, allowed)) B = B ∩ wp act B"
```
⟨*proof*⟩

**lemma** *wp_Un_subset:*
```
"wp act A ∪ wp act B ⊆ wp act (A ∪ B)"
```
⟨*proof*⟩

**lemma** *wp_Un_eq:*
```
"single_valued act ==> wp act (A ∪ B) = wp act A ∪ wp act B"
```
⟨*proof*⟩

**lemma** *wp_UN_subset:*
```
"(⋃i∈I. wp act (A i)) ⊆ wp act (⋃i∈I. A i)"
```
⟨*proof*⟩

**lemma** *wp_UN_eq:*
```
"[|single_valued act; I≠{}|]
 ==> wp act (⋃i∈I. A i) = (⋃i∈I. wp act (A i))"
```
⟨*proof*⟩

**lemma** *wens_single_eq:*
```
"wens (mk_program (init, {act}, allowed)) act B = B ∪ wp act B"
```
⟨*proof*⟩

Next, we express the *wens_set* for single-assignment programs

**constdefs**
```
  wens_single_finite :: "[('a*'a) set, 'a set, nat] => 'a set"
    "wens_single_finite act B k == ⋃i ∈ atMost k. ((wp act)^i) B"

  wens_single :: "[('a*'a) set, 'a set] => 'a set"
    "wens_single act B == ⋃i. ((wp act)^i) B"
```

**lemma** *wens_single_Un_eq:*
```
"single_valued act
 ==> wens_single act B ∪ wp act (wens_single act B) = wens_single act B"
```
⟨*proof*⟩

**lemma** *atMost_nat_nonempty:*
```
"atMost (k::nat) ≠ {}"
```
⟨*proof*⟩

**lemma** *wens_single_finite_0 [simp]:*
```
"wens_single_finite act B 0 = B"
```
⟨*proof*⟩

**lemma** *wens_single_finite_Suc:*
```
"single_valued act
 ==> wens_single_finite act B (Suc k) =
     wens_single_finite act B k ∪ wp act (wens_single_finite act B k)"
```
⟨*proof*⟩

**lemma** *wens_single_finite_Suc_eq_wens:*
```
"single_valued act
  ==> wens_single_finite act B (Suc k) =
      wens (mk_program (init, {act}, allowed)) act
           (wens_single_finite act B k)"
```
⟨*proof*⟩

**lemma** *def_wens_single_finite_Suc_eq_wens:*
```
"[|F = mk_program (init, {act}, allowed); single_valued act|]
  ==> wens_single_finite act B (Suc k) =
      wens F act (wens_single_finite act B k)"
```
⟨*proof*⟩

**lemma** *wens_single_finite_Un_eq:*
```
"single_valued act
 ==> wens_single_finite act B k ∪ wp act (wens_single_finite act B k)
     ∈ range (wens_single_finite act B)"
```
⟨*proof*⟩

**lemma** *wens_single_eq_Union:*
```
"wens_single act B = ⋃range (wens_single_finite act B)"
```
⟨*proof*⟩

**lemma** *wens_single_finite_eq_Union:*
```
"wens_single_finite act B n = (⋃k∈atMost n. wens_single_finite act B
k)"
```
⟨*proof*⟩

**lemma** *wens_single_finite_mono:*
```
"m ≤ n ==> wens_single_finite act B m ⊆ wens_single_finite act B n"
```
⟨*proof*⟩

**lemma** *wens_single_finite_subset_wens_single:*
```
"wens_single_finite act B k ⊆ wens_single act B"
```
⟨*proof*⟩

**lemma** *subset_wens_single_finite:*
```
"[|W ⊆ wens_single_finite act B ` (atMost k); single_valued act; W≠{}|]
 ==> ∃m. ⋃W = wens_single_finite act B m"
```
⟨*proof*⟩

lemma for Union case

**lemma** *Union_eq_wens_single:*
```
"⟦∀k. ¬ W ⊆ wens_single_finite act B ` {..k};
  W ⊆ insert (wens_single act B)
       (range (wens_single_finite act B))⟧
 ⟹ ⋃W = wens_single act B"
```
⟨*proof*⟩

**lemma** *wens_set_subset_single:*
```
"single_valued act
 ==> wens_set (mk_program (init, {act}, allowed)) B ⊆
     insert (wens_single act B) (range (wens_single_finite act B))"
```
⟨*proof*⟩

**lemma** *wens_single_finite_in_wens_set:*
```
"single_valued act ⟹
   wens_single_finite act B k
   ∈ wens_set (mk_program (init, {act}, allowed)) B"
```
⟨*proof*⟩

**lemma** *single_subset_wens_set:*
```
"single_valued act
 ==> insert (wens_single act B) (range (wens_single_finite act B)) ⊆

     wens_set (mk_program (init, {act}, allowed)) B"
```
⟨*proof*⟩

Theorem (4.29)

**theorem** *wens_set_single_eq:*
```
"[|F = mk_program (init, {act}, allowed); single_valued act|]
 ==> wens_set F B =
     insert (wens_single act B) (range (wens_single_finite act B))"
```
⟨*proof*⟩

Generalizing Misra's Fixed Point Union Theorem (4.41)

**lemma** *fp_leadsTo_Join:*
```
"[|T-B ⊆ awp F T; T-B ⊆ FP G; F ∈ A leadsTo B|] ==> F⊔G ∈ T∩A leadsTo
B"
```
⟨*proof*⟩

**end**

# 17 Progress Sets

**theory** *ProgressSets* **imports** *Transformers* **begin**

## 17.1 Complete Lattices and the Operator *cl*

**constdefs**

```
lattice :: "'a set set => bool"
```
— Meier calls them closure sets, but they are just complete lattices
```
"lattice L ==
      (∀M. M ⊆ L --> ⋂M ∈ L) & (∀M. M ⊆ L --> ⋃M ∈ L)"
```

```
cl :: "['a set set, 'a set] => 'a set"
```
— short for "closure"
```
"cl L r == ⋂{x. x∈L & r ⊆ x}"
```

**lemma** *UNIV_in_lattice:* `"lattice L ==> UNIV ∈ L"`
⟨*proof*⟩

**lemma** *empty_in_lattice:* `"lattice L ==> {} ∈ L"`
⟨*proof*⟩

**lemma** *Union_in_lattice:* `"[|M ⊆ L; lattice L|] ==> ⋃M ∈ L"`

⟨*proof*⟩

**lemma** *Inter_in_lattice:* "[|M ⊆ L; lattice L|] ==> ⋂M ∈ L"
⟨*proof*⟩

**lemma** *UN_in_lattice:*
"[|lattice L; !!i. i∈I ==> r i ∈ L|] ==> (⋃i∈I. r i) ∈ L"
⟨*proof*⟩

**lemma** *INT_in_lattice:*
"[|lattice L; !!i. i∈I ==> r i ∈ L|] ==> (⋂i∈I. r i) ∈ L"
⟨*proof*⟩

**lemma** *Un_in_lattice:* "[|x∈L; y∈L; lattice L|] ==> x∪y ∈ L"
⟨*proof*⟩

**lemma** *Int_in_lattice:* "[|x∈L; y∈L; lattice L|] ==> x∩y ∈ L"
⟨*proof*⟩

**lemma** *lattice_stable:* "lattice {X. F ∈ stable X}"
⟨*proof*⟩

The next three results state that `cl L r` is the minimal element of `L` that includes `r`.

**lemma** *cl_in_lattice:* "lattice L ==> cl L r ∈ L"
⟨*proof*⟩

**lemma** *cl_least:* "[|c∈L; r⊆c|] ==> cl L r ⊆ c"
⟨*proof*⟩

The next three lemmas constitute assertion (4.61)

**lemma** *cl_mono:* "r ⊆ r' ==> cl L r ⊆ cl L r'"
⟨*proof*⟩

**lemma** *subset_cl:* "r ⊆ cl L r"
⟨*proof*⟩

A reformulation of `?r ⊆ cl ?L ?r`

**lemma** *clI:* "x ∈ r ==> x ∈ cl L r"
⟨*proof*⟩

A reformulation of ⟦`?c ∈ ?L; ?r ⊆ ?c`⟧ ⟹ `cl ?L ?r ⊆ ?c`

**lemma** *clD:* "[|c ∈ cl L r; B ∈ L; r ⊆ B|] ==> c ∈ B"
⟨*proof*⟩

**lemma** *cl_UN_subset:* "(⋃i∈I. cl L (r i)) ⊆ cl L (⋃i∈I. r i)"
⟨*proof*⟩

**lemma** *cl_Un:* "lattice L ==> cl L (r∪s) = cl L r ∪ cl L s"
⟨*proof*⟩

**lemma** *cl_UN:* "lattice L ==> cl L (⋃i∈I. r i) = (⋃i∈I. cl L (r i))"
⟨*proof*⟩

**lemma** *cl_Int_subset: "cl L (r∩s) ⊆ cl L r ∩ cl L s"*
⟨*proof*⟩

**lemma** *cl_idem [simp]: "cl L (cl L r) = cl L r"*
⟨*proof*⟩

**lemma** *cl_ident: "r∈L ==> cl L r = r"*
⟨*proof*⟩

**lemma** *cl_empty [simp]: "lattice L ==> cl L {} = {}"*
⟨*proof*⟩

**lemma** *cl_UNIV [simp]: "lattice L ==> cl L UNIV = UNIV"*
⟨*proof*⟩

Assertion (4.62)

**lemma** *cl_ident_iff: "lattice L ==> (cl L r = r) = (r∈L)"*
⟨*proof*⟩

**lemma** *cl_subset_in_lattice: "[|cl L r ⊆ r; lattice L|] ==> r∈L"*
⟨*proof*⟩

## 17.2 Progress Sets and the Main Lemma

A progress set satisfies certain closure conditions and is a simple way of including the set *wens_set F B*.

**constdefs**
```
  closed :: "['a program, 'a set, 'a set,  'a set set] => bool"
   "closed F T B L == ∀M. ∀act ∈ Acts F. B⊆M & T∩M ∈ L -->
                              T ∩ (B ∪ wp act M) ∈ L"

  progress_set :: "['a program, 'a set, 'a set] => 'a set set set"
   "progress_set F T B ==
      {L. lattice L & B ∈ L & T ∈ L & closed F T B L}"
```

**lemma** *closedD:*
```
   "[|closed F T B L; act ∈ Acts F; B⊆M; T∩M ∈ L|]
    ==> T ∩ (B ∪ wp act M) ∈ L"
```
⟨*proof*⟩

Note: the formalization below replaces Meier's `q` by *B* and `m` by *X*.

Part of the proof of the claim at the bottom of page 97. It's proved separately because the argument requires a generalization over all *act ∈ Acts F*.

**lemma** *lattice_awp_lemma:*
**assumes** *TXC: "T∩X ∈ C"* — induction hypothesis in theorem below
**and** *BsubX: "B ⊆ X"* — holds in inductive step
**and** *latt: "lattice C"*
**and** *TC: "T ∈ C"*
**and** *BC: "B ∈ C"*
**and** *clos: "closed F T B C"*
**shows** *"T ∩ (B ∪ awp F (X ∪ cl C (T∩r))) ∈ C"*

⟨*proof*⟩

Remainder of the proof of the claim at the bottom of page 97.

**lemma** *lattice_lemma:*
  **assumes** *TXC:* "T∩X ∈ C" — induction hypothesis in theorem below
      **and** *BsubX:* "B ⊆ X" — holds in inductive step
      **and** *act:* "act ∈ Acts F"
      **and** *latt:* "lattice C"
      **and** *TC:* "T ∈ C"
      **and** *BC:* "B ∈ C"
      **and** *clos:* "closed F T B C"
    **shows** "T ∩ (wp act X ∩ awp F (X ∪ cl C (T∩r)) ∪ X) ∈ C"
⟨*proof*⟩

Induction step for the main lemma

**lemma** *progress_induction_step:*
  **assumes** *TXC:* "T∩X ∈ C" — induction hypothesis in theorem below
      **and** *act:* "act ∈ Acts F"
      **and** *Xwens:* "X ∈ wens_set F B"
      **and** *latt:* "lattice C"
      **and** *TC:* "T ∈ C"
      **and** *BC:* "B ∈ C"
      **and** *clos:* "closed F T B C"
      **and** *Fstable:* "F ∈ stable T"
  **shows** "T ∩ wens F act X ∈ C"
⟨*proof*⟩

Proved on page 96 of Meier's thesis. The special case when `T = UNIV` states that every progress set for the program `F` and set `B` includes the set `wens_set F B`.

**lemma** *progress_set_lemma:*
     "[|C ∈ progress_set F T B; r ∈ wens_set F B; F ∈ stable T|] ==> T∩r ∈ C"
⟨*proof*⟩

## 17.3 The Progress Set Union Theorem

**lemma** *closed_mono:*
  **assumes** *BB':* "B ⊆ B'"
      **and** *TBwp:* "T ∩ (B ∪ wp act M) ∈ C"
      **and** *B'C:* "B' ∈ C"
      **and** *TC:* "T ∈ C"
      **and** *latt:* "lattice C"
  **shows** "T ∩ (B' ∪ wp act M) ∈ C"
⟨*proof*⟩

**lemma** *progress_set_mono:*
    **assumes** *BB':* "B ⊆ B'"
    **shows**
     "[| B' ∈ C;  C ∈ progress_set F T B|]
      ==> C ∈ progress_set F T B'"
⟨*proof*⟩

**theorem** *progress_set_Union:*
**assumes** *leadsTo: "F ∈ A leadsTo B'"*
**and** *prog: "C ∈ progress_set F T B"*
**and** *Fstable: "F ∈ stable T"*
**and** *BB': "B ⊆ B'"*
**and** *B'C: "B' ∈ C"*
**and** *Gco: "!!X. X∈C ==> G ∈ X-B co X"*
**shows** *"F⊔G ∈ T∩A leadsTo B'"*
⟨*proof*⟩

## 17.4 Some Progress Sets

**lemma** *UNIV_in_progress_set: "UNIV ∈ progress_set F T B"*
⟨*proof*⟩

### 17.4.1 Lattices and Relations

From Meier's thesis, section 4.5.3

**constdefs**
*relcl :: "'a set set => ('a * 'a) set"*
— Derived relation from a lattice
*"relcl L == {(x,y). y ∈ cl L {x}}"*

*latticeof :: "('a * 'a) set => 'a set set"*
— Derived lattice from a relation: the set of upwards-closed sets
*"latticeof r == {X. ∀s t. s ∈ X & (s,t) ∈ r --> t ∈ X}"*

**lemma** *relcl_refl: "(a,a) ∈ relcl L"*
⟨*proof*⟩

**lemma** *relcl_trans:*
*"[| (a,b) ∈ relcl L; (b,c) ∈ relcl L; lattice L |] ==> (a,c) ∈ relcl L"*
⟨*proof*⟩

**lemma** *refl_relcl: "lattice L ==> refl UNIV (relcl L)"*
⟨*proof*⟩

**lemma** *trans_relcl: "lattice L ==> trans (relcl L)"*
⟨*proof*⟩

**lemma** *lattice_latticeof: "lattice (latticeof r)"*
⟨*proof*⟩

**lemma** *lattice_singletonI:*
*"[|lattice L; !!s. s ∈ X ==> {s} ∈ L|] ==> X ∈ L"*
⟨*proof*⟩

Equation (4.71) of Meier's thesis. He gives no proof.

**lemma** *cl_latticeof:*
*"[|refl UNIV r; trans r|]*
*==> cl (latticeof r) X = {t. ∃s. s∈X & (s,t) ∈ r}"*

⟨*proof*⟩

Related to (4.71).

**lemma** *cl_eq_Collect_relcl:*
```
"lattice L ==> cl L X = {t. ∃s. s∈X & (s,t) ∈ relcl L}"
```
⟨*proof*⟩

Meier's theorem of section 4.5.3

**theorem** *latticeof_relcl_eq:* `"lattice L ==> latticeof (relcl L) = L"`
⟨*proof*⟩

**theorem** *relcl_latticeof_eq:*
```
"[|refl UNIV r; trans r|] ==> relcl (latticeof r) = r"
```
⟨*proof*⟩

### 17.4.2 Decoupling Theorems

**constdefs**
```
  decoupled :: "['a program, 'a program] => bool"
   "decoupled F G ==
        ∀act ∈ Acts F. ∀B. G ∈ stable B --> G ∈ stable (wp act B)"
```

Rao's Decoupling Theorem

**lemma** *stableco:* `"F ∈ stable A ==> F ∈ A-B co A"`
⟨*proof*⟩

**theorem** *decoupling:*
**assumes** *leadsTo:* `"F ∈ A leadsTo B"`
**and** *Gstable:* `"G ∈ stable B"`
**and** *dec:* `"decoupled F G"`
**shows** `"F⊔G ∈ A leadsTo B"`
⟨*proof*⟩

Rao's Weak Decoupling Theorem

**theorem** *weak_decoupling:*
**assumes** *leadsTo:* `"F ∈ A leadsTo B"`
**and** *stable:* `"F⊔G ∈ stable B"`
**and** *dec:* `"decoupled F (F⊔G)"`
**shows** `"F⊔G ∈ A leadsTo B"`
⟨*proof*⟩

The "Decoupling via *G'* Union Theorem"

**theorem** *decoupling_via_aux:*
**assumes** *leadsTo:* `"F ∈ A leadsTo B"`
**and** *prog:* `"{X. G' ∈ stable X} ∈ progress_set F UNIV B"`
**and** *GG':* `"G ≤ G'"`
— Beware! This is the converse of the refinement relation!
**shows** `"F⊔G ∈ A leadsTo B"`
⟨*proof*⟩

## 17.5 Composition Theorems Based on Monotonicity and Commutativity

### 17.5.1 Commutativity of `cl L` and assignment.

**constdefs**

```
  commutes :: "['a program, 'a set, 'a set,  'a set set] => bool"
   "commutes F T B L ==
       ∀M. ∀act ∈ Acts F. B ⊆ M -->
           cl L (T ∩ wp act M) ⊆ T ∩ (B ∪ wp act (cl L (T∩M)))"
```

From Meier's thesis, section 4.5.6

**lemma** *commutativity1_lemma:*

```
  assumes commutes: "commutes F T B L"
      and lattice:  "lattice L"
      and BL: "B ∈ L"
      and TL: "T ∈ L"
  shows "closed F T B L"
```

⟨*proof*⟩

Version packaged with ⟦*?F ∈ ?A leadsTo ?B'; ?C ∈ progress_set ?F ?T ?B; ?F ∈ stable ?T; ?B ⊆ ?B'; ?B' ∈ ?C;* ⋀*X. X ∈ ?C* ⟹ *?G ∈ X - ?B co X*⟧ ⟹ *?F ⊔ ?G ∈ ?T ∩ ?A leadsTo ?B'*

**lemma** *commutativity1:*

```
  assumes leadsTo: "F ∈ A leadsTo B"
      and lattice:  "lattice L"
      and BL: "B ∈ L"
      and TL: "T ∈ L"
      and Fstable: "F ∈ stable T"
      and Gco: "!!X. X∈L ==> G ∈ X-B co X"
      and commutes: "commutes F T B L"
  shows "F⊔G ∈ T∩A leadsTo B"
```

⟨*proof*⟩

Possibly move to Relation.thy, after *single_valued*

**constdefs**

```
  funof :: "[('a*'b)set, 'a] => 'b"
   "funof r == (λx. THE y. (x,y) ∈ r)"
```

**lemma** *funof_eq: "[|single_valued r; (x,y) ∈ r|] ==> funof r x = y"*

⟨*proof*⟩

**lemma** *funof_Pair_in:*

```
     "[|single_valued r; x ∈ Domain r|] ==> (x, funof r x) ∈ r"
```

⟨*proof*⟩

**lemma** *funof_in:*

```
     "[|r``{x} ⊆ A; single_valued r; x ∈ Domain r|] ==> funof r x ∈ A"
```

⟨*proof*⟩

**lemma** *funof_imp_wp: "[|funof act t ∈ A; single_valued act|] ==> t ∈ wp act A"*

⟨*proof*⟩

### 17.5.2 Commutativity of Functions and Relation

Thesis, page 109

From Meier's thesis, section 4.5.6

**lemma** *commutativity2_lemma:*
**assumes** *dcommutes:*
"∀*act* ∈ *Acts F.*
∀*s* ∈ *T.* ∀*t. (s,t)* ∈ *relcl L* -->
*s* ∈ *B* | *t* ∈ *B* | *(funof act s, funof act t)* ∈ *relcl L"*
**and** *determ: "!!act. act* ∈ *Acts F ==> single_valued act"*
**and** *total: "!!act. act* ∈ *Acts F ==> Domain act = UNIV"*
**and** *lattice: "lattice L"*
**and** *BL: "B* ∈ *L"*
**and** *TL: "T* ∈ *L"*
**and** *Fstable: "F* ∈ *stable T"*
**shows** *"commutes F T B L"*
⟨*proof*⟩

Version packaged with ⟦*?F* ∈ *?A leadsTo ?B'; ?C* ∈ *progress_set ?F ?T ?B; ?F* ∈ *stable ?T; ?B* ⊆ *?B'; ?B'* ∈ *?C;* ⋀*X. X* ∈ *?C* ⟹ *?G* ∈ *X - ?B co X*⟧ ⟹ *?F* ⊔ *?G* ∈ *?T* ∩ *?A leadsTo ?B'*

**lemma** *commutativity2:*
**assumes** *leadsTo: "F* ∈ *A leadsTo B"*
**and** *dcommutes:*
"∀*act* ∈ *Acts F.*
∀*s* ∈ *T.* ∀*t. (s,t)* ∈ *relcl L* -->
*s* ∈ *B* | *t* ∈ *B* | *(funof act s, funof act t)* ∈ *relcl L"*
**and** *determ: "!!act. act* ∈ *Acts F ==> single_valued act"*
**and** *total: "!!act. act* ∈ *Acts F ==> Domain act = UNIV"*
**and** *lattice: "lattice L"*
**and** *BL: "B* ∈ *L"*
**and** *TL: "T* ∈ *L"*
**and** *Fstable: "F* ∈ *stable T"*
**and** *Gco: "!!X. X*∈*L ==> G* ∈ *X-B co X"*
**shows** *"F*⊔*G* ∈ *T*∩*A leadsTo B"*
⟨*proof*⟩

## 17.6 Monotonicity

From Meier's thesis, section 4.5.7, page 110

**end**

# 18 Comprehensive UNITY Theory

**theory** *UNITY_Main* **imports** *Detects PPROD Follows ProgressSets*
**uses** *"UNITY_tactics.ML"* **begin**

⟨*ML*⟩

**end**

**theory** *Deadlock* **imports** *UNITY* **begin**

**lemma** *"[| F ∈ (A ∩ B) co A;  F ∈ (B ∩ A) co B |] ==> F ∈ stable (A ∩ B)"*
⟨*proof*⟩

**lemma** *Collect_le_Int_equals:*
*"(⋂i ∈ atMost n. A(Suc i) ∩ A i) = (⋂i ∈ atMost (Suc n). A i)"*
⟨*proof*⟩

**lemma** *UN_Int_Compl_subset:*
*"(⋃i ∈ lessThan n. A i) ∩ (- A n) ⊆*
*(⋃i ∈ lessThan n. (A i) ∩ (- A (Suc i)))"*
⟨*proof*⟩

**lemma** *INT_Un_Compl_subset:*
*"(⋂i ∈ lessThan n. -A i ∪ A (Suc i))  ⊆*
*(⋂i ∈ lessThan n. -A i) ∪ A n"*
⟨*proof*⟩

**lemma** *INT_le_equals_Int_lemma:*
*"A 0 ∩ (-(A n) ∩ (⋂i ∈ lessThan n. -A i ∪ A (Suc i))) = {}"*
⟨*proof*⟩

**lemma** *INT_le_equals_Int:*
*"(⋂i ∈ atMost n. A i) =*
*A 0 ∩ (⋂i ∈ lessThan n. -A i ∪ A(Suc i))"*
⟨*proof*⟩

**lemma** *INT_le_Suc_equals_Int:*
*"(⋂i ∈ atMost (Suc n). A i) =*
*A 0 ∩ (⋂i ∈ atMost n. -A i ∪ A(Suc i))"*
⟨*proof*⟩

**lemma**
**assumes** *zeroprem: "F ∈ (A 0 ∩ A (Suc n)) co (A 0)"*
**and** *allprem:*
*"!!i. i ∈ atMost n ==> F ∈ (A(Suc i) ∩ A i) co (-A i ∪ A(Suc i))"*

**shows** *"F ∈ stable (⋂i ∈ atMost (Suc n). A i)"*
⟨*proof*⟩

**end**

**theory** *Common* **imports** *"../UNITY_Main"* **begin**

**consts**
*ftime :: "nat=>nat"*
*gtime :: "nat=>nat"*

**axioms**
*fmono: "m ≤ n ==> ftime m ≤ ftime n"*
*gmono: "m ≤ n ==> gtime m ≤ gtime n"*

*fasc: "m ≤ ftime n"*
*gasc: "m ≤ gtime n"*

**constdefs**
*common :: "nat set"*
*"common == {n. ftime n = n & gtime n = n}"*

*maxfg :: "nat => nat set"*
*"maxfg m == {t. t ≤ max (ftime m) (gtime m)}"*

**lemma** *common_stable:*
*"[| ∀m. F ∈ {m} Co (maxfg m); n ∈ common |]*
*==> F ∈ Stable (atMost n)"*
⟨*proof*⟩

**lemma** *common_safety:*
*"[| Init F ⊆ atMost n;*
*∀m. F ∈ {m} Co (maxfg m); n ∈ common |]*
*==> F ∈ Always (atMost n)"*
⟨*proof*⟩

**lemma** *"SKIP ∈ {m} co (maxfg m)"*
⟨*proof*⟩

**lemma** *"mk_total_program*
*(UNIV, {range(%t.(t,ftime t)), range(%t.(t,gtime t))}, UNIV)*
*∈ {m} co (maxfg m)"*
⟨*proof*⟩

**lemma** *"mk_total_program (UNIV, {range(%t.(t, max (ftime t) (gtime t)))},*

```
UNIV)
       ∈ {m} co (maxfg m)"
```
⟨*proof*⟩

**lemma** *"mk_total_program*
*(UNIV, { {(t, Suc t) | t. t < max (ftime t) (gtime t)} }, UNIV)*
*∈ {m} co (maxfg m)"*
⟨*proof*⟩

**declare** *atMost_Int_atLeast [simp]*

**lemma** *leadsTo_common_lemma:*
*"[| ∀m. F ∈ {m} Co (maxfg m);*
*∀m ∈ lessThan n. F ∈ {m} LeadsTo (greaterThan m);*
*n ∈ common |]*
*==> F ∈ (atMost n) LeadsTo common"*
⟨*proof*⟩

**lemma** *leadsTo_common:*
*"[| ∀m. F ∈ {m} Co (maxfg m);*
*∀m ∈ -common. F ∈ {m} LeadsTo (greaterThan m);*
*n ∈ common |]*
*==> F ∈ (atMost (LEAST n. n ∈ common)) LeadsTo common"*
⟨*proof*⟩

**end**

**theory** *Network* **imports** *UNITY* **begin**

**datatype** *pvar = Sent | Rcvd | Idle*

**datatype** *pname = Aproc | Bproc*

**types** *state = "pname * pvar => nat"*

**locale** *F_props =*
**fixes** *F*
**assumes** *rsA: "F ∈ stable {s. s(Bproc,Rcvd) ≤ s(Aproc,Sent)}"*
**and** *rsB: "F ∈ stable {s. s(Aproc,Rcvd) ≤ s(Bproc,Sent)}"*
**and** *sent_nondec: "F ∈ stable {s. m ≤ s(proc,Sent)}"*
**and** *rcvd_nondec: "F ∈ stable {s. n ≤ s(proc,Rcvd)}"*
**and** *rcvd_idle: "F ∈ {s. s(proc,Idle) = Suc 0 & s(proc,Rcvd) = m}*

```
                    co {s. s(proc,Rcvd) = m --> s(proc,Idle) = Suc 0}"
  and sent_idle: "F ∈ {s. s(proc,Idle) = Suc 0 & s(proc,Sent) = n}
                    co {s. s(proc,Sent) = n}"


lemmas (in F_props)
      sent_nondec_A = sent_nondec [of _ Aproc]
  and sent_nondec_B = sent_nondec [of _ Bproc]
  and rcvd_nondec_A = rcvd_nondec [of _ Aproc]
  and rcvd_nondec_B = rcvd_nondec [of _ Bproc]
  and rcvd_idle_A = rcvd_idle [of Aproc]
  and rcvd_idle_B = rcvd_idle [of Bproc]
  and sent_idle_A = sent_idle [of Aproc]
  and sent_idle_B = sent_idle [of Bproc]

  and rs_AB = stable_Int [OF rsA rsB]
  and sent_nondec_AB = stable_Int [OF sent_nondec_A sent_nondec_B]
  and rcvd_nondec_AB = stable_Int [OF rcvd_nondec_A rcvd_nondec_B]
  and rcvd_idle_AB = constrains_Int [OF rcvd_idle_A rcvd_idle_B]
  and sent_idle_AB = constrains_Int [OF sent_idle_A sent_idle_B]
  and nondec_AB = stable_Int [OF sent_nondec_AB rcvd_nondec_AB]
  and idle_AB = constrains_Int [OF rcvd_idle_AB sent_idle_AB]
  and nondec_idle = constrains_Int [OF nondec_AB [unfolded stable_def]
                                       idle_AB]

lemma (in F_props)
  shows "F ∈ stable {s. s(Aproc,Idle) = Suc 0 & s(Bproc,Idle) = Suc 0 &
                    s(Aproc,Sent) = s(Bproc,Rcvd) &
                    s(Bproc,Sent) = s(Aproc,Rcvd) &
                    s(Aproc,Rcvd) = m & s(Bproc,Rcvd) = n}"
⟨proof⟩

end
```

# 19 The Token Ring

```
theory Token
imports "../WFair"

begin
```

From Misra, "A Logic for Concurrent Programming" (1994), sections 5.2 and 13.2.

## 19.1 Definitions

```
datatype pstate = Hungry | Eating | Thinking
```
— process states

```
record state =
  token :: "nat"
  proc  :: "nat => pstate"
```

**constdefs**
```
  HasTok :: "nat => state set"
    "HasTok i == {s. token s = i}"

  H :: "nat => state set"
    "H i == {s. proc s i = Hungry}"

  E :: "nat => state set"
    "E i == {s. proc s i = Eating}"

  T :: "nat => state set"
    "T i == {s. proc s i = Thinking}"
```

**locale** *Token* =
```
  fixes N and F and nodeOrder and "next"
  defines nodeOrder_def:
       "nodeOrder j == measure(%i. ((j+N)-i) mod N) ∩ {..<N} × {..<N}"
      and next_def:
       "next i == (Suc i) mod N"
  assumes N_positive [iff]: "0<N"
      and TR2:  "F ∈ (T i) co (T i ∪ H i)"
      and TR3:  "F ∈ (H i) co (H i ∪ E i)"
      and TR4:  "F ∈ (H i - HasTok i) co (H i)"
      and TR5:  "F ∈ (HasTok i) co (HasTok i ∪ -(E i))"
      and TR6:  "F ∈ (H i ∩ HasTok i) leadsTo (E i)"
      and TR7:  "F ∈ (HasTok i) leadsTo (HasTok (next i))"
```

**lemma** *HasToK_partition: "[| s ∈ HasTok i; s ∈ HasTok j |] ==> i=j"*
⟨*proof*⟩

**lemma** *not_E_eq: "(s ∉ E i) = (s ∈ H i | s ∈ T i)"*
⟨*proof*⟩

**lemma** (**in** *Token*) *token_stable: "F ∈ stable (-(E i) ∪ (HasTok i))"*
⟨*proof*⟩

## 19.2 Progress under Weak Fairness

**lemma** (**in** *Token*) *wf_nodeOrder: "wf(nodeOrder j)"*
⟨*proof*⟩

**lemma** (**in** *Token*) *nodeOrder_eq:*
```
     "[| i<N; j<N |] ==> ((next i, i) ∈ nodeOrder j) = (i ≠ j)"
```
⟨*proof*⟩

From "A Logic for Concurrent Programming", but not used in Chapter 4. Note the use of `case_tac`. Reasoning about leadsTo takes practice!

**lemma** (**in** *Token*) *TR7_nodeOrder:*
```
     "[| i<N; j<N |] ==>
      F ∈ (HasTok i) leadsTo ({s. (token s, i) ∈ nodeOrder j} ∪ HasTok j)"
```

⟨*proof*⟩

Chapter 4 variant, the one actually used below.

**lemma** (**in** *Token*) *TR7_aux*: "[| i<N; j<N; i≠j |]
==> F ∈ (HasTok i) leadsTo {s. (token s, i) ∈ nodeOrder j}"
⟨*proof*⟩

**lemma** (**in** *Token*) *token_lemma*:
"({s. token s < N} ∩ token -‘ {m}) = (if m<N then token -‘ {m} else {})"
⟨*proof*⟩

Misra's TR9: the token reaches an arbitrary node

**lemma** (**in** *Token*) *leadsTo_j*: "j<N ==> F ∈ {s. token s < N} leadsTo (HasTok j)"
⟨*proof*⟩

Misra's TR8: a hungry process eventually eats

**lemma** (**in** *Token*) *token_progress*:
"j<N ==> F ∈ ({s. token s < N} ∩ H j) leadsTo (E j)"
⟨*proof*⟩

**end**

**theory** *Channel* **imports** "../UNITY_Main" **begin**

**types** *state* = "nat set"

**consts**
F :: "state program"

**constdefs**
minSet :: "nat set => nat option"
"minSet A == if A={} then None else Some (LEAST x. x ∈ A)"

**axioms**

UC1: "F ∈ (minSet -‘ {Some x}) co (minSet -‘ (Some‘atLeast x))"

UC2: "F ∈ (minSet -‘ {Some x}) leadsTo {s. x ∉ s}"

**lemma** *minSet_eq_SomeD*: "minSet A = Some x ==> x ∈ A"
⟨*proof*⟩

**lemma** *minSet_empty* [*simp*]: " minSet{} = None"
⟨*proof*⟩

**lemma** *minSet_nonempty: "x ∈ A ==> minSet A = Some (LEAST x. x ∈ A)"*
⟨*proof*⟩

**lemma** *minSet_greaterThan:*
*"F ∈ (minSet -‘ {Some x}) leadsTo (minSet -‘ (Some‘greaterThan x))"*
⟨*proof*⟩

**lemma** *Channel_progress_lemma:*
*"F ∈ (UNIV-{{}}) leadsTo (minSet -‘ (Some‘atLeast y))"*
⟨*proof*⟩

**lemma** *Channel_progress: "!!y::nat. F ∈ (UNIV-{{}}) leadsTo {s. y ∉ s}"*
⟨*proof*⟩

**end**

**theory** *Lift*
**imports** *"../UNITY_Main"*

**begin**

**record** *state =*
*floor :: "int"* — current position of the lift
*"open" :: "bool"* — whether the door is opened at floor
*stop :: "bool"* — whether the lift is stopped at floor
*req :: "int set"* — for each floor, whether the lift is requested
*up :: "bool"* — current direction of movement
*move :: "bool"* — whether moving takes precedence over opening

**consts**
*Min :: "int"* — least and greatest floors
*Max :: "int"* — least and greatest floors

**axioms**
*Min_le_Max [iff]: "Min ≤ Max"*

**constdefs**

— Abbreviations: the "always" part

*above :: "state set"*
*"above == {s. ∃i. floor s < i & i ≤ Max & i ∈ req s}"*

*below :: "state set"*
*"below == {s. ∃i. Min ≤ i & i < floor s & i ∈ req s}"*

*queueing :: "state set"*
*"queueing == above ∪ below"*

*goingup :: "state set"*

```
  "goingup   == above ∩ ({s. up s}  ∪ -below)"

goingdown :: "state set"
  "goingdown == below ∩ ({s. ~ up s} ∪ -above)"

ready :: "state set"
  "ready == {s. stop s & ~ open s & move s}"
```

— Further abbreviations

```
moving :: "state set"
  "moving ==  {s. ~ stop s & ~ open s}"

stopped :: "state set"
  "stopped == {s. stop s  & ~ open s & ~ move s}"

opened :: "state set"
  "opened ==  {s. stop s  &  open s  &  move s}"

closed :: "state set"
```

— but this is the same as ready!!

```
  "closed ==  {s. stop s  & ~ open s &  move s}"

atFloor :: "int => state set"
  "atFloor n ==  {s. floor s = n}"

Req :: "int => state set"
  "Req n ==  {s. n ∈ req s}"
```

— The program

```
request_act :: "(state*state) set"
  "request_act == {(s,s'). s' = s (|stop:=True, move:=False|)
                                & ~ stop s & floor s ∈ req s}"

open_act :: "(state*state) set"
  "open_act ==
       {(s,s'). s' = s (|open :=True,
                         req  := req s - {floor s},
                         move := True|)
                    & stop s & ~ open s & floor s ∈ req s
                    & ~(move s & s ∈ queueing)}"

close_act :: "(state*state) set"
  "close_act == {(s,s'). s' = s (|open := False|) & open s}"

req_up :: "(state*state) set"
  "req_up ==
       {(s,s'). s' = s (|stop  :=False,
                         floor := floor s + 1,
                         up    := True|)
                    & s ∈ (ready ∩ goingup)}"
```

```
req_down :: "(state*state) set"
  "req_down ==
       {(s,s'). s' = s (|stop  :=False,
                        floor := floor s - 1,
                        up    := False|)
                      & s ∈ (ready ∩ goingdown)}"

move_up :: "(state*state) set"
  "move_up ==
       {(s,s'). s' = s (|floor := floor s + 1|)
                      & ~ stop s & up s & floor s ∉ req s}"

move_down :: "(state*state) set"
  "move_down ==
       {(s,s'). s' = s (|floor := floor s - 1|)
                      & ~ stop s & ~ up s & floor s ∉ req s}"

button_press  :: "(state*state) set"
```

— This action is omitted from prior treatments, which therefore are unrealistic: nobody asks the lift to do anything! But adding this action invalidates many of the existing progress arguments: various "ensures" properties fail. Maybe it should be constrained to only allow button presses in the current direction of travel, like in a real lift.

```
   "button_press ==
       {(s,s'). ∃n. s' = s (|req := insert n (req s)|)
                     & Min ≤ n & n ≤ Max}"


Lift :: "state program"
```

— for the moment, we OMIT `button_press`

```
  "Lift == mk_total_program
             ({s. floor s = Min & ~ up s & move s & stop s &
                       ~ open s & req s = {}},
              {request_act, open_act, close_act,
               req_up, req_down, move_up, move_down},
              UNIV)"
```

— Invariants

```
bounded :: "state set"
  "bounded == {s. Min ≤ floor s & floor s ≤ Max}"

open_stop :: "state set"
  "open_stop == {s. open s --> stop s}"

open_move :: "state set"
  "open_move == {s. open s --> move s}"

stop_floor :: "state set"
  "stop_floor == {s. stop s & ~ move s --> floor s ∈ req s}"

moving_up :: "state set"
  "moving_up == {s. ~ stop s & up s -->
```

```
                (∃f. floor s ≤ f & f ≤ Max & f ∈ req s)}"

  moving_down :: "state set"
    "moving_down == {s. ~ stop s & ~ up s -->
                    (∃f. Min ≤ f & f ≤ floor s & f ∈ req s)}"

  metric :: "[int,state] => int"
    "metric ==
       %n s. if floor s < n then (if up s then n - floor s
                                  else (floor s - Min) + (n-Min))
             else
             if n < floor s then (if up s then (Max - floor s) + (Max-n)
                                  else floor s - n)
             else 0"

locale Floor =
  fixes n
  assumes Min_le_n [iff]: "Min ≤ n"
      and n_le_Max [iff]: "n ≤ Max"

lemma not_mem_distinct: "[| x ∉ A;  y ∈ A |] ==> x ≠ y"
⟨proof⟩


declare Lift_def [THEN def_prg_Init, simp]

declare request_act_def [THEN def_act_simp, simp]
declare open_act_def [THEN def_act_simp, simp]
declare close_act_def [THEN def_act_simp, simp]
declare req_up_def [THEN def_act_simp, simp]
declare req_down_def [THEN def_act_simp, simp]
declare move_up_def [THEN def_act_simp, simp]
declare move_down_def [THEN def_act_simp, simp]
declare button_press_def [THEN def_act_simp, simp]


declare above_def [THEN def_set_simp, simp]
declare below_def [THEN def_set_simp, simp]
declare queueing_def [THEN def_set_simp, simp]
declare goingup_def [THEN def_set_simp, simp]
declare goingdown_def [THEN def_set_simp, simp]
declare ready_def [THEN def_set_simp, simp]


declare bounded_def [simp]
        open_stop_def [simp]
        open_move_def [simp]
        stop_floor_def [simp]
        moving_up_def [simp]
        moving_down_def [simp]

lemma open_stop: "Lift ∈ Always open_stop"
⟨proof⟩
```

**lemma** *stop_floor: "Lift ∈ Always stop_floor"*
⟨*proof*⟩

**lemma** *open_move: "Lift ∈ Always open_move"*
⟨*proof*⟩

**lemma** *moving_up: "Lift ∈ Always moving_up"*
⟨*proof*⟩

**lemma** *moving_down: "Lift ∈ Always moving_down"*
⟨*proof*⟩

**lemma** *bounded: "Lift ∈ Always bounded"*
⟨*proof*⟩

## 19.3 Progress

**declare** *moving_def [THEN def_set_simp, simp]*
**declare** *stopped_def [THEN def_set_simp, simp]*
**declare** *opened_def [THEN def_set_simp, simp]*
**declare** *closed_def [THEN def_set_simp, simp]*
**declare** *atFloor_def [THEN def_set_simp, simp]*
**declare** *Req_def [THEN def_set_simp, simp]*

The HUG'93 paper mistakenly omits the Req n from these!

**lemma** *E_thm01: "Lift ∈ (stopped ∩ atFloor n) LeadsTo (opened ∩ atFloor n)"*
⟨*proof*⟩

**lemma** *E_thm02: "Lift ∈ (Req n ∩ stopped - atFloor n) LeadsTo (Req n ∩ opened - atFloor n)"*
⟨*proof*⟩

**lemma** *E_thm03: "Lift ∈ (Req n ∩ opened - atFloor n) LeadsTo (Req n ∩ closed - (atFloor n - queueing))"*
⟨*proof*⟩

**lemma** *E_thm04: "Lift ∈ (Req n ∩ closed ∩ (atFloor n - queueing)) LeadsTo (opened ∩ atFloor n)"*
⟨*proof*⟩

**lemmas** *linorder_leI = linorder_not_less [THEN iffD1]*

**lemmas** (**in** *Floor*) *le_MinD = Min_le_n [THEN order_antisym]*
**and** *Max_leD = n_le_Max [THEN [2] order_antisym]*

**declare** (**in** *Floor*) *le_MinD [dest!]*

```
              and linorder_leI [THEN le_MinD, dest!]
              and Max_leD [dest!]
              and linorder_leI [THEN Max_leD, dest!]
```

**lemma** (**in** *Floor*) *E_thm05c:*

```
    "Lift ∈ (Req n ∩ closed - (atFloor n - queueing))
            LeadsTo ((closed ∩ goingup ∩ Req n)  ∪
                     (closed ∩ goingdown ∩ Req n))"
```

⟨*proof*⟩

**lemma** (**in** *Floor*) *lift_2:* "*Lift ∈ (Req n ∩ closed - (atFloor n - queueing))*

```
            LeadsTo (moving ∩ Req n)"
```

⟨*proof*⟩

**declare** *split_if_asm [split]*

**lemma** (**in** *Floor*) *E_thm12a:*

```
     "0 < N ==>
      Lift ∈ (moving ∩ Req n ∩ {s. metric n s = N} ∩
              {s. floor s ∉ req s} ∩ {s. up s})
             LeadsTo
               (moving ∩ Req n ∩ {s. metric n s < N})"
```

⟨*proof*⟩

**lemma** (**in** *Floor*) *E_thm12b:* "*0 < N ==>*

```
      Lift ∈ (moving ∩ Req n ∩ {s. metric n s = N} ∩
              {s. floor s ∉ req s} - {s. up s})
             LeadsTo (moving ∩ Req n ∩ {s. metric n s < N})"
```

⟨*proof*⟩

**lemma** (**in** *Floor*) *lift_4:*

```
     "0<N ==> Lift ∈ (moving ∩ Req n ∩ {s. metric n s = N} ∩
                              {s. floor s ∉ req s}) LeadsTo
                             (moving ∩ Req n ∩ {s. metric n s < N})"
```

⟨*proof*⟩

**lemma** (**in** *Floor*) *E_thm16a:* "*0<N*

```
  ==> Lift ∈ (closed ∩ Req n ∩ {s. metric n s = N} ∩ goingup) LeadsTo
```

```
            (moving ∩ Req n ∩ {s. metric n s < N})"
```
⟨*proof*⟩

**lemma** (**in** *Floor*) *E_thm16b*: "0<N ==>
```
     Lift ∈ (closed ∩ Req n ∩ {s. metric n s = N} ∩ goingdown) LeadsTo

                  (moving ∩ Req n ∩ {s. metric n s < N})"
```
⟨*proof*⟩

**lemma** (**in** *Floor*) *E_thm16c*:
```
    "0<N ==> Req n ∩ {s. metric n s = N} ⊆ goingup ∪ goingdown"
```
⟨*proof*⟩

**lemma** (**in** *Floor*) *lift_5*:
```
    "0<N ==> Lift ∈ (closed ∩ Req n ∩ {s. metric n s = N}) LeadsTo
                    (moving ∩ Req n ∩ {s. metric n s < N})"
```
⟨*proof*⟩

**lemma** (**in** *Floor*) *metric_eq_0D* [*dest*]:
```
    "[| metric n s = 0;  Min ≤ floor s;  floor s ≤ Max |] ==> floor s =
n"
```
⟨*proof*⟩

**lemma** (**in** *Floor*) *E_thm11*: "*Lift* ∈ (*moving* ∩ *Req n* ∩ {*s. metric n s = 0*})
*LeadsTo*
```
                      (stopped ∩ atFloor n)"
```
⟨*proof*⟩

**lemma** (**in** *Floor*) *E_thm13*:
```
  "Lift ∈ (moving ∩ Req n ∩ {s. metric n s = N} ∩ {s. floor s ∈ req s})

  LeadsTo (stopped ∩ Req n ∩ {s. metric n s = N} ∩ {s. floor s ∈ req s})"
```
⟨*proof*⟩

**lemma** (**in** *Floor*) *E_thm14*: "0 < N ==>
```
     Lift ∈
       (stopped ∩ Req n ∩ {s. metric n s = N} ∩ {s. floor s ∈ req s})
       LeadsTo (opened ∩ Req n ∩ {s. metric n s = N})"
```
⟨*proof*⟩

**lemma** (**in** *Floor*) *E_thm15*: "*Lift* ∈ (*opened* ∩ *Req n* ∩ {*s*. *metric n s* = *N*})

LeadsTo (*closed* ∩ *Req n* ∩ {*s*. *metric n s* = *N*})"
⟨*proof*⟩

**lemma** (**in** *Floor*) *lift_3_Req*: "0 < *N* ==>
*Lift* ∈
(*moving* ∩ *Req n* ∩ {*s*. *metric n s* = *N*} ∩ {*s*. *floor s* ∈ *req s*})
*LeadsTo* (*moving* ∩ *Req n* ∩ {*s*. *metric n s* < *N*})"
⟨*proof*⟩

**lemma** (**in** *Floor*) *Always_nonneg*: "*Lift* ∈ *Always* {*s*. 0 ≤ *metric n s*}"
⟨*proof*⟩

**lemmas** (**in** *Floor*) *R_thm11* = *Always_LeadsTo_weaken* [*OF Always_nonneg E_thm11*]

**lemma** (**in** *Floor*) *lift_3*:
"*Lift* ∈ (*moving* ∩ *Req n*) *LeadsTo* (*stopped* ∩ *atFloor n*)"
⟨*proof*⟩

**lemma** (**in** *Floor*) *lift_1*: "*Lift* ∈ (*Req n*) *LeadsTo* (*opened* ∩ *atFloor n*)"
⟨*proof*⟩

**end**

**theory** *Mutex* **imports** "../*UNITY_Main*" **begin**

**record** *state* =
*p* :: *bool*
*m* :: *int*
*n* :: *int*
*u* :: *bool*
*v* :: *bool*

**types** *command* = "(*state***state*) *set*"

**constdefs**

*U0* :: *command*
"*U0* == {(*s*,*s'*). *s'* = *s* (|*u*:=*True*, *m*:=1|) & *m s* = 0}"

*U1* :: *command*

```
    "U1 == {(s,s'). s' = s (|p:= v s, m:=2|) & m s = 1}"

  U2 :: command
    "U2 == {(s,s'). s' = s (|m:=3|) & ~ p s & m s = 2}"

  U3 :: command
    "U3 == {(s,s'). s' = s (|u:=False, m:=4|) & m s = 3}"

  U4 :: command
    "U4 == {(s,s'). s' = s (|p:=True, m:=0|) & m s = 4}"



  V0 :: command
    "V0 == {(s,s'). s' = s (|v:=True, n:=1|) & n s = 0}"

  V1 :: command
    "V1 == {(s,s'). s' = s (|p:= ~ u s, n:=2|) & n s = 1}"

  V2 :: command
    "V2 == {(s,s'). s' = s (|n:=3|) & p s & n s = 2}"

  V3 :: command
    "V3 == {(s,s'). s' = s (|v:=False, n:=4|) & n s = 3}"

  V4 :: command
    "V4 == {(s,s'). s' = s (|p:=False, n:=0|) & n s = 4}"

  Mutex :: "state program"
    "Mutex == mk_total_program
                ({s. ~ u s & ~ v s & m s = 0 & n s = 0},
                 {U0, U1, U2, U3, U4, V0, V1, V2, V3, V4},
                 UNIV)"



  IU :: "state set"
    "IU == {s. (u s = (1 ≤ m s & m s ≤ 3)) & (m s = 3 --> ~ p s)}"

  IV :: "state set"
    "IV == {s. (v s = (1 ≤ n s & n s ≤ 3)) & (n s = 3 --> p s)}"



  bad_IU :: "state set"
    "bad_IU == {s. (u s = (1 ≤ m s & m s ≤ 3)) &
                   (3 ≤ m s & m s ≤ 4 --> ~ p s)}"
```

**declare** *Mutex_def [THEN def_prg_Init, simp]*

**declare** *U0_def [THEN def_act_simp, simp]*
**declare** *U1_def [THEN def_act_simp, simp]*

**declare** *U2_def [THEN def_act_simp, simp]*
**declare** *U3_def [THEN def_act_simp, simp]*
**declare** *U4_def [THEN def_act_simp, simp]*
**declare** *V0_def [THEN def_act_simp, simp]*
**declare** *V1_def [THEN def_act_simp, simp]*
**declare** *V2_def [THEN def_act_simp, simp]*
**declare** *V3_def [THEN def_act_simp, simp]*
**declare** *V4_def [THEN def_act_simp, simp]*

**declare** *IU_def [THEN def_set_simp, simp]*
**declare** *IV_def [THEN def_set_simp, simp]*
**declare** *bad_IU_def [THEN def_set_simp, simp]*

**lemma** *IU: "Mutex ∈ Always IU"*
⟨*proof*⟩

**lemma** *IV: "Mutex ∈ Always IV"*
⟨*proof*⟩

**lemma** *mutual_exclusion: "Mutex ∈ Always {s. ~ (m s = 3 & n s = 3)}"*
⟨*proof*⟩

**lemma** *"Mutex ∈ Always bad_IU"*
⟨*proof*⟩

**lemma** *eq_123: "((1::int) ≤ i & i ≤ 3) = (i = 1 | i = 2 | i = 3)"*
⟨*proof*⟩

**lemma** *U_F0: "Mutex ∈ {s. m s=2} Unless {s. m s=3}"*
⟨*proof*⟩

**lemma** *U_F1: "Mutex ∈ {s. m s=1} LeadsTo {s. p s = v s & m s = 2}"*
⟨*proof*⟩

**lemma** *U_F2: "Mutex ∈ {s. ~ p s & m s = 2} LeadsTo {s. m s = 3}"*
⟨*proof*⟩

**lemma** *U_F3: "Mutex ∈ {s. m s = 3} LeadsTo {s. p s}"*
⟨*proof*⟩

**lemma** *U_lemma2: "Mutex ∈ {s. m s = 2} LeadsTo {s. p s}"*
⟨*proof*⟩

**lemma** *U_lemma1: "Mutex ∈ {s. m s = 1} LeadsTo {s. p s}"*
⟨*proof*⟩

**lemma** *U_lemma123: "Mutex ∈ {s. 1 ≤ m s & m s ≤ 3} LeadsTo {s. p s}"*
⟨*proof*⟩

**lemma** *u_Leadsto_p: "Mutex ∈ {s. u s} LeadsTo {s. p s}"*
⟨*proof*⟩

**lemma** *V_F0: "Mutex ∈ {s. n s=2} Unless {s. n s=3}"*
⟨*proof*⟩

**lemma** *V_F1: "Mutex ∈ {s. n s=1} LeadsTo {s. p s = (~ u s) & n s = 2}"*
⟨*proof*⟩

**lemma** *V_F2: "Mutex ∈ {s. p s & n s = 2} LeadsTo {s. n s = 3}"*
⟨*proof*⟩

**lemma** *V_F3: "Mutex ∈ {s. n s = 3} LeadsTo {s. ~ p s}"*
⟨*proof*⟩

**lemma** *V_lemma2: "Mutex ∈ {s. n s = 2} LeadsTo {s. ~ p s}"*
⟨*proof*⟩

**lemma** *V_lemma1: "Mutex ∈ {s. n s = 1} LeadsTo {s. ~ p s}"*
⟨*proof*⟩

**lemma** *V_lemma123: "Mutex ∈ {s. 1 ≤ n s & n s ≤ 3} LeadsTo {s. ~ p s}"*
⟨*proof*⟩

**lemma** *v_Leadsto_not_p: "Mutex ∈ {s. v s} LeadsTo {s. ~ p s}"*
⟨*proof*⟩

**lemma** *m1_Leadsto_3: "Mutex ∈ {s. m s = 1} LeadsTo {s. m s = 3}"*
⟨*proof*⟩

**lemma** *n1_Leadsto_3: "Mutex ∈ {s. n s = 1} LeadsTo {s. n s = 3}"*
⟨*proof*⟩

**end**

**theory** *Reach* **imports** *"../UNITY_Main"* **begin**

**typedecl** *vertex*

**types** *state = "vertex=>bool"*

**consts**
*init :: "vertex"*

*edges :: "(vertex*vertex) set"*

**constdefs**

*asgt :: "[vertex,vertex] => (state*state) set"*
*"asgt u v == {(s,s'). s' = s(v:= s u | s v)}"*

*Rprg :: "state program"*
*"Rprg == mk_total_program ({%v. v=init}, ⋃(u,v)∈edges. {asgt u v}, UNIV)"*

*reach_invariant :: "state set"*
*"reach_invariant == {s. (∀v. s v --> (init, v) ∈ edges^*) & s init}"*

*fixedpoint :: "state set"*
*"fixedpoint == {s. ∀(u,v)∈edges. s u --> s v}"*

*metric :: "state => nat"*
*"metric s == card {v. ~ s v}"*

*We assume that the set of vertices is finite

**axioms**
*finite_graph: "finite (UNIV :: vertex set)"*

**lemma** *ifE [elim!]:*
*"[| if P then Q else R;*
*[| P; Q |] ==> S;*
*[| ~ P; R |] ==> S |] ==> S"*
⟨*proof*⟩

**declare** *Rprg_def [THEN def_prg_Init, simp]*

**declare** *asgt_def [THEN def_act_simp,simp]*

All vertex sets are finite

**declare** *finite_subset [OF subset_UNIV finite_graph, iff]*

**declare** *reach_invariant_def [THEN def_set_simp, simp]*

**lemma** *reach_invariant: "Rprg ∈ Always reach_invariant"*
⟨*proof*⟩

**lemma** *fixedpoint_invariant_correct:*
*"fixedpoint ∩ reach_invariant = { %v. (init, v) ∈ edges^* }"*
⟨*proof*⟩

**lemma** *lemma1:*
*"FP Rprg ⊆ fixedpoint"*
⟨*proof*⟩

**lemma** *lemma2:*
*"fixedpoint ⊆ FP Rprg"*
⟨*proof*⟩

**lemma** *FP_fixedpoint: "FP Rprg = fixedpoint"*
⟨*proof*⟩

**lemma** *Compl_fixedpoint: "- fixedpoint = (⋃(u,v)∈edges. {s. s u & ~ s v})"*
⟨*proof*⟩

**lemma** *Diff_fixedpoint:*
*"A - fixedpoint = (⋃(u,v)∈edges. A ∩ {s. s u & ~ s v})"*
⟨*proof*⟩

**lemma** *Suc_metric: "~ s x ==> Suc (metric (s(x:=True))) = metric s"*
⟨*proof*⟩

**lemma** *metric_less [intro!]: "~ s x ==> metric (s(x:=True)) < metric s"*
⟨*proof*⟩

**lemma** *metric_le: "metric (s(y:=s x | s y)) ≤ metric s"*
⟨*proof*⟩

**lemma** *LeadsTo_Diff_fixedpoint:*
*"Rprg ∈ ((metric-‘{m}) - fixedpoint) LeadsTo (metric-‘(lessThan m))"*
⟨*proof*⟩

**lemma** *LeadsTo_Un_fixedpoint:*
*"Rprg ∈ (metric-‘{m}) LeadsTo (metric-‘(lessThan m) ∪ fixedpoint)"*
⟨*proof*⟩

**lemma** *LeadsTo_fixedpoint: "Rprg ∈ UNIV LeadsTo fixedpoint"*
⟨*proof*⟩

**lemma** *LeadsTo_correct: "Rprg ∈ UNIV LeadsTo { %v. (init, v) ∈ edges^* }"*

⟨*proof*⟩

**end**

```
theory Reachability imports "../Detects" Reach begin

types  edge = "(vertex*vertex)"

record state =
  reach :: "vertex => bool"
  nmsg  :: "edge => nat"

consts root :: "vertex"
       E :: "edge set"
       V :: "vertex set"

inductive_set REACHABLE :: "edge set"
  where
    base: "v ∈ V ==> ((v,v) ∈ REACHABLE)"
  | step: "((u,v) ∈ REACHABLE) & (v,w) ∈ E ==> ((u,w) ∈ REACHABLE)"

constdefs
  reachable :: "vertex => state set"
  "reachable p == {s. reach s p}"

  nmsg_eq :: "nat => edge  => state set"
  "nmsg_eq k == %e. {s. nmsg s e = k}"

  nmsg_gt :: "nat => edge  => state set"
  "nmsg_gt k  == %e. {s. k < nmsg s e}"

  nmsg_gte :: "nat => edge => state set"
  "nmsg_gte k == %e. {s. k ≤ nmsg s e}"

  nmsg_lte  :: "nat => edge => state set"
  "nmsg_lte k  == %e. {s. nmsg s e ≤ k}"

  final :: "state set"
  "final == (⋂v∈V. reachable v <==> {s. (root, v) ∈ REACHABLE}) ∩
            (INTER E (nmsg_eq 0))"

axioms

    Graph1: "root ∈ V"

    Graph2: "(v,w) ∈ E ==> (v ∈ V) & (w ∈ V)"

    MA1:  "F ∈ Always (reachable root)"

    MA2:  "v ∈ V ==> F ∈ Always (- reachable v ∪ {s. ((root,v) ∈ REACHABLE)})"

    MA3:  "[|v ∈ V;w ∈ V|] ==> F ∈ Always (-(nmsg_gt 0 (v,w)) ∪ (reachable
```

```
v))"

    MA4:  "(v,w) ∈ E ==>
           F ∈ Always (-(reachable v) ∪ (nmsg_gt 0 (v,w)) ∪ (reachable w))"

    MA5:  "[|v ∈ V; w ∈ V|]
           ==> F ∈ Always (nmsg_gte 0 (v,w) ∩ nmsg_lte (Suc 0) (v,w))"

    MA6:  "[|v ∈ V|] ==> F ∈ Stable (reachable v)"

    MA6b: "[|v ∈ V;w ∈ W|] ==> F ∈ Stable (reachable v ∩ nmsg_lte k (v,w))"

    MA7:  "[|v ∈ V;w ∈ V|] ==> F ∈ UNIV LeadsTo nmsg_eq 0 (v,w)"
```

**lemmas** *E_imp_in_V_L = Graph2 [THEN conjunct1, standard]*
**lemmas** *E_imp_in_V_R = Graph2 [THEN conjunct2, standard]*

**lemma** *lemma2:*
*"(v,w) ∈ E ==> F ∈ reachable v LeadsTo nmsg_eq 0 (v,w) ∩ reachable v"*
⟨*proof*⟩

**lemma** *Induction_base: "(v,w) ∈ E ==> F ∈ reachable v LeadsTo reachable w"*
⟨*proof*⟩

**lemma** *REACHABLE_LeadsTo_reachable:*
*"(v,w) ∈ REACHABLE ==> F ∈ reachable v LeadsTo reachable w"*
⟨*proof*⟩

**lemma** *Detects_part1: "F ∈ {s. (root,v) ∈ REACHABLE} LeadsTo reachable v"*
⟨*proof*⟩

**lemma** *Reachability_Detected:*
*"v ∈ V ==> F ∈ (reachable v) Detects {s. (root,v) ∈ REACHABLE}"*
⟨*proof*⟩

**lemma** *LeadsTo_Reachability:*
*"v ∈ V ==> F ∈ UNIV LeadsTo (reachable v <==> {s. (root,v) ∈ REACHABLE})"*
⟨*proof*⟩

**lemma** *Eq_lemma1:*
*"(reachable v <==> {s. (root,v) ∈ REACHABLE}) =*
*{s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))}"*
⟨*proof*⟩

**lemma** *Eq_lemma2:*

```
     "(reachable v <==> (if (root,v) ∈ REACHABLE then UNIV else {})) =
      {s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))}"
```
⟨*proof*⟩

**lemma** *final_lemma1:*
```
     "(⋂v ∈ V. ⋂w ∈ V. {s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))
&
                                s ∈ nmsg_eq 0 (v,w)})
      ⊆ final"
```
⟨*proof*⟩

**lemma** *final_lemma2:*
```
 "E≠{}
  ==> (⋂v ∈ V. ⋂e ∈ E. {s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))}

                            ∩ nmsg_eq 0 e)    ⊆  final"
```
⟨*proof*⟩

**lemma** *final_lemma3:*
```
     "E≠{}
      ==> (⋂v ∈ V. ⋂e ∈ E.
           (reachable v <==> {s. (root,v) ∈ REACHABLE}) ∩ nmsg_eq 0 e)
          ⊆ final"
```
⟨*proof*⟩

**lemma** *final_lemma4:*
```
     "E≠{}
      ==> (⋂v ∈ V. ⋂e ∈ E.
           {s. ((s ∈ reachable v) = ((root,v) ∈ REACHABLE))} ∩ nmsg_eq 0
e)
          = final"
```
⟨*proof*⟩

**lemma** *final_lemma5:*
```
     "E≠{}
      ==> (⋂v ∈ V. ⋂e ∈ E.
           ((reachable v) <==> {s. (root,v) ∈ REACHABLE}) ∩ nmsg_eq 0 e)

          = final"
```
⟨*proof*⟩

**lemma** *final_lemma6:*
```
     "(⋂v ∈ V. ⋂w ∈ V.
       (reachable v <==> {s. (root,v) ∈ REACHABLE}) ∩ nmsg_eq 0 (v,w))
      ⊆ final"
```
⟨*proof*⟩

**lemma** *final_lemma7:*
```
"final =
 (⋂v ∈ V. ⋂w ∈ V.
  ((reachable v) <==> {s. (root,v) ∈ REACHABLE}) ∩
  (-{s. (v,w) ∈ E} ∪ (nmsg_eq 0 (v,w))))"
```
⟨*proof*⟩

**lemma** *not_REACHABLE_imp_Stable_not_reachable:*
```
"[| v ∈ V; (root,v) ∉ REACHABLE |] ==> F ∈ Stable (- reachable v)"
```
⟨*proof*⟩

**lemma** *Stable_reachable_EQ_R:*
```
"v ∈ V ==> F ∈ Stable (reachable v <==> {s. (root,v) ∈ REACHABLE})"
```
⟨*proof*⟩

**lemma** *lemma4:*
```
"((nmsg_gte 0 (v,w) ∩ nmsg_lte (Suc 0) (v,w)) ∩
  (- nmsg_gt 0 (v,w) ∪ A))
 ⊆ A ∪ nmsg_eq 0 (v,w)"
```
⟨*proof*⟩

**lemma** *lemma5:*
```
"reachable v ∩ nmsg_eq 0 (v,w) =
 ((nmsg_gte 0 (v,w) ∩ nmsg_lte (Suc 0) (v,w)) ∩
  (reachable v ∩ nmsg_lte 0 (v,w)))"
```
⟨*proof*⟩

**lemma** *lemma6:*
```
"- nmsg_gt 0 (v,w) ∪ reachable v ⊆ nmsg_eq 0 (v,w) ∪ reachable v"
```
⟨*proof*⟩

**lemma** *Always_reachable_OR_nmsg_0:*
```
"[|v ∈ V; w ∈ V|] ==> F ∈ Always (reachable v ∪ nmsg_eq 0 (v,w))"
```
⟨*proof*⟩

**lemma** *Stable_reachable_AND_nmsg_0:*
```
"[|v ∈ V; w ∈ V|] ==> F ∈ Stable (reachable v ∩ nmsg_eq 0 (v,w))"
```
⟨*proof*⟩

**lemma** *Stable_nmsg_0_OR_reachable:*
```
"[|v ∈ V; w ∈ V|] ==> F ∈ Stable (nmsg_eq 0 (v,w) ∪ reachable v)"
```
⟨*proof*⟩

**lemma** *not_REACHABLE_imp_Stable_not_reachable_AND_nmsg_0:*
```
"[| v ∈ V; w ∈ V; (root,v) ∉ REACHABLE |]
```

```
     ==> F ∈ Stable (- reachable v ∩ nmsg_eq 0 (v,w))"
```
⟨*proof*⟩

**lemma** *Stable_reachable_EQ_R_AND_nmsg_0:*
```
     "[| v ∈ V; w ∈ V |]
      ==> F ∈ Stable ((reachable v <==> {s. (root,v) ∈ REACHABLE}) ∩
                      nmsg_eq 0 (v,w))"
```
⟨*proof*⟩

**lemma** *UNIV_lemma: "UNIV ⊆ (⋂v ∈ V. UNIV)"*
⟨*proof*⟩

**lemmas** *UNIV_LeadsTo_completion =*
```
    LeadsTo_weaken_L [OF Finite_stable_completion UNIV_lemma]
```

**lemma** *LeadsTo_final_E_empty: "E={} ==> F ∈ UNIV LeadsTo final"*
⟨*proof*⟩

**lemma** *Leadsto_reachability_AND_nmsg_0:*
```
     "[| v ∈ V; w ∈ V |]
      ==> F ∈ UNIV LeadsTo
             ((reachable v <==> {s. (root,v): REACHABLE}) ∩ nmsg_eq 0 (v,w))"
```
⟨*proof*⟩

**lemma** *LeadsTo_final_E_NOT_empty: "E≠{} ==> F ∈ UNIV LeadsTo final"*
⟨*proof*⟩

**lemma** *LeadsTo_final: "F ∈ UNIV LeadsTo final"*
⟨*proof*⟩

**lemma** *Stable_final_E_empty: "E={} ==> F ∈ Stable final"*
⟨*proof*⟩

**lemma** *Stable_final_E_NOT_empty: "E≠{} ==> F ∈ Stable final"*
⟨*proof*⟩

**lemma** *Stable_final: "F ∈ Stable final"*
⟨*proof*⟩

**end**

# 20 Analyzing the Needham-Schroeder Public-Key Protocol in UNITY

**theory** *NSP_Bad* **imports** *"../../Auth/Public" "../UNITY_Main"* **begin**

This is the flawed version, vulnerable to Lowe's attack. From page 260 of Burrows, Abadi and Needham. A Logic of Authentication. Proc. Royal Soc. 426 (1989).

**types** *state = "event list"*

**constdefs**

```
  Fake :: "(state*state) set"
    "Fake == {(s,s'). 
              ∃B X. s' = Says Spy B X # s
                    & X ∈ synth (analz (spies s))}"



  NS1 :: "(state*state) set"
    "NS1 == {(s1,s'). 
             ∃A1 B NA.
                 s' = Says A1 B (Crypt (pubK B) {|Nonce NA, Agent A1|}) # s1
              & Nonce NA ∉ used s1}"


  NS2 :: "(state*state) set"
    "NS2 == {(s2,s'). 
             ∃A' A2 B NA NB.
                 s' = Says B A2 (Crypt (pubK A2) {|Nonce NA, Nonce NB|}) #
s2
              & Says A' B (Crypt (pubK B) {|Nonce NA, Agent A2|}) ∈ set s2
              & Nonce NB ∉ used s2}"


  NS3 :: "(state*state) set"
    "NS3 == {(s3,s'). 
             ∃A3 B' B NA NB.
                 s' = Says A3 B (Crypt (pubK B) (Nonce NB)) # s3
              & Says A3  B (Crypt (pubK B) {|Nonce NA, Agent A3|}) ∈ set
s3
              & Says B' A3 (Crypt (pubK A3) {|Nonce NA, Nonce NB|}) ∈ set
s3}"


constdefs
  Nprg :: "state program"
```

```
    "Nprg == mk_total_program({[]}, {Fake, NS1, NS2, NS3}, UNIV)"

declare spies_partsEs [elim]
declare analz_into_parts [dest]
declare Fake_parts_insert_in_Un  [dest]
```

For other theories, e.g. Mutex and Lift, using [iff] slows proofs down. Here, it facilitates re-use of the Auth proofs.

```
declare Fake_def [THEN def_act_simp, iff]
declare NS1_def [THEN def_act_simp, iff]
declare NS2_def [THEN def_act_simp, iff]
declare NS3_def [THEN def_act_simp, iff]

declare Nprg_def [THEN def_prg_Init, simp]
```

A "possibility property": there are traces that reach the end. Replace by LEADSTO proof!

```
lemma "A ≠ B ==>
       ∃NB. ∃s ∈ reachable Nprg. Says A B (Crypt (pubK B) (Nonce NB)) ∈
set s"
```
⟨*proof*⟩

## 20.1 Inductive Proofs about *ns_public*

```
lemma ns_constrainsI:
     "(!!act s s'. [| act ∈ {Id, Fake, NS1, NS2, NS3};
                      (s,s') ∈ act;  s ∈ A |] ==> s' ∈ A')
      ==> Nprg ∈ A co A'"
```
⟨*proof*⟩

This ML code does the inductions directly.

⟨*ML*⟩

Converts invariants into statements about reachable states

```
lemmas Always_Collect_reachableD =
     Always_includes_reachable [THEN subsetD, THEN CollectD]
```

Spy never sees another agent's private key! (unless it's bad at start)

```
lemma Spy_see_priK:
     "Nprg ∈ Always {s. (Key (priK A) ∈ parts (spies s)) = (A ∈ bad)}"
```
⟨*proof*⟩
```
declare Spy_see_priK [THEN Always_Collect_reachableD, simp]

lemma Spy_analz_priK:
     "Nprg ∈ Always {s. (Key (priK A) ∈ analz (spies s)) = (A ∈ bad)}"
```
⟨*proof*⟩
```
declare Spy_analz_priK [THEN Always_Collect_reachableD, simp]
```

## 20.2 Authenticity properties obtained from NS2

It is impossible to re-use a nonce in both NS1 and NS2 provided the nonce is secret. (Honest users generate fresh nonces.)

**lemma** *no_nonce_NS1_NS2:*
```
 "Nprg
  ∈ Always {s. Crypt (pubK C) {|NA', Nonce NA|} ∈ parts (spies s) -->
               Crypt (pubK B) {|Nonce NA, Agent A|} ∈ parts (spies s) -->
               Nonce NA ∈ analz (spies s)}"
```
⟨*proof*⟩

Adding it to the claset slows down proofs...

**lemmas** *no_nonce_NS1_NS2_reachable =*
```
       no_nonce_NS1_NS2 [THEN Always_Collect_reachableD, rule_format]
```

Unicity for NS1: nonce NA identifies agents A and B

**lemma** *unique_NA_lemma:*
```
     "Nprg
  ∈ Always {s. Nonce NA ∉ analz (spies s) -->
               Crypt(pubK B) {|Nonce NA, Agent A|} ∈ parts(spies s) -->
               Crypt(pubK B') {|Nonce NA, Agent A'|} ∈ parts(spies s) -->
               A=A' & B=B'}"
```
⟨*proof*⟩

Unicity for NS1: nonce NA identifies agents A and B

**lemma** *unique_NA:*
```
     "[| Crypt(pubK B)  {|Nonce NA, Agent A|} ∈ parts(spies s);
         Crypt(pubK B') {|Nonce NA, Agent A'|} ∈ parts(spies s);
         Nonce NA ∉ analz (spies s);
         s ∈ reachable Nprg |]
      ==> A=A' & B=B'"
```
⟨*proof*⟩

Secrecy: Spy does not see the nonce sent in msg NS1 if A and B are secure

**lemma** *Spy_not_see_NA:*
```
     "[| A ∉ bad;  B ∉ bad |]
  ==> Nprg ∈ Always
              {s. Says A B (Crypt(pubK B) {|Nonce NA, Agent A|}) ∈ set s
                  --> Nonce NA ∉ analz (spies s)}"
```
⟨*proof*⟩

Authentication for A: if she receives message 2 and has used NA to start a run, then B has sent message 2.

**lemma** *A_trusts_NS2:*
```
 "[| A ∉ bad;  B ∉ bad |]
  ==> Nprg ∈ Always
              {s. Says A B (Crypt(pubK B) {|Nonce NA, Agent A|}) ∈ set s &
                  Crypt(pubK A) {|Nonce NA, Nonce NB|} ∈ parts (knows Spy
s)
         --> Says B A (Crypt(pubK A) {|Nonce NA, Nonce NB|}) ∈ set s}"
```

⟨*proof*⟩

If the encrypted message appears then it originated with Alice in NS1

**lemma** *B_trusts_NS1:*
```
     "Nprg ∈ Always
```

```
             {s. Nonce NA ∉ analz (spies s) -->
                 Crypt (pubK B) {|Nonce NA, Agent A|} ∈ parts (spies s)
         --> Says A B (Crypt (pubK B) {|Nonce NA, Agent A|}) ∈ set s}"
```

⟨*proof*⟩

## 20.3 Authenticity properties obtained from NS2

Unicity for NS2: nonce NB identifies nonce NA and agent A. Proof closely follows that of `unique_NA`.

**lemma** *unique_NB_lemma:*

```
 "Nprg
  ∈ Always {s. Nonce NB ∉ analz (spies s)  -->
               Crypt (pubK A) {|Nonce NA, Nonce NB|} ∈ parts (spies s) -->
               Crypt(pubK A'){|Nonce NA', Nonce NB|} ∈ parts(spies s) -->
               A=A' & NA=NA'}"
```

⟨*proof*⟩

**lemma** *unique_NB:*

```
     "[| Crypt(pubK A) {|Nonce NA, Nonce NB|} ∈ parts(spies s);
         Crypt(pubK A'){|Nonce NA', Nonce NB|} ∈ parts(spies s);
         Nonce NB ∉ analz (spies s);
         s ∈ reachable Nprg |]
      ==> A=A' & NA=NA'"
```

⟨*proof*⟩

NB remains secret PROVIDED Alice never responds with round 3

**lemma** *Spy_not_see_NB:*

```
     "[| A ∉ bad;  B ∉ bad |]
  ==> Nprg ∈ Always
              {s. Says B A (Crypt (pubK A) {|Nonce NA, Nonce NB|}) ∈ set s
&
                  (ALL C. Says A C (Crypt (pubK C) (Nonce NB)) ∉ set s)
                  --> Nonce NB ∉ analz (spies s)}"
```

⟨*proof*⟩

Authentication for B: if he receives message 3 and has used NB in message 2, then A has sent message 3–to somebody....

**lemma** *B_trusts_NS3:*

```
     "[| A ∉ bad;  B ∉ bad |]
  ==> Nprg ∈ Always
              {s. Crypt (pubK B) (Nonce NB) ∈ parts (spies s) &
                  Says B A  (Crypt (pubK A) {|Nonce NA, Nonce NB|}) ∈ set
s
                  --> (∃C. Says A C (Crypt (pubK C) (Nonce NB)) ∈ set s)}"
```

⟨*proof*⟩

Can we strengthen the secrecy theorem? NO

**lemma** *"[| A ∉ bad;  B ∉ bad |]*

```
  ==> Nprg ∈ Always
              {s. Says B A (Crypt (pubK A) {|Nonce NA, Nonce NB|}) ∈ set s
                  --> Nonce NB ∉ analz (spies s)}"
```

⟨*proof*⟩

**end**

**theory** *Handshake* **imports** *"../UNITY_Main"* **begin**

**record** *state =*
*BB :: bool*
*NF :: nat*
*NG :: nat*

**constdefs**

*cmdF :: "(state*state) set"*
*"cmdF == {(s,s'). s' = s (|NF:= Suc(NF s), BB:=False|) & BB s}"*

*F :: "state program"*
*"F == mk_total_program ({s. NF s = 0 & BB s}, {cmdF}, UNIV)"*

*cmdG :: "(state*state) set"*
*"cmdG == {(s,s'). s' = s (|NG:= Suc(NG s), BB:=True|) & ~ BB s}"*

*G :: "state program"*
*"G == mk_total_program ({s. NG s = 0 & BB s}, {cmdG}, UNIV)"*

*invFG :: "state set"*
*"invFG == {s. NG s <= NF s & NF s <= Suc (NG s) & (BB s = (NF s = NG s))}"*

**declare** *F_def [THEN def_prg_Init, simp]*
*G_def [THEN def_prg_Init, simp]*

*cmdF_def [THEN def_act_simp, simp]*
*cmdG_def [THEN def_act_simp, simp]*

*invFG_def [THEN def_set_simp, simp]*

**lemma** *invFG: "(F Join G) : Always invFG"*
⟨*proof*⟩

**lemma** *lemma2_1: "(F Join G) : ({s. NF s = k} - {s. BB s}) LeadsTo*
*({s. NF s = k} Int {s. BB s})"*
⟨*proof*⟩

**lemma** *lemma2_2: "(F Join G) : ({s. NF s = k} Int {s. BB s}) LeadsTo*
*{s. k < NF s}"*

⟨*proof*⟩

**lemma** *progress: "(F Join G) : UNIV LeadsTo {s. m < NF s}"*
⟨*proof*⟩

**end**

# 21 A Family of Similar Counters: Original Version

**theory** *Counter* **imports** *"../UNITY_Main"* **begin**

**datatype** *name = C | c nat*
**types** *state = "name=>int"*

**consts**
  *sum  :: "[nat,state]=>int"*
  *sumj :: "[nat, nat, state]=>int"*

**primrec**
  *"sum 0 s = 0"*
  *"sum (Suc i) s = s (c i) + sum i s"*

**primrec**
  *"sumj 0 i s = 0"*
  *"sumj (Suc n) i s = (if n=i then sum n s else s (c n) + sumj n i s)"*

**types** *command = "(state*state)set"*

**constdefs**
  *a :: "nat=>command"*
 *"a i == {(s, s'). s'=s(c i:= s (c i) + 1, C:= s C + 1)}"*

  *Component :: "nat => state program"*
  *"Component i ==*
    *mk_total_program({s. s C = 0 & s (c i) = 0}, {a i},*
                    *⋃G ∈ preserves (%s. s (c i)). Acts G)"*

**declare** *Component_def [THEN def_prg_Init, simp]*
**declare** *a_def [THEN def_act_simp, simp]*

**lemma** *sum_upd_gt [rule_format]: "∀n. I<n --> sum I (s(c n := x)) = sum I s"*
⟨*proof*⟩

**lemma** *sum_upd_eq: "sum I (s(c I := x)) = sum I s"*
⟨*proof*⟩

**lemma** *sum_upd_C: "sum I (s(C := x)) = sum I s"*
⟨*proof*⟩

**lemma** *sumj_upd_ci: "sumj I i (s(c i := x)) = sumj I i s"*
⟨*proof*⟩

**lemma** *sumj_upd_C: "sumj I i (s(C := x)) = sumj I i s"*
⟨*proof*⟩

**lemma** *sumj_sum_gt [rule_format]: "∀i. I<i--> (sumj I i s = sum I s)"*
⟨*proof*⟩

**lemma** *sumj_sum_eq: "(sumj I I s = sum I s)"*
⟨*proof*⟩

**lemma** *sum_sumj [rule_format]: "∀i. i<I-->(sum I s = s (c i) + sumj I i s)"*
⟨*proof*⟩

**lemma** *p2: "Component i ∈ stable {s. s C = s (c i) + k}"*
⟨*proof*⟩

**lemma** *p3: "Component i ∈ stable {s. ∀v. v≠c i & v≠C --> s v = k v}"*
⟨*proof*⟩

**lemma** *p2_p3_lemma1:*
*"(∀k. Component i ∈ stable ({s. s C = s (c i) + sumj I i k}*
*∩ {s. ∀v. v≠c i & v≠C --> s v = k v}))*
*= (Component i ∈ stable {s. s C = s (c i) + sumj I i s})"*
⟨*proof*⟩

**lemma** *p2_p3_lemma2:*
*"∀k. Component i ∈ stable ({s. s C = s (c i) + sumj I i k} Int*
*{s. ∀v. v≠c i & v≠C --> s v = k v})"*
⟨*proof*⟩

**lemma** *p2_p3: "Component i ∈ stable {s. s C = s (c i) + sumj I i s}"*
⟨*proof*⟩

**lemma** *sum_0' [rule_format]: "(∀i. i < I --> s (c i) = 0) --> sum I s = 0"*
⟨*proof*⟩

**lemma** *safety:*
*"0<I ==> (⨅i ∈ {i. i<I}. Component i) ∈ invariant {s. s C = sum I s}"*
⟨*proof*⟩

**end**

# 22 A Family of Similar Counters: Version with Compatibility

**theory** *Counterc* **imports** *"../UNITY_Main"* **begin**

**typedecl** *state*

**consts**
```
  C :: "state=>int"
  c :: "state=>nat=>int"
```

**consts**
```
  sum  :: "[nat,state]=>int"
  sumj :: "[nat, nat, state]=>int"
```

**primrec**
```
  "sum 0 s = 0"
  "sum (Suc i) s = (c s) i + sum i s"
```

**primrec**
```
  "sumj 0 i s = 0"
  "sumj (Suc n) i s = (if n=i then sum n s else (c s) n + sumj n i s)"
```

**types** *command = "(state*state)set"*

**constdefs**
```
  a :: "nat=>command"
 "a i == {(s, s'). (c s') i = (c s) i + 1 & (C s') = (C s) + 1}"

  Component :: "nat => state program"
  "Component i == mk_total_program({s. C s = 0 & (c s) i = 0},
                                   {a i},
                                   ⋃G ∈ preserves (%s. (c s) i). Acts G)"
```

**declare** *Component_def [THEN def_prg_Init, simp]*
**declare** *Component_def [THEN def_prg_AllowedActs, simp]*
**declare** *a_def [THEN def_act_simp, simp]*

**lemma** *sum_sumj_eq1 [rule_format]: "∀i. I<i--> (sum I s = sumj I i s)"*
⟨*proof*⟩

**lemma** *sum_sumj_eq2 [rule_format]: "i<I --> sum I s  = c s i + sumj I i s"*
⟨*proof*⟩

**lemma** *sum_ext [rule_format]:*
```
     "(∀i. i<I --> c s' i = c s i) --> (sum I s' = sum I s)"
```
⟨*proof*⟩

**lemma** *sumj_ext [rule_format]:*

```
    "(∀j. j<I & j≠i --> c s' j =  c s j) --> (sumj I i s' = sumj I i s)"
```
⟨*proof*⟩

**lemma** *sum0 [rule_format]: "(∀i. i<I --> c s i = 0) -->  sum I s = 0"*
⟨*proof*⟩

**lemma** *Component_ok_iff:*
```
    "(Component i ok G) =
     (G ∈ preserves (%s. c s i) & Component i ∈ Allowed G)"
```
⟨*proof*⟩
**declare** *Component_ok_iff [iff]*
**declare** *OK_iff_ok [iff]*
**declare** *preserves_def [simp]*

**lemma** *p2: "Component i ∈ stable {s. C s = (c s) i + k}"*
⟨*proof*⟩

**lemma** *p3:*
```
    "[| OK I Component; i∈I |]
     ==> Component i ∈ stable {s. ∀j∈I. j≠i --> c s j = c k j}"
```
⟨*proof*⟩

**lemma** *p2_p3_lemma1:*
```
    "[| OK {i. i<I} Component; i<I |] ==>
     ∀k. Component i ∈ stable ({s. C s = c s i + sumj I i k} Int
                               {s. ∀j∈{i. i<I}. j≠i --> c s j = c k j})"
```
⟨*proof*⟩

**lemma** *p2_p3_lemma2:*
```
    "(∀k. F ∈ stable ({s. C s = (c s) i + sumj I i k} Int
                      {s. ∀j∈{i. i<I}. j≠i --> c s j = c k j}))
     ==> (F ∈ stable {s. C s = c s i + sumj I i s})"
```
⟨*proof*⟩

**lemma** *p2_p3:*
```
    "[| OK {i. i<I} Component; i<I |]
     ==> Component i ∈ stable {s. C s = c s i + sumj I i s}"
```
⟨*proof*⟩

**lemma** *safety:*
```
    "[| 0<I; OK {i. i<I} Component |]
     ==> (⨅i∈{i. i<I}. (Component i)) ∈ invariant {s. C s = sum I s}"
```
⟨*proof*⟩

**end**

```
theory PriorityAux
imports "../UNITY_Main"
begin

typedecl vertex

constdefs
  symcl :: "(vertex*vertex)set=>(vertex*vertex)set"
  "symcl r == r ∪ (r^-1)"
    — symmetric closure: removes the orientation of a relation

  neighbors :: "[vertex, (vertex*vertex)set]=>vertex set"
  "neighbors i r == ((r ∪ r^-1)``{i}) - {i}"
    — Neighbors of a vertex i

  R :: "[vertex, (vertex*vertex)set]=>vertex set"
  "R i r == r``{i}"

  A :: "[vertex, (vertex*vertex)set]=>vertex set"
  "A i r == (r^-1)``{i}"

  reach :: "[vertex, (vertex*vertex)set]=> vertex set"
  "reach i r == (r^+)``{i}"
    — reachable and above vertices: the original notation was R* and A*

  above :: "[vertex, (vertex*vertex)set]=> vertex set"
  "above i r == ((r^-1)^+)``{i}"

  reverse :: "[vertex, (vertex*vertex) set]=>(vertex*vertex)set"
  "reverse i r == (r - {(x,y). x=i | y=i} ∩ r) ∪ ({(x,y). x=i|y=i} ∩ r)^-1"

  derive1 :: "[vertex, (vertex*vertex)set, (vertex*vertex)set]=>bool"
    — The original definition
  "derive1 i r q == symcl r = symcl q &
                    (∀k k'. k≠i & k'≠i -->((k,k'):r) = ((k,k'):q)) &
                    A i r = {} & R i q = {}"

  derive :: "[vertex, (vertex*vertex)set, (vertex*vertex)set]=>bool"
    — Our alternative definition
  "derive i r q == A i r = {} & (q = reverse i r)"

axioms
  finite_vertex_univ:  "finite (UNIV :: vertex set)"
    — we assume that the universe of vertices is finite

declare derive_def [simp] derive1_def [simp] symcl_def [simp]
        A_def [simp] R_def [simp]
        above_def [simp] reach_def [simp]
        reverse_def [simp] neighbors_def [simp]
```

All vertex sets are finite

**declare** *finite_subset [OF subset_UNIV finite_vertex_univ, iff]*

and relatons over vertex are finite too

**lemmas** *finite_UNIV_Prod =*
*finite_Prod_UNIV [OF finite_vertex_univ finite_vertex_univ]*

**declare** *finite_subset [OF subset_UNIV finite_UNIV_Prod, iff]*

**lemma** *image0_trancl_iff_image0_r: "((r^+)``{i} = {}) = (r``{i} = {})"*
⟨*proof*⟩

**lemma** *image0_r_iff_image0_trancl: "(r``{i}={}) = (ALL x. ((i,x):r^+) = False)"*
⟨*proof*⟩

**lemma** *acyclic_eq_wf: "!!r::(vertex*vertex)set. acyclic r = wf r"*
⟨*proof*⟩

**lemma** *derive_derive1_eq: "derive i r q = derive1 i r q"*
⟨*proof*⟩

**lemma** *lemma1_a:*
*"[| x ∈ reach i q; derive1 k r q |] ==> x≠k --> x ∈ reach i r"*
⟨*proof*⟩

**lemma** *reach_lemma: "derive k r q ==> reach i q ⊆ (reach i r ∪ {k})"*
⟨*proof*⟩

**lemma** *reach_above_lemma:*
*"(∀i. reach i q ⊆ (reach i r ∪ {k})) =*
*(∀x. x≠k --> (∀i. i ∉ above x r --> i ∉ above x q))"*
⟨*proof*⟩

**lemma** *maximal_converse_image0:*
*"(z, i):r^+ ==> (∀y. (y, z):r --> (y,i) ∉ r^+) = ((r^-1)``{z}={})"*
⟨*proof*⟩

**lemma** *above_lemma_a:*
*"acyclic r ==> A i r≠{}-->(∃j ∈ above i r. A j r = {})"*
⟨*proof*⟩

**lemma** *above_lemma_b:*
*"acyclic r ==> above i r≠{}-->(∃j ∈ above i r. above j r = {})"*
⟨*proof*⟩

**end**

# 23 The priority system

**theory** *Priority* **imports** *PriorityAux* **begin**

From Charpentier and Chandy, Examples of Program Composition Illustrating the Use of Universal Properties In J. Rolim (editor), Parallel and Distributed Processing, Spriner LNCS 1586 (1999), pages 1215-1227.

**types** *state = "(vertex*vertex)set"*
**types** *command = "vertex=>(state*state)set"*

**consts**
*init :: "(vertex*vertex)set"*
— the initial state

Following the definitions given in section 4.4

**constdefs**
*highest :: "[vertex, (vertex*vertex)set]=>bool"*
*"highest i r == A i r = {}"*
— i has highest priority in r

*lowest :: "[vertex, (vertex*vertex)set]=>bool"*
*"lowest i r == R i r = {}"*
— i has lowest priority in r

*act :: command*
*"act i == {(s, s'). s'=reverse i s & highest i s}"*

*Component :: "vertex=>state program"*
*"Component i == mk_total_program({init}, {act i}, UNIV)"*
— All components start with the same initial state

Some Abbreviations

**constdefs**
*Highest :: "vertex=>state set"*
*"Highest i == {s. highest i s}"*

*Lowest :: "vertex=>state set"*
*"Lowest i == {s. lowest i s}"*

*Acyclic :: "state set"*
*"Acyclic == {s. acyclic s}"*

*Maximal :: "state set"*
— Every "above" set has a maximal vertex
*"Maximal == ⋂i. {s. ~highest i s-->(∃j ∈ above i  s. highest j s)}"*

*Maximal' :: "state set"*
— Maximal vertex: equivalent definition

```
"Maximal' == ⋂i. Highest i Un (⋃j. {s. j ∈ above i s} Int Highest j)"


Safety :: "state set"
"Safety == ⋂i. {s. highest i s --> (∀j ∈ neighbors i s. ~highest j s)}"




system :: "state program"
"system == JN i. Component i"
```

**declare** *highest_def [simp] lowest_def [simp]*
**declare** *Highest_def [THEN def_set_simp, simp]*
**and** *Lowest_def [THEN def_set_simp, simp]*

**declare** *Component_def [THEN def_prg_Init, simp]*
**declare** *act_def [THEN def_act_simp, simp]*

## 23.1 Component correctness proofs

neighbors is stable

**lemma** *Component_neighbors_stable: "Component i ∈ stable {s. neighbors k s = n}"*
⟨*proof*⟩

property 4

**lemma** *Component_waits_priority: "Component i: {s. ((i,j):s) = b} Int (- Highest i) co {s. ((i,j):s)=b}"*
⟨*proof*⟩

property 5: charpentier and Chandy mistakenly express it as 'transient Highest i'. Consider the case where i has neighbors

**lemma** *Component_yields_priority:*
*"Component i: {s. neighbors i s ≠ {}} Int Highest i*
*ensures - Highest i"*
⟨*proof*⟩

or better

**lemma** *Component_yields_priority': "Component i ∈ Highest i ensures Lowest i"*
⟨*proof*⟩

property 6: Component doesn't introduce cycle

**lemma** *Component_well_behaves: "Component i ∈ Highest i co Highest i Un Lowest i"*
⟨*proof*⟩

property 7: local axiom

**lemma** *locality: "Component i ∈ stable {s. ∀j k. j≠i & k≠i--> ((j,k):s) = b j k}"*
⟨*proof*⟩

## 23.2 System properties

property 8: strictly universal

**lemma** *Safety: "system ∈ stable Safety"*
⟨*proof*⟩

property 13: universal

**lemma** *p13: "system ∈ {s. s = q} co {s. s=q} Un {s. ∃i. derive i q s}"*
⟨*proof*⟩

property 14: the 'above set' of a Component that hasn't got priority doesn't increase

**lemma** *above_not_increase:*
*"system ∈ -Highest i Int {s. j∉above i s} co {s. j∉above i s}"*
⟨*proof*⟩

**lemma** *above_not_increase':*
*"system ∈ -Highest i Int {s. above i s = x} co {s. above i s <= x}"*
⟨*proof*⟩

p15: universal property: all Components well behave

**lemma** *system_well_behaves [rule_format]:*
*"∀i. system ∈ Highest i co Highest i Un Lowest i"*
⟨*proof*⟩

**lemma** *Acyclic_eq: "Acyclic = (⋂i. {s. i∉above i s})"*
⟨*proof*⟩

**lemmas** *system_co =*
*constrains_Un [OF above_not_increase [rule_format] system_well_behaves]*

**lemma** *Acyclic_stable: "system ∈ stable Acyclic"*
⟨*proof*⟩

**lemma** *Acyclic_subset_Maximal: "Acyclic <= Maximal"*
⟨*proof*⟩

property 17: original one is an invariant

**lemma** *Acyclic_Maximal_stable: "system ∈ stable (Acyclic Int Maximal)"*
⟨*proof*⟩

property 5: existential property

**lemma** *Highest_leadsTo_Lowest: "system ∈ Highest i leadsTo Lowest i"*
⟨*proof*⟩

a lowest i can never be in any abover set

**lemma** *Lowest_above_subset: "Lowest i <= (⋂k. {s. i∉above k s})"*
⟨*proof*⟩

property 18: a simpler proof than the original, one which uses psp

**lemma** *Highest_escapes_above: "system ∈ Highest i leadsTo (⋂k. {s. i∉above k s})"*
⟨*proof*⟩

**lemma** *Highest_escapes_above':*
*"system ∈ Highest j Int {s. j ∈ above i s} leadsTo {s. j∉above i s}"*
⟨*proof*⟩

## 23.3 The main result: above set decreases

The original proof of the following formula was wrong

**lemma** *Highest_iff_above0: "Highest i = {s. above i s ={}}"*
⟨*proof*⟩

**lemmas** *above_decreases_lemma =*
*psp [THEN leadsTo_weaken, OF Highest_escapes_above' above_not_increase']*

**lemma** *above_decreases:*
*"system ∈ (⋃j. {s. above i s = x} Int {s. j ∈ above i s} Int Highest j)*
*leadsTo {s. above i s < x}"*
⟨*proof*⟩

**lemma** *Maximal_eq_Maximal': "Maximal = Maximal'"*
⟨*proof*⟩

**lemma** *Acyclic_subset:*
*"x≠{} ==>*
*Acyclic Int {s. above i s = x} <=*
*(⋃j. {s. above i s = x} Int {s. j ∈ above i s} Int Highest j)"*
⟨*proof*⟩

**lemmas** *above_decreases' = leadsTo_weaken_L [OF above_decreases Acyclic_subset]*
**lemmas** *above_decreases_psp = psp_stable [OF above_decreases' Acyclic_stable]*

**lemma** *above_decreases_psp':*
*"x≠{}==> system ∈ Acyclic Int {s. above i s = x} leadsTo*
*Acyclic Int {s. above i s < x}"*
⟨*proof*⟩

**lemmas** *finite_psubset_induct = wf_finite_psubset [THEN leadsTo_wf_induct]*

**lemma** *Progress: "system ∈ Acyclic leadsTo Highest i"*
⟨*proof*⟩

We have proved all (relevant) theorems given in the paper. We didn't assume any thing about the relation `r`. It is not necessary that `r` be a priority relation

as assumed in the original proof. It suffices that we start from a state which is finite and acyclic.

**end**

**theory** *TimerArray* **imports** *"../UNITY_Main"* **begin**

**types** *'a state = "nat * 'a"*

**constdefs**
```
  count  :: "'a state => nat"
    "count s == fst s"

  decr  :: "('a state * 'a state) set"
    "decr == UN n uu. {((Suc n, uu), (n,uu))}"

  Timer :: "'a state program"
    "Timer == mk_total_program (UNIV, {decr}, UNIV)"
```

**declare** *Timer_def [THEN def_prg_Init, simp]*

**declare** *count_def [simp] decr_def [simp]*

**lemma** *Timer_leadsTo_zero: "Timer : UNIV leadsTo {s. count s = 0}"*
⟨*proof*⟩

**lemma** *Timer_preserves_snd [iff]: "Timer : preserves snd"*
⟨*proof*⟩

**declare** *PLam_stable [simp]*

**lemma** *TimerArray_leadsTo_zero:*
```
     "finite I
      ==> (plam i: I. Timer) : UNIV leadsTo {(s,uu). ALL i:I. s i = 0}"
```
⟨*proof*⟩

**end**

# 24 Projections of State Sets

**theory** *Project* **imports** *Extend* **begin**

**constdefs**
```
  projecting :: "['c program => 'c set, 'a*'b => 'c,
                  'a program, 'c program set, 'a program set] => bool"
    "projecting C h F X' X ==
       ∀G. extend h F⊔G ∈ X' --> F⊔project h (C G) G ∈ X"
```

```
  extending :: "['c program => 'c set, 'a*'b => 'c, 'a program,
                 'c program set, 'a program set] => bool"
    "extending C h F Y' Y ==
       ∀G. extend h F  ok G --> F⊔project h (C G) G ∈ Y
              --> extend h F⊔G ∈ Y'"

  subset_closed :: "'a set set => bool"
    "subset_closed U == ∀A ∈ U. Pow A ⊆ U"
```

**lemma** (**in** *Extend*) *project_extend_constrains_I:*

```
     "F ∈ A co B ==> project h C (extend h F) ∈ A co B"
```

⟨*proof*⟩

## 24.1 Safety

**lemma** (**in** *Extend*) *project_unless [rule_format]:*

```
     "[| G ∈ stable C;  project h C G ∈ A unless B |]
      ==> G ∈ (C ∩ extend_set h A) unless (extend_set h B)"
```

⟨*proof*⟩

**lemma** (**in** *Extend*) *Join_project_constrains:*

```
     "(F⊔project h C G ∈ A co B)  =
        (extend h F⊔G ∈ (C ∩ extend_set h A) co (extend_set h B) &
         F ∈ A co B)"
```

⟨*proof*⟩

**lemma** (**in** *Extend*) *Join_project_stable:*

```
     "extend h F⊔G ∈ stable C
      ==> (F⊔project h C G ∈ stable A)  =
          (extend h F⊔G ∈ stable (C ∩ extend_set h A) &
           F ∈ stable A)"
```

⟨*proof*⟩

**lemma** (**in** *Extend*) *project_constrains_I:*

```
     "extend h F⊔G ∈ extend_set h A co extend_set h B
      ==> F⊔project h C G ∈ A co B"
```

⟨*proof*⟩

**lemma** (**in** *Extend*) *project_increasing_I:*

```
     "extend h F⊔G ∈ increasing (func o f)
      ==> F⊔project h C G ∈ increasing func"
```

⟨*proof*⟩

**lemma** (**in** *Extend*) *Join_project_increasing:*

```
     "(F⊔project h UNIV G ∈ increasing func)  =
      (extend h F⊔G ∈ increasing (func o f))"
```

⟨*proof*⟩

**lemma** (**in** *Extend*) *project_constrains_D:*

```
"F⊔project h UNIV G ∈ A co B
 ==> extend h F⊔G ∈ extend_set h A co extend_set h B"
```
⟨*proof*⟩

## 24.2 "projecting" and union/intersection (no converses)

**lemma** *projecting_Int:*
```
"[| projecting C h F XA' XA;  projecting C h F XB' XB |]
 ==> projecting C h F (XA' ∩ XB') (XA ∩ XB)"
```
⟨*proof*⟩

**lemma** *projecting_Un:*
```
"[| projecting C h F XA' XA;  projecting C h F XB' XB |]
 ==> projecting C h F (XA' ∪ XB') (XA ∪ XB)"
```
⟨*proof*⟩

**lemma** *projecting_INT:*
```
"[| !!i. i ∈ I ==> projecting C h F (X' i) (X i) |]
 ==> projecting C h F (⋂i ∈ I. X' i) (⋂i ∈ I. X i)"
```
⟨*proof*⟩

**lemma** *projecting_UN:*
```
"[| !!i. i ∈ I ==> projecting C h F (X' i) (X i) |]
 ==> projecting C h F (⋃i ∈ I. X' i) (⋃i ∈ I. X i)"
```
⟨*proof*⟩

**lemma** *projecting_weaken:*
```
"[| projecting C h F X' X;  U'<=X';  X ⊆ U |] ==> projecting C h F U'
U"
```
⟨*proof*⟩

**lemma** *projecting_weaken_L:*
```
"[| projecting C h F X' X;  U'<=X' |] ==> projecting C h F U' X"
```
⟨*proof*⟩

**lemma** *extending_Int:*
```
"[| extending C h F YA' YA;  extending C h F YB' YB |]
 ==> extending C h F (YA' ∩ YB') (YA ∩ YB)"
```
⟨*proof*⟩

**lemma** *extending_Un:*
```
"[| extending C h F YA' YA;  extending C h F YB' YB |]
 ==> extending C h F (YA' ∪ YB') (YA ∪ YB)"
```
⟨*proof*⟩

**lemma** *extending_INT:*
```
"[| !!i. i ∈ I ==> extending C h F (Y' i) (Y i) |]
 ==> extending C h F (⋂i ∈ I. Y' i) (⋂i ∈ I. Y i)"
```
⟨*proof*⟩

**lemma** *extending_UN:*
```
"[| !!i. i ∈ I ==> extending C h F (Y' i) (Y i) |]
 ==> extending C h F (⋃i ∈ I. Y' i) (⋃i ∈ I. Y i)"
```
⟨*proof*⟩

**lemma** *extending_weaken:*
```
"[| extending C h F Y' Y;  Y'<=V';  V ⊆ Y |] ==> extending C h F V' V"
```
⟨*proof*⟩

**lemma** *extending_weaken_L:*
```
"[| extending C h F Y' Y;  Y'<=V' |] ==> extending C h F V' Y"
```
⟨*proof*⟩

**lemma** *projecting_UNIV: "projecting C h F X' UNIV"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *projecting_constrains:*
```
"projecting C h F (extend_set h A co extend_set h B) (A co B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *projecting_stable:*
```
"projecting C h F (stable (extend_set h A)) (stable A)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *projecting_increasing:*
```
"projecting C h F (increasing (func o f)) (increasing func)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extending_UNIV: "extending C h F UNIV Y"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extending_constrains:*
```
"extending (%G. UNIV) h F (extend_set h A co extend_set h B) (A co B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extending_stable:*
```
"extending (%G. UNIV) h F (stable (extend_set h A)) (stable A)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extending_increasing:*
```
"extending (%G. UNIV) h F (increasing (func o f)) (increasing func)"
```
⟨*proof*⟩

## 24.3 Reachability and project

**lemma** (**in** *Extend*) *reachable_imp_reachable_project:*
```
"[| reachable (extend h F⊔G) ⊆ C;
    z ∈ reachable (extend h F⊔G) |]
 ==> f z ∈ reachable (F⊔project h C G)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_Constrains_D:*
```
"F⊔project h (reachable (extend h F⊔G)) G ∈ A Co B
 ==> extend h F⊔G ∈ (extend_set h A) Co (extend_set h B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_Stable_D:*
```
"F⊔project h (reachable (extend h F⊔G)) G ∈ Stable A
```

```
     ==> extend h F⊔G ∈ Stable (extend_set h A)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_Always_D:*
```
     "F⊔project h (reachable (extend h F⊔G)) G ∈ Always A
      ==> extend h F⊔G ∈ Always (extend_set h A)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_Increasing_D:*
```
     "F⊔project h (reachable (extend h F⊔G)) G ∈ Increasing func
      ==> extend h F⊔G ∈ Increasing (func o f)"
```
⟨*proof*⟩

## 24.4 Converse results for weak safety: benefits of the argument C

**lemma** (**in** *Extend*) *reachable_project_imp_reachable:*
```
     "[| C ⊆ reachable(extend h F⊔G);
         x ∈ reachable (F⊔project h C G) |]
      ==> ∃y. h(x,y) ∈ reachable (extend h F⊔G)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_set_reachable_extend_eq:*
```
     "project_set h (reachable (extend h F⊔G)) =
      reachable (F⊔project h (reachable (extend h F⊔G)) G)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *reachable_extend_Join_subset:*
```
     "reachable (extend h F⊔G) ⊆ C
      ==> reachable (extend h F⊔G) ⊆
          extend_set h (reachable (F⊔project h C G))"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_Constrains_I:*
```
     "extend h F⊔G ∈ (extend_set h A) Co (extend_set h B)
      ==> F⊔project h (reachable (extend h F⊔G)) G ∈ A Co B"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_Stable_I:*
```
     "extend h F⊔G ∈ Stable (extend_set h A)
      ==> F⊔project h (reachable (extend h F⊔G)) G ∈ Stable A"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_Always_I:*
```
     "extend h F⊔G ∈ Always (extend_set h A)
      ==> F⊔project h (reachable (extend h F⊔G)) G ∈ Always A"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_Increasing_I:*
```
    "extend h F⊔G ∈ Increasing (func o f)
     ==> F⊔project h (reachable (extend h F⊔G)) G ∈ Increasing func"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_Constrains:*
```
"(F⊔project h (reachable (extend h F⊔G)) G ∈ A Co B)  =
 (extend h F⊔G ∈ (extend_set h A) Co (extend_set h B))"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_Stable:*
```
"(F⊔project h (reachable (extend h F⊔G)) G ∈ Stable A)  =
 (extend h F⊔G ∈ Stable (extend_set h A))"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_Increasing:*
```
"(F⊔project h (reachable (extend h F⊔G)) G ∈ Increasing func)  =
 (extend h F⊔G ∈ Increasing (func o f))"
```
⟨*proof*⟩

## 24.5 A lot of redundant theorems: all are proved to facilitate reasoning about guarantees.

**lemma** (**in** *Extend*) *projecting_Constrains:*
```
"projecting (%G. reachable (extend h F⊔G)) h F
            (extend_set h A Co extend_set h B) (A Co B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *projecting_Stable:*
```
"projecting (%G. reachable (extend h F⊔G)) h F
            (Stable (extend_set h A)) (Stable A)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *projecting_Always:*
```
"projecting (%G. reachable (extend h F⊔G)) h F
            (Always (extend_set h A)) (Always A)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *projecting_Increasing:*
```
"projecting (%G. reachable (extend h F⊔G)) h F
            (Increasing (func o f)) (Increasing func)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extending_Constrains:*
```
"extending (%G. reachable (extend h F⊔G)) h F
             (extend_set h A Co extend_set h B) (A Co B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extending_Stable:*
```
"extending (%G. reachable (extend h F⊔G)) h F
             (Stable (extend_set h A)) (Stable A)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extending_Always:*
```
"extending (%G. reachable (extend h F⊔G)) h F
             (Always (extend_set h A)) (Always A)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extending_Increasing:*
```
"extending (%G. reachable (extend h F⊔G)) h F
             (Increasing (func o f)) (Increasing func)"
```
⟨*proof*⟩

## 24.6 leadsETo in the precondition (??)

### 24.6.1 transient

**lemma** (**in** *Extend*) *transient_extend_set_imp_project_transient:*
```
"[| G ∈ transient (C ∩ extend_set h A);  G ∈ stable C |]
 ==> project h C G ∈ transient (project_set h C ∩ A)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_extend_transient_D:*
```
"project h C (extend h F) ∈ transient (project_set h C ∩ D)
 ==> F ∈ transient (project_set h C ∩ D)"
```
⟨*proof*⟩

### 24.6.2 ensures – a primitive combining progress with safety

**lemma** (**in** *Extend*) *ensures_extend_set_imp_project_ensures:*
```
"[| extend h F ∈ stable C;  G ∈ stable C;
    extend h F⊔G ∈ A ensures B;  A-B = C ∩ extend_set h D |]
 ==> F⊔project h C G
       ∈ (project_set h C ∩ project_set h A) ensures (project_set h B)"
```
⟨*proof*⟩

Transferring a transient property upwards

**lemma** (**in** *Extend*) *project_transient_extend_set:*
```
"project h C G ∈ transient (project_set h C ∩ A - B)
 ==> G ∈ transient (C ∩ extend_set h A - extend_set h B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_unless2 [rule_format]:*
```
"[| G ∈ stable C;  project h C G ∈ (project_set h C ∩ A) unless B |]

 ==> G ∈ (C ∩ extend_set h A) unless (extend_set h B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_unless:*
```
"[|extend h F ∈ stable C; F ∈ A unless B|]
 ==> extend h F ∈ C ∩ extend_set h A unless extend_set h B"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *Join_project_ensures [rule_format]:*
```
"[| extend h F⊔G ∈ stable C;
    F⊔project h C G ∈ A ensures B |]
 ==> extend h F⊔G ∈ (C ∩ extend_set h A) ensures (extend_set h B)"
```
⟨*proof*⟩

Lemma useful for both STRONG and WEAK progress, but the transient condition's very strong

**lemma** (**in** *Extend*) *PLD_lemma:*
```
"[| extend h F⊔G ∈ stable C;
    F⊔project h C G ∈ (project_set h C ∩ A) leadsTo B |]
 ==> extend h F⊔G ∈
      C ∩ extend_set h (project_set h C ∩ A) leadsTo (extend_set h B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_leadsTo_D_lemma:*
```
"[| extend h F⊔G ∈ stable C;
    F⊔project h C G ∈ (project_set h C ∩ A) leadsTo B |]
 ==> extend h F⊔G ∈ (C ∩ extend_set h A) leadsTo (extend_set h B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *Join_project_LeadsTo:*
```
"[| C = (reachable (extend h F⊔G));
    F⊔project h C G ∈ A LeadsTo B |]
 ==> extend h F⊔G ∈ (extend_set h A) LeadsTo (extend_set h B)"
```
⟨*proof*⟩

## 24.7 Towards the theorem *project_Ensures_D*

**lemma** (**in** *Extend*) *project_ensures_D_lemma:*
```
"[| G ∈ stable ((C ∩ extend_set h A) - (extend_set h B));
    F⊔project h C G ∈ (project_set h C ∩ A) ensures B;
    extend h F⊔G ∈ stable C |]
 ==> extend h F⊔G ∈ (C ∩ extend_set h A) ensures (extend_set h B)"
```

⟨*proof*⟩

**lemma** (**in** *Extend*) *project_ensures_D:*
```
"[| F⊔project h UNIV G ∈ A ensures B;
    G ∈ stable (extend_set h A - extend_set h B) |]
 ==> extend h F⊔G ∈ (extend_set h A) ensures (extend_set h B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_Ensures_D:*
```
"[| F⊔project h (reachable (extend h F⊔G)) G ∈ A Ensures B;
    G ∈ stable (reachable (extend h F⊔G) ∩ extend_set h A -
                extend_set h B) |]
 ==> extend h F⊔G ∈ (extend_set h A) Ensures (extend_set h B)"
```
⟨*proof*⟩

## 24.8 Guarantees

**lemma** (**in** *Extend*) *project_act_Restrict_subset_project_act:*
```
"project_act h (Restrict C act) ⊆ project_act h act"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *subset_closed_ok_extend_imp_ok_project:*
```
"[| extend h F ok G; subset_closed (AllowedActs F) |]
 ==> F ok project h C G"
```

⟨*proof*⟩

**lemma** (**in** *Extend*) *project_guarantees_raw:*
**assumes** *xguary:* "F ∈ X guarantees Y"
**and** *closed:* "subset_closed (AllowedActs F)"
**and** *project:* "!!G. extend h F⊔G ∈ X'
==> F⊔project h (C G) G ∈ X"
**and** *extend:* "!!G. [| F⊔project h (C G) G ∈ Y |]
==> extend h F⊔G ∈ Y'"
**shows** "extend h F ∈ X' guarantees Y'"
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_guarantees:*
"[| F ∈ X guarantees Y; subset_closed (AllowedActs F);
projecting C h F X' X; extending C h F Y' Y |]
==> extend h F ∈ X' guarantees Y'"
⟨*proof*⟩

## 24.9 guarantees corollaries

### 24.9.1 Some could be deleted: the required versions are easy to prove

**lemma** (**in** *Extend*) *extend_guar_increasing:*
"[| F ∈ UNIV guarantees increasing func;
subset_closed (AllowedActs F) |]
==> extend h F ∈ X' guarantees increasing (func o f)"
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_guar_Increasing:*
"[| F ∈ UNIV guarantees Increasing func;
subset_closed (AllowedActs F) |]
==> extend h F ∈ X' guarantees Increasing (func o f)"
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_guar_Always:*
"[| F ∈ Always A guarantees Always B;
subset_closed (AllowedActs F) |]
==> extend h F
∈ Always(extend_set h A) guarantees Always(extend_set h B)"
⟨*proof*⟩

### 24.9.2 Guarantees with a leadsTo postcondition

**lemma** (**in** *Extend*) *project_leadsTo_D:*
"F⊔project h UNIV G ∈ A leadsTo B
==> extend h F⊔G ∈ (extend_set h A) leadsTo (extend_set h B)"
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_LeadsTo_D:*
"F⊔project h (reachable (extend h F⊔G)) G ∈ A LeadsTo B

```
      ==> extend h F⊔G ∈ (extend_set h A) LeadsTo (extend_set h B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extending_leadsTo:*
```
     "extending (%G. UNIV) h F
                (extend_set h A leadsTo extend_set h B) (A leadsTo B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *extending_LeadsTo:*
```
     "extending (%G. reachable (extend h F⊔G)) h F
                (extend_set h A LeadsTo extend_set h B) (A LeadsTo B)"
```
⟨*proof*⟩

**end**

# 25 Progress Under Allowable Sets

**theory** *ELT* **imports** *Project* **begin**

**inductive_set**

```
  elt :: "['a set set, 'a program] => ('a set * 'a set) set"
  for CC :: "'a set set" and F :: "'a program"
 where

   Basis:  "[| F : A ensures B;  A-B : (insert {} CC) |] ==> (A,B) : elt CC
F"

 | Trans:  "[| (A,B) : elt CC F;  (B,C) : elt CC F |] ==> (A,C) : elt CC F"

 | Union:  "ALL A: S. (A,B) : elt CC F ==> (Union S, B) : elt CC F"
```

**constdefs**

```
  givenBy :: "['a => 'b] => 'a set set"
    "givenBy f == range (%B. f-` B)"


  leadsETo :: "['a set, 'a set set, 'a set] => 'a program set"
                                        ("(3_/ leadsTo[_]/ _)" [80,0,80] 80)
    "leadsETo A CC B == {F. (A,B) : elt CC F}"

  LeadsETo :: "['a set, 'a set set, 'a set] => 'a program set"
                                        ("(3_/ LeadsTo[_]/ _)" [80,0,80] 80)
    "LeadsETo A CC B ==
      {F. F : (reachable F Int A) leadsTo[(%C. reachable F Int C) ` CC] B}"
```

**lemma** *givenBy_id [simp]: "givenBy id = UNIV"*
⟨*proof*⟩

**lemma** *givenBy_eq_all: "(givenBy v) = {A. ALL x:A. ALL y. v x = v y --> y: A}"*
⟨*proof*⟩

**lemma** *givenByI: "(!!x y. [| x:A;  v x = v y |] ==> y: A) ==> A: givenBy v"*
⟨*proof*⟩

**lemma** *givenByD: "[| A: givenBy v;  x:A;  v x = v y |] ==> y: A"*
⟨*proof*⟩

**lemma** *empty_mem_givenBy [iff]: "{} : givenBy v"*
⟨*proof*⟩

**lemma** *givenBy_imp_eq_Collect: "A: givenBy v ==> EX P. A = {s. P(v s)}"*
⟨*proof*⟩

**lemma** *Collect_mem_givenBy: "{s. P(v s)} : givenBy v"*
⟨*proof*⟩

**lemma** *givenBy_eq_Collect: "givenBy v = {A. EX P. A = {s. P(v s)}}"*
⟨*proof*⟩

**lemma** *preserves_givenBy_imp_stable:*
*"[| F : preserves v;  D : givenBy v |] ==> F : stable D"*
⟨*proof*⟩

**lemma** *givenBy_o_subset: "givenBy (w o v) <= givenBy v"*
⟨*proof*⟩

**lemma** *givenBy_DiffI:*
*"[| A : givenBy v;  B : givenBy v |] ==> A-B : givenBy v"*
⟨*proof*⟩

**lemma** *leadsETo_Basis [intro]:*
*"[| F: A ensures B;  A-B: insert {} CC |] ==> F : A leadsTo[CC] B"*
⟨*proof*⟩

**lemma** *leadsETo_Trans:*
*"[| F : A leadsTo[CC] B;  F : B leadsTo[CC] C |] ==> F : A leadsTo[CC] C"*
⟨*proof*⟩

**lemma** *leadsETo_Un_duplicate:*
*"F : A leadsTo[CC] (A' Un A') ==> F : A leadsTo[CC] A'"*
⟨*proof*⟩

**lemma** *leadsETo_Un_duplicate2:*
```
"F : A leadsTo[CC] (A' Un C Un C) ==> F : A leadsTo[CC] (A' Un C)"
```
⟨*proof*⟩

**lemma** *leadsETo_Union:*
```
"(!!A. A : S ==> F : A leadsTo[CC] B) ==> F : (Union S) leadsTo[CC] B"
```
⟨*proof*⟩

**lemma** *leadsETo_UN:*
```
"(!!i. i : I ==> F : (A i) leadsTo[CC] B)
 ==> F : (UN i:I. A i) leadsTo[CC] B"
```
⟨*proof*⟩

**lemma** *leadsETo_induct:*
```
"[| F : za leadsTo[CC] zb;
    !!A B. [| F : A ensures B;  A-B : insert {} CC |] ==> P A B;
    !!A B C. [| F : A leadsTo[CC] B; P A B; F : B leadsTo[CC] C; P B C |]

             ==> P A C;
    !!B S. ALL A:S. F : A leadsTo[CC] B & P A B ==> P (Union S) B
 |] ==> P za zb"
```
⟨*proof*⟩

**lemma** *leadsETo_mono: "CC' <= CC ==> (A leadsTo[CC'] B) <= (A leadsTo[CC] B)"*
⟨*proof*⟩

**lemma** *leadsETo_Trans_Un:*
```
"[| F : A leadsTo[CC] B;  F : B leadsTo[DD] C |]
 ==> F : A leadsTo[CC Un DD] C"
```
⟨*proof*⟩

**lemma** *leadsETo_Union_Int:*
```
"(!!A. A : S ==> F : (A Int C) leadsTo[CC] B)
 ==> F : (Union S Int C) leadsTo[CC] B"
```
⟨*proof*⟩

**lemma** *leadsETo_Un:*
```
"[| F : A leadsTo[CC] C; F : B leadsTo[CC] C |]
 ==> F : (A Un B) leadsTo[CC] C"
```
⟨*proof*⟩

**lemma** *single_leadsETo_I:*
```
"(!!x. x : A ==> F : {x} leadsTo[CC] B) ==> F : A leadsTo[CC] B"
```
⟨*proof*⟩

**lemma** *subset_imp_leadsETo: "A<=B ==> F : A leadsTo[CC] B"*
⟨*proof*⟩

**lemmas** *empty_leadsETo = empty_subsetI [THEN subset_imp_leadsETo, simp]*

**lemma** *leadsETo_weaken_R:*
*"[| F : A leadsTo[CC] A';  A'<=B' |] ==> F : A leadsTo[CC] B'"*
⟨*proof*⟩

**lemma** *leadsETo_weaken_L [rule_format]:*
*"[| F : A leadsTo[CC] A'; B<=A |] ==> F : B leadsTo[CC] A'"*
⟨*proof*⟩

**lemma** *leadsETo_Un_distrib:*
*"F : (A Un B) leadsTo[CC] C  =*
*(F : A leadsTo[CC] C & F : B leadsTo[CC] C)"*
⟨*proof*⟩

**lemma** *leadsETo_UN_distrib:*
*"F : (UN i:I. A i) leadsTo[CC] B  =*
*(ALL i : I. F : (A i) leadsTo[CC] B)"*
⟨*proof*⟩

**lemma** *leadsETo_Union_distrib:*
*"F : (Union S) leadsTo[CC] B  =  (ALL A : S. F : A leadsTo[CC] B)"*
⟨*proof*⟩

**lemma** *leadsETo_weaken:*
*"[| F : A leadsTo[CC'] A'; B<=A; A'<=B';  CC' <= CC |]*
*==> F : B leadsTo[CC] B'"*
⟨*proof*⟩

**lemma** *leadsETo_givenBy:*
*"[| F : A leadsTo[CC] A';  CC <= givenBy v |]*
*==> F : A leadsTo[givenBy v] A'"*
⟨*proof*⟩

**lemma** *leadsETo_Diff:*
*"[| F : (A-B) leadsTo[CC] C; F : B leadsTo[CC] C |]*
*==> F : A leadsTo[CC] C"*
⟨*proof*⟩

**lemma** *leadsETo_Un_Un:*
*"[| F : A leadsTo[CC] A';  F : B leadsTo[CC] B' |]*
*==> F : (A Un B) leadsTo[CC] (A' Un B')"*

⟨*proof*⟩

**lemma** *leadsETo_cancel2:*
```
"[| F : A leadsTo[CC] (A' Un B); F : B leadsTo[CC] B' |]
 ==> F : A leadsTo[CC] (A' Un B')"
```
⟨*proof*⟩

**lemma** *leadsETo_cancel1:*
```
"[| F : A leadsTo[CC] (B Un A'); F : B leadsTo[CC] B' |]
==> F : A leadsTo[CC] (B' Un A')"
```
⟨*proof*⟩

**lemma** *leadsETo_cancel_Diff1:*
```
"[| F : A leadsTo[CC] (B Un A'); F : (B-A') leadsTo[CC] B' |]
==> F : A leadsTo[CC] (B' Un A')"
```
⟨*proof*⟩

**lemma** *e_psp_stable:*
```
"[| F : A leadsTo[CC] A';  F : stable B;  ALL C:CC. C Int B : CC |]
 ==> F : (A Int B) leadsTo[CC] (A' Int B)"
```
⟨*proof*⟩

**lemma** *e_psp_stable2:*
```
"[| F : A leadsTo[CC] A'; F : stable B;  ALL C:CC. C Int B : CC |]
 ==> F : (B Int A) leadsTo[CC] (B Int A')"
```
⟨*proof*⟩

**lemma** *e_psp:*
```
"[| F : A leadsTo[CC] A'; F : B co B';
    ALL C:CC. C Int B Int B' : CC |]
 ==> F : (A Int B') leadsTo[CC] ((A' Int B) Un (B' - B))"
```
⟨*proof*⟩

**lemma** *e_psp2:*
```
"[| F : A leadsTo[CC] A'; F : B co B';
    ALL C:CC. C Int B Int B' : CC |]
 ==> F : (B' Int A) leadsTo[CC] ((B Int A') Un (B' - B))"
```
⟨*proof*⟩

**lemma** *gen_leadsETo_imp_Join_leadsETo:*
```
"[| F: (A leadsTo[givenBy v] B);  G : preserves v;
    F⊔G : stable C |]
 ==> F⊔G : ((C Int A) leadsTo[(%D. C Int D) ` givenBy v] B)"
```

⟨*proof*⟩

**lemma** *leadsETo_subset_leadsTo: "(A leadsTo[CC] B) <= (A leadsTo B)"*
⟨*proof*⟩

**lemma** *leadsETo_UNIV_eq_leadsTo: "(A leadsTo[UNIV] B) = (A leadsTo B)"*
⟨*proof*⟩

**lemma** *LeadsETo_eq_leadsETo:*
```
"A LeadsTo[CC] B =
   {F. F : (reachable F Int A) leadsTo[(%C. reachable F Int C) ` CC]
   (reachable F Int B)}"
```
⟨*proof*⟩

**lemma** *LeadsETo_Trans:*
```
"[| F : A LeadsTo[CC] B;  F : B LeadsTo[CC] C |]
 ==> F : A LeadsTo[CC] C"
```
⟨*proof*⟩

**lemma** *LeadsETo_Union:*
```
"(!!A. A : S ==> F : A LeadsTo[CC] B) ==> F : (Union S) LeadsTo[CC] B"
```
⟨*proof*⟩

**lemma** *LeadsETo_UN:*
```
"(!!i. i : I ==> F : (A i) LeadsTo[CC] B)
 ==> F : (UN i:I. A i) LeadsTo[CC] B"
```
⟨*proof*⟩

**lemma** *LeadsETo_Un:*
```
"[| F : A LeadsTo[CC] C; F : B LeadsTo[CC] C |]
 ==> F : (A Un B) LeadsTo[CC] C"
```
⟨*proof*⟩

**lemma** *single_LeadsETo_I:*
```
"(!!s. s : A ==> F : {s} LeadsTo[CC] B) ==> F : A LeadsTo[CC] B"
```
⟨*proof*⟩

**lemma** *subset_imp_LeadsETo:*
```
"A <= B ==> F : A LeadsTo[CC] B"
```
⟨*proof*⟩

**lemmas** *empty_LeadsETo = empty_subsetI [THEN subset_imp_LeadsETo, standard]*

**lemma** *LeadsETo_weaken_R [rule_format]:*

```
    "[| F : A LeadsTo[CC] A';  A' <= B' |] ==> F : A LeadsTo[CC] B'"
```
⟨*proof*⟩

**lemma** *LeadsETo_weaken_L [rule_format]:*
```
    "[| F : A LeadsTo[CC] A';  B <= A |] ==> F : B LeadsTo[CC] A'"
```
⟨*proof*⟩

**lemma** *LeadsETo_weaken:*
```
    "[| F : A LeadsTo[CC'] A';
        B <= A;  A' <= B';  CC' <= CC |]
     ==> F : B LeadsTo[CC] B'"
```
⟨*proof*⟩

**lemma** *LeadsETo_subset_LeadsTo: "(A LeadsTo[CC] B) <= (A LeadsTo B)"*
⟨*proof*⟩

**lemma** *reachable_ensures:*
```
    "F : A ensures B ==> F : (reachable F Int A) ensures B"
```
⟨*proof*⟩

**lemma** *lel_lemma:*
```
    "F : A leadsTo B ==> F : (reachable F Int A) leadsTo[Pow(reachable F)]
B"
```
⟨*proof*⟩

**lemma** *LeadsETo_UNIV_eq_LeadsTo: "(A LeadsTo[UNIV] B) = (A LeadsTo B)"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *givenBy_o_eq_extend_set:*
```
    "givenBy (v o f) = extend_set h ` (givenBy v)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *givenBy_eq_extend_set: "givenBy f = range (extend_set h)"*
⟨*proof*⟩

**lemma** (**in** *Extend*) *extend_set_givenBy_I:*
```
    "D : givenBy v ==> extend_set h D : givenBy (v o f)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *leadsETo_imp_extend_leadsETo:*
```
    "F : A leadsTo[CC] B
     ==> extend h F : (extend_set h A) leadsTo[extend_set h ` CC]
                      (extend_set h B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *Join_project_ensures_strong:*
```
    "[| project h C G ~: transient (project_set h C Int (A-B)) |
        project_set h C Int (A - B) = {};
```

```
         extend h F⊔G : stable C;
         F⊔project h C G : (project_set h C Int A) ensures B |]
      ==> extend h F⊔G : (C Int extend_set h A) ensures (extend_set h B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *pli_lemma:*
```
     "[| extend h F⊔G : stable C;
         F⊔project h C G
           : project_set h C Int project_set h A leadsTo project_set h B |]

      ==> F⊔project h C G
            : project_set h C Int project_set h A leadsTo
              project_set h C Int project_set h B"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_leadsETo_I_lemma:*
```
     "[| extend h F⊔G : stable C;
         extend h F⊔G :
           (C Int A) leadsTo[(%D. C Int D)‘givenBy f]  B |]
  ==> F⊔project h C G
    : (project_set h C Int project_set h (C Int A)) leadsTo (project_set h
B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_leadsETo_I:*
```
     "extend h F⊔G : (extend_set h A) leadsTo[givenBy f] (extend_set h B)
      ==> F⊔project h UNIV G : A leadsTo B"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *project_LeadsETo_I:*
```
     "extend h F⊔G : (extend_set h A) LeadsTo[givenBy f] (extend_set h B)

      ==> F⊔project h (reachable (extend h F⊔G)) G
           : A LeadsTo B"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *projecting_leadsTo:*
```
     "projecting (%G. UNIV) h F
                 (extend_set h A leadsTo[givenBy f] extend_set h B)
                 (A leadsTo B)"
```
⟨*proof*⟩

**lemma** (**in** *Extend*) *projecting_LeadsTo:*
```
     "projecting (%G. reachable (extend h F⊔G)) h F
                 (extend_set h A LeadsTo[givenBy f] extend_set h B)
                 (A LeadsTo B)"
```
⟨*proof*⟩

**end**

# 26 Common Declarations for Chandy and Charpentier's Allocator

**theory** *AllocBase* **imports** *"../UNITY_Main"* **begin**

**consts**
```
  NbT      :: nat
  Nclients :: nat
```

**axioms**
```
  NbT_pos:  "0 < NbT"
```

**consts** *tokens* `:: "nat list => nat"`
**primrec**
```
  "tokens [] = 0"
  "tokens (x#xs) = x + tokens xs"
```

**consts**
```
  bag_of :: "'a list => 'a multiset"
```

**primrec**
```
  "bag_of []     = {#}"
  "bag_of (x#xs) = {#x#} + bag_of xs"
```

**lemma** *setsum_fun_mono [rule_format]:*
```
     "!!f :: nat=>nat.
      (ALL i. i<n --> f i <= g i) -->
      setsum f (lessThan n) <= setsum g (lessThan n)"
```
⟨*proof*⟩

**lemma** *tokens_mono_prefix [rule_format]:*
```
     "ALL xs. xs <= ys --> tokens xs <= tokens ys"
```
⟨*proof*⟩

**lemma** *mono_tokens: "mono tokens"*
⟨*proof*⟩

**lemma** *bag_of_append [simp]: "bag_of (l@l') = bag_of l + bag_of l'"*
⟨*proof*⟩

**lemma** *mono_bag_of: "mono (bag_of :: 'a list => ('a::order) multiset)"*
⟨*proof*⟩

**declare** *setsum_cong [cong]*

**lemma** *bag_of_sublist_lemma:*
```
"(∑ i∈ A Int lessThan k. {#if i<k then f i else g i#}) =
 (∑ i∈ A Int lessThan k. {#f i#})"
```
⟨*proof*⟩

**lemma** *bag_of_sublist:*
```
"bag_of (sublist l A) =
 (∑ i∈ A Int lessThan (length l). {# l!i #})"
```
⟨*proof*⟩

**lemma** *bag_of_sublist_Un_Int:*
```
"bag_of (sublist l (A Un B)) + bag_of (sublist l (A Int B)) =
 bag_of (sublist l A) + bag_of (sublist l B)"
```
⟨*proof*⟩

**lemma** *bag_of_sublist_Un_disjoint:*
```
"A Int B = {}
 ==> bag_of (sublist l (A Un B)) =
     bag_of (sublist l A) + bag_of (sublist l B)"
```
⟨*proof*⟩

**lemma** *bag_of_sublist_UN_disjoint [rule_format]:*
```
"[| finite I; ALL i:I. ALL j:I. i~=j --> A i Int A j = {} |]
 ==> bag_of (sublist l (UNION I A)) =
     (∑ i∈I. bag_of (sublist l (A i)))"
```
⟨*proof*⟩

**end**

**theory** *Alloc*
**imports** *AllocBase "../PPROD"*
**begin**

## 26.1 State definitions. OUTPUT variables are locals

**record** *clientState =*
```
  giv :: "nat list"   — client's INPUT history: tokens GRANTED
  ask :: "nat list"   — client's OUTPUT history: tokens REQUESTED
  rel :: "nat list"   — client's OUTPUT history: tokens RELEASED
```

**record** *'a clientState_d =*
```
  clientState +
  dummy :: 'a         — dummy field for new variables
```

**constdefs**
```
  — DUPLICATED FROM Client.thy, but with "tok" removed
  — Maybe want a special theory section to declare such maps
  non_dummy :: "'a clientState_d => clientState"
    "non_dummy s == (|giv = giv s, ask = ask s, rel = rel s|)"
```

```
  — Renaming map to put a Client into the standard form
  client_map :: "'a clientState_d => clientState*'a"
    "client_map == funPair non_dummy dummy"


record allocState =
  allocGiv :: "nat => nat list"   — OUTPUT history: source of "giv" for i
  allocAsk :: "nat => nat list"   — INPUT: allocator's copy of "ask" for i
  allocRel :: "nat => nat list"   — INPUT: allocator's copy of "rel" for i

record 'a allocState_d =
  allocState +
  dummy    :: 'a                 — dummy field for new variables

record 'a systemState =
  allocState +
  client :: "nat => clientState"  — states of all clients
  dummy  :: 'a                    — dummy field for new variables


constdefs

— * Resource allocation system specification *

  — spec (1)
  system_safety :: "'a systemState program set"
    "system_safety ==
     Always {s. (SUM i: lessThan Nclients. (tokens o giv o sub i o client)s)
     ≤ NbT + (SUM i: lessThan Nclients. (tokens o rel o sub i o client)s)}"

  — spec (2)
  system_progress :: "'a systemState program set"
    "system_progress == INT i : lessThan Nclients.
                        INT h.
                          {s. h ≤ (ask o sub i o client)s} LeadsTo
                          {s. h pfixLe (giv o sub i o client) s}"

  system_spec :: "'a systemState program set"
    "system_spec == system_safety Int system_progress"

— * Client specification (required) **

  — spec (3)
  client_increasing :: "'a clientState_d program set"
    "client_increasing ==
         UNIV guarantees  Increasing ask Int Increasing rel"

  — spec (4)
  client_bounded :: "'a clientState_d program set"
    "client_bounded ==
         UNIV guarantees  Always {s. ALL elt : set (ask s). elt ≤ NbT}"

  — spec (5)
  client_progress :: "'a clientState_d program set"
```

```
  "client_progress ==
       Increasing giv  guarantees
       (INT h. {s. h ≤ giv s & h pfixGe ask s}
               LeadsTo {s. tokens h ≤ (tokens o rel) s})"
```

— spec: preserves part

```
client_preserves :: "'a clientState_d program set"
  "client_preserves == preserves giv Int preserves clientState_d.dummy"
```

— environmental constraints

```
client_allowed_acts :: "'a clientState_d program set"
  "client_allowed_acts ==
     {F. AllowedActs F =
          insert Id (UNION (preserves (funPair rel ask)) Acts)}"

client_spec :: "'a clientState_d program set"
  "client_spec == client_increasing Int client_bounded Int client_progress
                  Int client_allowed_acts Int client_preserves"
```

— * Allocator specification (required) *

— spec (6)

```
alloc_increasing :: "'a allocState_d program set"
  "alloc_increasing ==
       UNIV  guarantees
       (INT i : lessThan Nclients. Increasing (sub i o allocGiv))"
```

— spec (7)

```
alloc_safety :: "'a allocState_d program set"
  "alloc_safety ==
       (INT i : lessThan Nclients. Increasing (sub i o allocRel))
       guarantees
       Always {s. (SUM i: lessThan Nclients. (tokens o sub i o allocGiv)s)
       ≤ NbT + (SUM i: lessThan Nclients. (tokens o sub i o allocRel)s)}"
```

— spec (8)

```
alloc_progress :: "'a allocState_d program set"
  "alloc_progress ==
       (INT i : lessThan Nclients. Increasing (sub i o allocAsk) Int
                                   Increasing (sub i o allocRel))
       Int
       Always {s. ALL i<Nclients.
                   ALL elt : set ((sub i o allocAsk) s). elt ≤ NbT}
       Int
       (INT i : lessThan Nclients.
        INT h. {s. h ≤ (sub i o allocGiv)s & h pfixGe (sub i o allocAsk)s}
               LeadsTo
               {s. tokens h ≤ (tokens o sub i o allocRel)s})
       guarantees
           (INT i : lessThan Nclients.
            INT h. {s. h ≤ (sub i o allocAsk) s}
                   LeadsTo
                   {s. h pfixLe (sub i o allocGiv) s})"
```

```
  — spec: preserves part
  alloc_preserves :: "'a allocState_d program set"
    "alloc_preserves == preserves allocRel Int preserves allocAsk Int
                        preserves allocState_d.dummy"

  — environmental constraints
  alloc_allowed_acts :: "'a allocState_d program set"
    "alloc_allowed_acts ==
       {F. AllowedActs F =
            insert Id (UNION (preserves allocGiv) Acts)}"

  alloc_spec :: "'a allocState_d program set"
    "alloc_spec == alloc_increasing Int alloc_safety Int alloc_progress Int
                   alloc_allowed_acts Int alloc_preserves"

— * Network specification *

  — spec (9.1)
  network_ask :: "'a systemState program set"
    "network_ask == INT i : lessThan Nclients.
                        Increasing (ask o sub i o client)  guarantees
                        ((sub i o allocAsk) Fols (ask o sub i o client))"

  — spec (9.2)
  network_giv :: "'a systemState program set"
    "network_giv == INT i : lessThan Nclients.
                        Increasing (sub i o allocGiv)
                        guarantees
                        ((giv o sub i o client) Fols (sub i o allocGiv))"

  — spec (9.3)
  network_rel :: "'a systemState program set"
    "network_rel == INT i : lessThan Nclients.
                        Increasing (rel o sub i o client)
                        guarantees
                        ((sub i o allocRel) Fols (rel o sub i o client))"

  — spec: preserves part
  network_preserves :: "'a systemState program set"
    "network_preserves ==
       preserves allocGiv  Int
       (INT i : lessThan Nclients. preserves (rel o sub i o client)  Int
                                   preserves (ask o sub i o client))"

  — environmental constraints
  network_allowed_acts :: "'a systemState program set"
    "network_allowed_acts ==
       {F. AllowedActs F =
           insert Id
            (UNION (preserves allocRel Int
                    (INT i: lessThan Nclients. preserves(giv o sub i o client)))
                  Acts)}"
```

```
  network_spec :: "'a systemState program set"
    "network_spec == network_ask Int network_giv Int
                     network_rel Int network_allowed_acts Int
                     network_preserves"
```

— * State mappings *

```
  sysOfAlloc :: "((nat => clientState) * 'a) allocState_d => 'a systemState"
    "sysOfAlloc == %s. let (cl,xtr) = allocState_d.dummy s
                       in (| allocGiv = allocGiv s,
                             allocAsk = allocAsk s,
                             allocRel = allocRel s,
                             client   = cl,
                             dummy    = xtr|)"


  sysOfClient :: "(nat => clientState) * 'a allocState_d => 'a systemState"
    "sysOfClient == %(cl,al). (| allocGiv = allocGiv al,
                                 allocAsk = allocAsk al,
                                 allocRel = allocRel al,
                                 client   = cl,
                                 systemState.dummy = allocState_d.dummy al|)"
```

**consts**

```
    Alloc   :: "'a allocState_d program"
    Client  :: "'a clientState_d program"
    Network :: "'a systemState program"
    System  :: "'a systemState program"
```

**axioms**

```
    Alloc:   "Alloc   : alloc_spec"
    Client:  "Client  : client_spec"
    Network: "Network : network_spec"
```

**defs**

```
    System_def:
      "System == rename sysOfAlloc Alloc Join Network Join
                  (rename sysOfClient
                   (plam x: lessThan Nclients. rename client_map Client))"
```

**declare** *image_Collect [simp del]*

**declare** *subset_preserves_o [THEN [2] rev_subsetD, intro]*
**declare** *subset_preserves_o [THEN [2] rev_subsetD, simp]*
**declare** *funPair_o_distrib [simp]*
**declare** *Always_INT_distrib [simp]*
**declare** *o_apply [simp del]*

**lemmas** *[simp] =*
*rename_image_constrains*
*rename_image_stable*
*rename_image_increasing*
*rename_image_invariant*
*rename_image_Constrains*
*rename_image_Stable*
*rename_image_Increasing*
*rename_image_Always*
*rename_image_leadsTo*
*rename_image_LeadsTo*
*rename_preserves*
*rename_image_preserves*
*lift_image_preserves*
*bij_image_INT*
*bij_is_inj [THEN image_Int]*
*bij_image_Collect_eq*

⟨*ML*⟩

**lemmas** *lessThanBspec = lessThan_iff [THEN iffD2, THEN [2] bspec]*

⟨*ML*⟩

**lemma** *inj_sysOfAlloc [iff]: "inj sysOfAlloc"*
⟨*proof*⟩

We need the inverse; also having it simplifies the proof of surjectivity

**lemma** *inv_sysOfAlloc_eq [simp]: "!!s. inv sysOfAlloc s =*
*(| allocGiv = allocGiv s,*
*allocAsk = allocAsk s,*
*allocRel = allocRel s,*
*allocState_d.dummy = (client s, dummy s) |)"*
⟨*proof*⟩

**lemma** *surj_sysOfAlloc [iff]: "surj sysOfAlloc"*
⟨*proof*⟩

**lemma** *bij_sysOfAlloc [iff]: "bij sysOfAlloc"*
⟨*proof*⟩

### 26.1.1 bijectivity of *sysOfClient*

**lemma** *inj_sysOfClient [iff]: "inj sysOfClient"*
⟨*proof*⟩

**lemma** *inv_sysOfClient_eq [simp]: "!!s. inv sysOfClient s =*
*(client s,*
*(| allocGiv = allocGiv s,*
*allocAsk = allocAsk s,*
*allocRel = allocRel s,*
*allocState_d.dummy = systemState.dummy s|) )"*
⟨*proof*⟩

**lemma** *surj_sysOfClient [iff]: "surj sysOfClient"*
⟨*proof*⟩

**lemma** *bij_sysOfClient [iff]: "bij sysOfClient"*
⟨*proof*⟩

### 26.1.2 bijectivity of *client_map*

**lemma** *inj_client_map [iff]: "inj client_map"*
⟨*proof*⟩

**lemma** *inv_client_map_eq [simp]: "!!s. inv client_map s =*
*(%(x,y).(|giv = giv x, ask = ask x, rel = rel x,*
*clientState_d.dummy = y|)) s"*
⟨*proof*⟩

**lemma** *surj_client_map [iff]: "surj client_map"*
⟨*proof*⟩

**lemma** *bij_client_map [iff]: "bij client_map"*
⟨*proof*⟩

o-simprules for *client_map*

**lemma** *fst_o_client_map: "fst o client_map = non_dummy"*
⟨*proof*⟩

⟨*ML*⟩
**declare** *fst_o_client_map' [simp]*

**lemma** *snd_o_client_map: "snd o client_map = clientState_d.dummy"*
⟨*proof*⟩

⟨*ML*⟩
**declare** *snd_o_client_map' [simp]*

## 26.2 o-simprules for *sysOfAlloc* [MUST BE AUTOMATED]

**lemma** *client_o_sysOfAlloc: "client o sysOfAlloc = fst o allocState_d.dummy*
*"*
⟨*proof*⟩

⟨*ML*⟩
**declare** *client_o_sysOfAlloc' [simp]*

**lemma** *allocGiv_o_sysOfAlloc_eq: "allocGiv o sysOfAlloc = allocGiv"*
⟨*proof*⟩

⟨*ML*⟩
**declare** *allocGiv_o_sysOfAlloc_eq' [simp]*

**lemma** *allocAsk_o_sysOfAlloc_eq: "allocAsk o sysOfAlloc = allocAsk"*
⟨*proof*⟩

⟨*ML*⟩

**declare** *allocAsk_o_sysOfAlloc_eq' [simp]*

**lemma** *allocRel_o_sysOfAlloc_eq: "allocRel o sysOfAlloc = allocRel"*
⟨*proof*⟩

⟨*ML*⟩
**declare** *allocRel_o_sysOfAlloc_eq' [simp]*

## 26.3 o-simprules for *sysOfClient* [MUST BE AUTOMATED]

**lemma** *client_o_sysOfClient: "client o sysOfClient = fst"*
⟨*proof*⟩

⟨*ML*⟩
**declare** *client_o_sysOfClient' [simp]*

**lemma** *allocGiv_o_sysOfClient_eq: "allocGiv o sysOfClient = allocGiv o snd*
*"*
⟨*proof*⟩

⟨*ML*⟩
**declare** *allocGiv_o_sysOfClient_eq' [simp]*

**lemma** *allocAsk_o_sysOfClient_eq: "allocAsk o sysOfClient = allocAsk o snd*
*"*
⟨*proof*⟩

⟨*ML*⟩
**declare** *allocAsk_o_sysOfClient_eq' [simp]*

**lemma** *allocRel_o_sysOfClient_eq: "allocRel o sysOfClient = allocRel o snd*
*"*
⟨*proof*⟩

⟨*ML*⟩
**declare** *allocRel_o_sysOfClient_eq' [simp]*

**lemma** *allocGiv_o_inv_sysOfAlloc_eq: "allocGiv o inv sysOfAlloc = allocGiv"*
⟨*proof*⟩

⟨*ML*⟩
**declare** *allocGiv_o_inv_sysOfAlloc_eq' [simp]*

**lemma** *allocAsk_o_inv_sysOfAlloc_eq: "allocAsk o inv sysOfAlloc = allocAsk"*
⟨*proof*⟩

⟨*ML*⟩
**declare** *allocAsk_o_inv_sysOfAlloc_eq' [simp]*

**lemma** *allocRel_o_inv_sysOfAlloc_eq: "allocRel o inv sysOfAlloc = allocRel"*
⟨*proof*⟩

⟨*ML*⟩
**declare** *allocRel_o_inv_sysOfAlloc_eq' [simp]*

**lemma** *rel_inv_client_map_drop_map: "(rel o inv client_map o drop_map i o inv sysOfClient) =*
*rel o sub i o client"*
⟨*proof*⟩

⟨*ML*⟩
**declare** *rel_inv_client_map_drop_map [simp]*

**lemma** *ask_inv_client_map_drop_map: "(ask o inv client_map o drop_map i o inv sysOfClient) =*
*ask o sub i o client"*
⟨*proof*⟩

⟨*ML*⟩
**declare** *ask_inv_client_map_drop_map [simp]*

**declare** *finite_lessThan [iff]*

Client : ¡unfolded specification¿

**lemmas** *client_spec_simps =*
*client_spec_def client_increasing_def client_bounded_def*
*client_progress_def client_allowed_acts_def client_preserves_def*
*guarantees_Int_right*

⟨*ML*⟩

**declare**
*Client_Increasing_ask [iff]*
*Client_Increasing_rel [iff]*
*Client_Bounded [iff]*
*Client_preserves_giv [iff]*
*Client_preserves_dummy [iff]*

Network : ¡unfolded specification¿

**lemmas** *network_spec_simps =*
*network_spec_def network_ask_def network_giv_def*
*network_rel_def network_allowed_acts_def network_preserves_def*
*ball_conj_distrib*

⟨*ML*⟩

**declare** *Network_preserves_allocGiv [iff]*

**declare**
*Network_preserves_rel [simp]*
*Network_preserves_ask [simp]*

**declare**
*Network_preserves_rel [simplified o_def, simp]*
*Network_preserves_ask [simplified o_def, simp]*

Alloc : ¡unfolded specification¿

**lemmas** *alloc_spec_simps =*
*alloc_spec_def alloc_increasing_def alloc_safety_def*
*alloc_progress_def alloc_allowed_acts_def alloc_preserves_def*

⟨*ML*⟩

Strip off the INT in the guarantees postcondition

**lemmas** *Alloc_Increasing = Alloc_Increasing_0 [normalized]*

**declare**
*Alloc_preserves_allocRel [iff]*
*Alloc_preserves_allocAsk [iff]*
*Alloc_preserves_dummy [iff]*

## 26.4 Components Lemmas [MUST BE AUTOMATED]

**lemma** *Network_component_System: "Network Join*
*((rename sysOfClient*
*(plam x: (lessThan Nclients). rename client_map Client)) Join*
*rename sysOfAlloc Alloc)*
*= System"*
⟨*proof*⟩

**lemma** *Client_component_System: "(rename sysOfClient*
*(plam x: (lessThan Nclients). rename client_map Client)) Join*
*(Network Join rename sysOfAlloc Alloc) = System"*
⟨*proof*⟩

**lemma** *Alloc_component_System: "rename sysOfAlloc Alloc Join*
*((rename sysOfClient (plam x: (lessThan Nclients). rename client_map*
*Client)) Join*
*Network) = System"*
⟨*proof*⟩

**declare**
*Client_component_System [iff]*
*Network_component_System [iff]*
*Alloc_component_System [iff]*

* These preservation laws should be generated automatically *

**lemma** *Client_Allowed [simp]: "Allowed Client = preserves rel Int preserves*
*ask"*
⟨*proof*⟩

**lemma** *Network_Allowed [simp]: "Allowed Network =*
*preserves allocRel Int*
*(INT i: lessThan Nclients. preserves(giv o sub i o client))"*
⟨*proof*⟩

**lemma** *Alloc_Allowed [simp]: "Allowed Alloc = preserves allocGiv"*
⟨*proof*⟩

needed in *rename_client_map_tac*

**lemma** *OK_lift_rename_Client [simp]: "OK I (%i. lift i (rename client_map Client))"*
⟨*proof*⟩

**lemma** *fst_lift_map_eq_fst [simp]: "fst (lift_map i x) i = fst x"*
⟨*proof*⟩

**lemma** *fst_o_lift_map' [simp]:*
*"(f ∘ sub i ∘ fst ∘ lift_map i ∘ g) = f o fst o g"*
⟨*proof*⟩

⟨*ML*⟩

Lifting *Client_Increasing* to *systemState*

**lemma** *rename_Client_Increasing: "i : I*
*==> rename sysOfClient (plam x: I. rename client_map Client) :*
*UNIV guarantees*
*Increasing (ask o sub i o client) Int*
*Increasing (rel o sub i o client)"*
⟨*proof*⟩

**lemma** *preserves_sub_fst_lift_map: "[| F : preserves w; i ~= j |]*
*==> F : preserves (sub i o fst o lift_map j o funPair v w)"*
⟨*proof*⟩

**lemma** *client_preserves_giv_oo_client_map: "[| i < Nclients; j < Nclients |]*
*==> Client : preserves (giv o sub i o fst o lift_map j o client_map)"*
⟨*proof*⟩

**lemma** *rename_sysOfClient_ok_Network:*
*"rename sysOfClient (plam x: lessThan Nclients. rename client_map Client) ok Network"*
⟨*proof*⟩

**lemma** *rename_sysOfClient_ok_Alloc:*
*"rename sysOfClient (plam x: lessThan Nclients. rename client_map Client) ok rename sysOfAlloc Alloc"*
⟨*proof*⟩

**lemma** *rename_sysOfAlloc_ok_Network: "rename sysOfAlloc Alloc ok Network"*
⟨*proof*⟩

**declare**
*rename_sysOfClient_ok_Network [iff]*
*rename_sysOfClient_ok_Alloc [iff]*
*rename_sysOfAlloc_ok_Network [iff]*

The "ok" laws, re-oriented. But not sure this works: theorem *ok_commute* is needed below

**declare**
*rename_sysOfClient_ok_Network [THEN ok_sym, iff]*

```
  rename_sysOfClient_ok_Alloc [THEN ok_sym, iff]
  rename_sysOfAlloc_ok_Network [THEN ok_sym]

lemma System_Increasing: "i < Nclients
     ==> System : Increasing (ask o sub i o client) Int
                  Increasing (rel o sub i o client)"
```
⟨*proof*⟩

```
lemmas rename_guarantees_sysOfAlloc_I =
  bij_sysOfAlloc [THEN rename_rename_guarantees_eq, THEN iffD2, standard]



lemmas rename_Alloc_Increasing =
  Alloc_Increasing
    [THEN rename_guarantees_sysOfAlloc_I,
     simplified surj_rename [THEN surj_range] o_def sub_apply
                rename_image_Increasing bij_sysOfAlloc
                allocGiv_o_inv_sysOfAlloc_eq']

lemma System_Increasing_allocGiv:
    "i < Nclients ==> System : Increasing (sub i o allocGiv)"
```
⟨*proof*⟩

⟨*ML*⟩

```
declare System_Increasing' [intro!]
```

Follows consequences. The "Always (INT ...) formulation expresses the general safety property and allows it to be combined using `Always_Int_rule` below.

```
lemma System_Follows_rel:
  "i < Nclients ==> System : ((sub i o allocRel) Fols (rel o sub i o client))"
```
⟨*proof*⟩

```
lemma System_Follows_ask:
  "i < Nclients ==> System : ((sub i o allocAsk) Fols (ask o sub i o client))"
```
⟨*proof*⟩

```
lemma System_Follows_allocGiv:
  "i < Nclients ==> System : (giv o sub i o client) Fols (sub i o allocGiv)"
```
⟨*proof*⟩

```
lemma Always_giv_le_allocGiv: "System : Always (INT i: lessThan Nclients.
                      {s. (giv o sub i o client) s ≤ (sub i o allocGiv) s})"
```
⟨*proof*⟩

```
lemma Always_allocAsk_le_ask: "System : Always (INT i: lessThan Nclients.
                      {s. (sub i o allocAsk) s ≤ (ask o sub i o client) s})"
```
⟨*proof*⟩

**lemma** *Always_allocRel_le_rel: "System : Always (INT i: lessThan Nclients.*
*{s. (sub i o allocRel) s ≤ (rel o sub i o client) s})"*
⟨*proof*⟩

## 26.5 Proof of the safety property (1)

safety (1), step 1 is *System_Follows_rel*

safety (1), step 2

**lemmas** *System_Increasing_allocRel = System_Follows_rel [THEN Follows_Increasing1, standard]*

safety (1), step 3

**lemma** *System_sum_bounded:*
*"System : Always {s. (∑ i ∈ lessThan Nclients. (tokens o sub i o allocGiv) s)*
*≤ NbT + (∑ i ∈ lessThan Nclients. (tokens o sub i o allocRel) s)}"*
⟨*proof*⟩

Follows reasoning

**lemma** *Always_tokens_giv_le_allocGiv: "System : Always (INT i: lessThan Nclients.*
*{s. (tokens o giv o sub i o client) s*
*≤ (tokens o sub i o allocGiv) s})"*
⟨*proof*⟩

**lemma** *Always_tokens_allocRel_le_rel: "System : Always (INT i: lessThan Nclients.*
*{s. (tokens o sub i o allocRel) s*
*≤ (tokens o rel o sub i o client) s})"*
⟨*proof*⟩

safety (1), step 4 (final result!)

**theorem** *System_safety: "System : system_safety"*
⟨*proof*⟩

## 26.6 Proof of the progress property (2)

progress (2), step 1 is *System_Follows_ask* and *System_Follows_rel*

progress (2), step 2; see also *System_Increasing_allocRel*

**lemmas** *System_Increasing_allocAsk = System_Follows_ask [THEN Follows_Increasing1, standard]*

progress (2), step 3: lifting *Client_Bounded* to systemState

**lemma** *rename_Client_Bounded: "i : I*
*==> rename sysOfClient (plam x: I. rename client_map Client) :*
*UNIV guarantees*
*Always {s. ALL elt : set ((ask o sub i o client) s). elt ≤ NbT}"*
⟨*proof*⟩

**lemma** *System_Bounded_ask: "i < Nclients*

```
    ==> System : Always
                  {s. ALL elt : set ((ask o sub i o client) s). elt ≤ NbT}"
```
⟨*proof*⟩

**lemma** `Collect_all_imp_eq: "{x. ALL y. P y --> Q x y} = (INT y: {y. P y}. {x. Q x y})"`
⟨*proof*⟩

progress (2), step 4

**lemma** `System_Bounded_allocAsk: "System : Always {s. ALL i<Nclients. ALL elt : set ((sub i o allocAsk) s). elt ≤ NbT}"`
⟨*proof*⟩

progress (2), step 5 is `System_Increasing_allocGiv`

progress (2), step 6

**lemmas** `System_Increasing_giv = System_Follows_allocGiv [THEN Follows_Increasing1, standard]`

**lemma** `rename_Client_Progress: "i: I`
```
   ==> rename sysOfClient (plam x: I. rename client_map Client)
       : Increasing (giv o sub i o client)
         guarantees
         (INT h. {s. h ≤ (giv o sub i o client) s &
                       h pfixGe (ask o sub i o client) s}
                 LeadsTo {s. tokens h ≤ (tokens o rel o sub i o client) s})"
```
⟨*proof*⟩

progress (2), step 7

**lemma** `System_Client_Progress:`
```
  "System : (INT i : (lessThan Nclients).
             INT h. {s. h ≤ (giv o sub i o client) s &
                        h pfixGe (ask o sub i o client) s}
                 LeadsTo {s. tokens h ≤ (tokens o rel o sub i o client) s})"
```
⟨*proof*⟩

**lemmas** `System_lemma1 =`
```
  Always_LeadsToD [OF System_Follows_ask [THEN Follows_Bounded]
                      System_Follows_allocGiv [THEN Follows_LeadsTo]]
```

**lemmas** `System_lemma2 =`
```
  PSP_Stable [OF System_lemma1
              System_Follows_ask [THEN Follows_Increasing1, THEN IncreasingD]]
```

**lemma** `System_lemma3: "i < Nclients`
```
      ==> System : {s. h ≤ (sub i o allocGiv) s &
                       h pfixGe (sub i o allocAsk) s}
                   LeadsTo
                   {s. h ≤ (giv o sub i o client) s &
                       h pfixGe (ask o sub i o client) s}"
```

⟨*proof*⟩

progress (2), step 8: Client i's "release" action is visible system-wide

**lemma** *System_Alloc_Client_Progress: "i < Nclients*
*==> System : {s. h ≤ (sub i o allocGiv) s &*
*h pfixGe (sub i o allocAsk) s}*
*LeadsTo {s. tokens h ≤ (tokens o sub i o allocRel) s}"*
⟨*proof*⟩

Lifting *Alloc_Progress* up to the level of systemState

progress (2), step 9

**lemma** *System_Alloc_Progress:*
*"System : (INT i : (lessThan Nclients).*
*INT h. {s. h ≤ (sub i o allocAsk) s}*
*LeadsTo {s. h pfixLe (sub i o allocGiv) s})"*
⟨*proof*⟩

progress (2), step 10 (final result!)

**lemma** *System_Progress: "System : system_progress"*
⟨*proof*⟩

**theorem** *System_correct: "System : system_spec"*
⟨*proof*⟩

Some obsolete lemmas

**lemma** *non_dummy_eq_o_funPair: "non_dummy = (% (g,a,r). (| giv = g, ask = a, rel = r |)) o*
*(funPair giv (funPair ask rel))"*
⟨*proof*⟩

**lemma** *preserves_non_dummy_eq: "(preserves non_dummy) =*
*(preserves rel Int preserves ask Int preserves giv)"*
⟨*proof*⟩

Could go to Extend.ML

**lemma** *bij_fst_inv_inv_eq: "bij f ==> fst (inv (%(x, u). inv f x) z) = f z"*
⟨*proof*⟩

**end**

# 27 Implementation of a multiple-client allocator from a single-client allocator

**theory** *AllocImpl* **imports** *AllocBase Follows PPROD* **begin**

```
record 'b merge =
  In   :: "nat => 'b list"
  Out  :: "'b list"
  iOut :: "nat list"

record ('a,'b) merge_d =
  "'b merge" +
  dummy :: 'a

constdefs
  non_dummy :: "('a,'b) merge_d => 'b merge"
    "non_dummy s == (|In = In s, Out = Out s, iOut = iOut s|)"

record 'b distr =
  In  :: "'b list"
  iIn :: "nat list"
  Out :: "nat => 'b list"

record ('a,'b) distr_d =
  "'b distr" +
  dummy :: 'a

record allocState =
  giv :: "nat list"
  ask :: "nat list"
  rel :: "nat list"

record 'a allocState_d =
  allocState +
  dummy    :: 'a

record 'a systemState =
  allocState +
  mergeRel :: "nat merge"
  mergeAsk :: "nat merge"
  distr    :: "nat distr"
  dummy    :: 'a
```

**constdefs**

```
  merge_increasing :: "('a,'b) merge_d program set"
    "merge_increasing ==
         UNIV guarantees (Increasing merge.Out) Int (Increasing merge.iOut)"


  merge_eqOut :: "('a,'b) merge_d program set"
    "merge_eqOut ==
         UNIV guarantees
         Always {s. length (merge.Out s) = length (merge.iOut s)}"
```

```
merge_bounded :: "('a,'b) merge_d program set"
  "merge_bounded ==
       UNIV guarantees
       Always {s. ∀elt ∈ set (merge.iOut s). elt < Nclients}"


merge_follows :: "('a,'b) merge_d program set"
  "merge_follows ==
       (⋂i ∈ lessThan Nclients. Increasing (sub i o merge.In))
       guarantees
       (⋂i ∈ lessThan Nclients.
        (%s. sublist (merge.Out s)
                     {k. k < size(merge.iOut s) & merge.iOut s! k = i})
        Fols (sub i o merge.In))"


merge_preserves :: "('a,'b) merge_d program set"
  "merge_preserves == preserves merge.In Int preserves merge_d.dummy"


merge_allowed_acts :: "('a,'b) merge_d program set"
  "merge_allowed_acts ==
     {F. AllowedActs F =
          insert Id (UNION (preserves (funPair merge.Out merge.iOut)) Acts)}"

merge_spec :: "('a,'b) merge_d program set"
  "merge_spec == merge_increasing Int merge_eqOut Int merge_bounded Int
                 merge_follows Int merge_allowed_acts Int merge_preserves"




distr_follows :: "('a,'b) distr_d program set"
  "distr_follows ==
       Increasing distr.In Int Increasing distr.iIn Int
       Always {s. ∀elt ∈ set (distr.iIn s). elt < Nclients}
       guarantees
       (⋂i ∈ lessThan Nclients.
        (sub i o distr.Out) Fols
        (%s. sublist (distr.In s)
                     {k. k < size(distr.iIn s) & distr.iIn s ! k = i}))"

distr_allowed_acts :: "('a,'b) distr_d program set"
  "distr_allowed_acts ==
     {D. AllowedActs D = insert Id (UNION (preserves distr.Out) Acts)}"

distr_spec :: "('a,'b) distr_d program set"
  "distr_spec == distr_follows Int distr_allowed_acts"
```

```
  alloc_increasing :: "'a allocState_d program set"
    "alloc_increasing == UNIV  guarantees  Increasing giv"


  alloc_safety :: "'a allocState_d program set"
    "alloc_safety ==
         Increasing rel
         guarantees  Always {s. tokens (giv s) ≤ NbT + tokens (rel s)}"


  alloc_progress :: "'a allocState_d program set"
    "alloc_progress ==
         Increasing ask Int Increasing rel Int
         Always {s. ∀elt ∈ set (ask s). elt ≤ NbT}
         Int
         (⋂h. {s. h ≤ giv s & h pfixGe (ask s)}
                 LeadsTo
                 {s. tokens h ≤ tokens (rel s)})
         guarantees  (⋂h. {s. h ≤ ask s} LeadsTo {s. h pfixLe giv s})"


  alloc_preserves :: "'a allocState_d program set"
    "alloc_preserves == preserves rel Int
                        preserves ask Int
                        preserves allocState_d.dummy"



  alloc_allowed_acts :: "'a allocState_d program set"
    "alloc_allowed_acts ==
       {F. AllowedActs F = insert Id (UNION (preserves giv) Acts)}"

  alloc_spec :: "'a allocState_d program set"
    "alloc_spec == alloc_increasing Int alloc_safety Int alloc_progress Int
                   alloc_allowed_acts Int alloc_preserves"

locale Merge =
  fixes M :: "('a,'b::order) merge_d program"
  assumes
    Merge_spec:  "M  ∈ merge_spec"

locale Distrib =
  fixes D :: "('a,'b::order) distr_d program"
  assumes
    Distrib_spec:  "D ∈ distr_spec"




declare subset_preserves_o [THEN subsetD, intro]
declare funPair_o_distrib [simp]
declare Always_INT_distrib [simp]
declare o_apply [simp del]
```

## 27.1 Theorems for Merge

**lemma** (**in** *Merge*) *Merge_Allowed:*
```
"Allowed M = (preserves merge.Out) Int (preserves merge.iOut)"
```
⟨*proof*⟩

**lemma** (**in** *Merge*) *M_ok_iff [iff]:*
```
"M ok G = (G ∈ preserves merge.Out & G ∈ preserves merge.iOut &
                  M ∈ Allowed G)"
```
⟨*proof*⟩

**lemma** (**in** *Merge*) *Merge_Always_Out_eq_iOut:*
```
"[| G ∈ preserves merge.Out; G ∈ preserves merge.iOut; M ∈ Allowed G
|]
 ==> M Join G ∈ Always {s. length (merge.Out s) = length (merge.iOut
s)}"
```
⟨*proof*⟩

**lemma** (**in** *Merge*) *Merge_Bounded:*
```
"[| G ∈ preserves merge.iOut; G ∈ preserves merge.Out; M ∈ Allowed G
|]
 ==> M Join G ∈ Always {s. ∀elt ∈ set (merge.iOut s). elt < Nclients}"
```
⟨*proof*⟩

**lemma** (**in** *Merge*) *Merge_Bag_Follows_lemma:*
```
"[| G ∈ preserves merge.iOut; G ∈ preserves merge.Out; M ∈ Allowed G
|]
  ==> M Join G ∈ Always
        {s. (∑i ∈ lessThan Nclients. bag_of (sublist (merge.Out s)
                                {k. k < length (iOut s) & iOut s ! k = i}))
=
            (bag_of o merge.Out) s}"
```
⟨*proof*⟩

**lemma** (**in** *Merge*) *Merge_Bag_Follows:*
```
"M ∈ (⋂i ∈ lessThan Nclients. Increasing (sub i o merge.In))
     guarantees
        (bag_of o merge.Out) Fols
        (%s. ∑i ∈ lessThan Nclients. (bag_of o sub i o merge.In) s)"
```
⟨*proof*⟩

## 27.2 Theorems for Distributor

**lemma** (**in** *Distrib*) *Distr_Increasing_Out:*
```
"D ∈ Increasing distr.In Int Increasing distr.iIn Int
     Always {s. ∀elt ∈ set (distr.iIn s). elt < Nclients}
     guarantees
     (⋂i ∈ lessThan Nclients. Increasing (sub i o distr.Out))"
```
⟨*proof*⟩

**lemma** (**in** *Distrib*) *Distr_Bag_Follows_lemma:*
```
"[| G ∈ preserves distr.Out;
    D Join G ∈ Always {s. ∀elt ∈ set (distr.iIn s). elt < Nclients}
|]
```

```
  ==> D Join G ∈ Always
         {s. (∑ i ∈ lessThan Nclients. bag_of (sublist (distr.In s)
                                  {k. k < length (iIn s) & iIn s ! k = i}))
=
              bag_of (sublist (distr.In s) (lessThan (length (iIn s))))}"
```
⟨*proof*⟩

**lemma** (**in** *Distrib*) *D_ok_iff [iff]:*
```
     "D ok G = (G ∈ preserves distr.Out & D ∈ Allowed G)"
```
⟨*proof*⟩

**lemma** (**in** *Distrib*) *Distr_Bag_Follows:*
```
 "D ∈ Increasing distr.In Int Increasing distr.iIn Int
      Always {s. ∀elt ∈ set (distr.iIn s). elt < Nclients}
      guarantees
       (⋂i ∈ lessThan Nclients.
        (%s. ∑ i ∈ lessThan Nclients. (bag_of o sub i o distr.Out) s)
        Fols
        (%s. bag_of (sublist (distr.In s) (lessThan (length(distr.iIn s))))))"
```
⟨*proof*⟩

## 27.3 Theorems for Allocator

**lemma** *alloc_refinement_lemma:*
```
     "!!f::nat=>nat. (⋂i ∈ lessThan n. {s. f i ≤ g i s})
      ⊆ {s. (SUM x: lessThan n. f x) ≤ (SUM x: lessThan n. g x s)}"
```
⟨*proof*⟩

**lemma** *alloc_refinement:*
```
"(⋂i ∈ lessThan Nclients. Increasing (sub i o allocAsk) Int
                          Increasing (sub i o allocRel))
  Int
  Always {s. ∀i. i<Nclients -->
             (∀elt ∈ set ((sub i o allocAsk) s). elt ≤ NbT)}
  Int
  (⋂i ∈ lessThan Nclients.
   ⋂h. {s. h ≤ (sub i o allocGiv)s & h pfixGe (sub i o allocAsk)s}
       LeadsTo {s. tokens h ≤ (tokens o sub i o allocRel)s})
  ⊆
 (⋂i ∈ lessThan Nclients. Increasing (sub i o allocAsk) Int
                          Increasing (sub i o allocRel))
  Int
  Always {s. ∀i. i<Nclients -->
             (∀elt ∈ set ((sub i o allocAsk) s). elt ≤ NbT)}
  Int
  (⋂hf. (⋂i ∈ lessThan Nclients.
        {s. hf i ≤ (sub i o allocGiv)s & hf i pfixGe (sub i o allocAsk)s})
  LeadsTo {s. (∑ i ∈ lessThan Nclients. tokens (hf i)) ≤
             (∑ i ∈ lessThan Nclients. (tokens o sub i o allocRel)s)})"
```
⟨*proof*⟩

**end**

# 28 Distributed Resource Management System: the Client

**theory** *Client* **imports** *Rename AllocBase* **begin**

**types**
*tokbag = nat* — tokbags could be multisets...or any ordered type?

**record** *state =*
*giv :: "tokbag list"* — input history: tokens granted
*ask :: "tokbag list"* — output history: tokens requested
*rel :: "tokbag list"* — output history: tokens released
*tok :: tokbag* — current token request

**record** *'a state_d =*
*state +*
*dummy :: 'a* — new variables

**constdefs**

```
rel_act :: "('a state_d * 'a state_d) set"
  "rel_act == {(s,s').
               ∃nrel. nrel = size (rel s) &
                      s' = s (| rel := rel s @ [giv s!nrel] |) &
                      nrel < size (giv s) &
                      ask s!nrel ≤ giv s!nrel}"
```

```
tok_act :: "('a state_d * 'a state_d) set"
   "tok_act == {(s,s'). s'=s | s' = s (|tok := Suc (tok s mod NbT) |)}"

ask_act :: "('a state_d * 'a state_d) set"
  "ask_act == {(s,s'). s'=s |
                       (s' = s (|ask := ask s @ [tok s]|))}"

Client :: "'a state_d program"
  "Client ==
     mk_total_program
          ({s. tok s ∈ atMost NbT &
               giv s = [] & ask s = [] & rel s = []},
           {rel_act, tok_act, ask_act},
           ⋃G ∈ preserves rel Int preserves ask Int preserves tok.
                 Acts G)"
```

```
non_dummy :: "'a state_d => state"
```

```
    "non_dummy s == (|giv = giv s, ask = ask s, rel = rel s, tok = tok s|)"


  client_map :: "'a state_d => state*'a"
    "client_map == funPair non_dummy dummy"


declare Client_def [THEN def_prg_Init, simp]
declare Client_def [THEN def_prg_AllowedActs, simp]
declare rel_act_def [THEN def_act_simp, simp]
declare tok_act_def [THEN def_act_simp, simp]
declare ask_act_def [THEN def_act_simp, simp]

lemma Client_ok_iff [iff]:
     "(Client ok G) =
      (G ∈ preserves rel & G ∈ preserves ask & G ∈ preserves tok &
       Client ∈ Allowed G)"
```
⟨*proof*⟩

Safety property 1: ask, rel are increasing

```
lemma increasing_ask_rel:
     "Client ∈ UNIV guarantees Increasing ask Int Increasing rel"
```
⟨*proof*⟩

```
declare nth_append [simp] append_one_prefix [simp]
```

Safety property 2: the client never requests too many tokens. With no Substitution Axiom, we must prove the two invariants simultaneously.

```
lemma ask_bounded_lemma:
     "Client ok G
      ==> Client Join G ∈
              Always ({s. tok s ≤ NbT}  Int
                      {s. ∀elt ∈ set (ask s). elt ≤ NbT})"
```
⟨*proof*⟩

export version, with no mention of tok in the postcondition, but unfortunately tok must be declared local.

```
lemma ask_bounded:
     "Client ∈ UNIV guarantees Always {s. ∀elt ∈ set (ask s). elt ≤ NbT}"
```
⟨*proof*⟩

** Towards proving the liveness property **

```
lemma stable_rel_le_giv: "Client ∈ stable {s. rel s ≤ giv s}"
```
⟨*proof*⟩

```
lemma Join_Stable_rel_le_giv:
     "[| Client Join G ∈ Increasing giv;  G ∈ preserves rel |]
      ==> Client Join G ∈ Stable {s. rel s ≤ giv s}"
```
⟨*proof*⟩

```
lemma Join_Always_rel_le_giv:
     "[| Client Join G ∈ Increasing giv;  G ∈ preserves rel |]
      ==> Client Join G ∈ Always {s. rel s ≤ giv s}"
```

⟨*proof*⟩

**lemma** *transient_lemma:*
```
"Client ∈ transient {s. rel s = k & k<h & h ≤ giv s & h pfixGe ask s}"
```
⟨*proof*⟩

**lemma** *induct_lemma:*
```
"[| Client Join G ∈ Increasing giv;  Client ok G |]
 ==> Client Join G ∈ {s. rel s = k & k<h & h ≤ giv s & h pfixGe ask s}
                   LeadsTo {s. k < rel s & rel s ≤ giv s &
                               h ≤ giv s & h pfixGe ask s}"
```
⟨*proof*⟩

**lemma** *rel_progress_lemma:*
```
"[| Client Join G ∈ Increasing giv;  Client ok G |]
 ==> Client Join G ∈ {s. rel s < h & h ≤ giv s & h pfixGe ask s}
                   LeadsTo {s. h ≤ rel s}"
```
⟨*proof*⟩

**lemma** *client_progress_lemma:*
```
"[| Client Join G ∈ Increasing giv;  Client ok G |]
 ==> Client Join G ∈ {s. h ≤ giv s & h pfixGe ask s}
                     LeadsTo {s. h ≤ rel s}"
```
⟨*proof*⟩

Progress property: all tokens that are given will be released

**lemma** *client_progress:*
```
"Client ∈
  Increasing giv  guarantees
  (INT h. {s. h ≤ giv s & h pfixGe ask s} LeadsTo {s. h ≤ rel s})"
```
⟨*proof*⟩

This shows that the Client won't alter other variables in any state that it is combined with

**lemma** *client_preserves_dummy:* `"Client ∈ preserves dummy"`
⟨*proof*⟩

* Obsolete lemmas from first version of the Client *

**lemma** *stable_size_rel_le_giv:*
```
"Client ∈ stable {s. size (rel s) ≤ size (giv s)}"
```
⟨*proof*⟩

clients return the right number of tokens

**lemma** *ok_guar_rel_prefix_giv:*
```
"Client ∈ Increasing giv  guarantees  Always {s. rel s ≤ giv s}"
```
⟨*proof*⟩

**end**