The Isabelle/HOL Algebra Library

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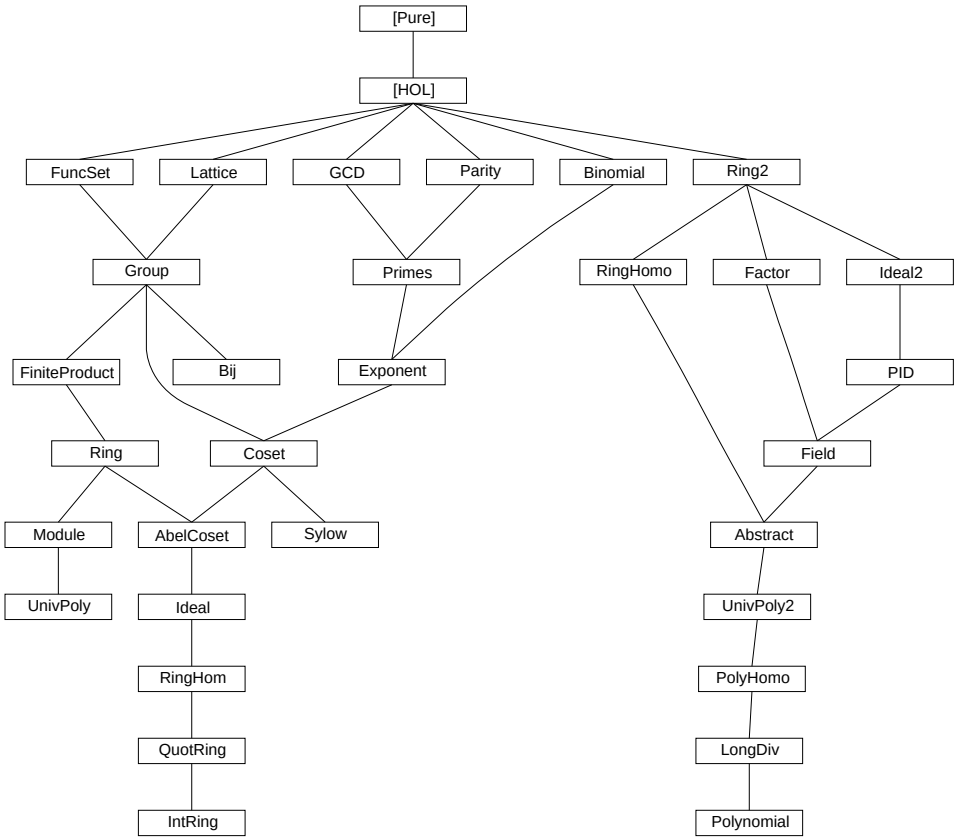
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theory *Lattice* **imports** *Main* **begin**

1 Orders and Lattices

Object with a carrier set.

record *'a partial-object* =
  carrier :: *'a set*

1.1 Partial Orders

record *'a order* = *'a partial-object* +
  le :: [*'a*, *'a*] => *bool* (**infixl** $\sqsubseteq_1$ *50*)

locale *partial-order* =
  fixes *L* (**structure**)
  assumes *refl* [*intro*, *simp*]:
    $x \in$ *carrier L* ==> $x \sqsubseteq x$
  and *anti-sym* [*intro*]:
    [| $x \sqsubseteq y$; $y \sqsubseteq x$; $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x = y$
  and *trans* [*trans*]:
    [| $x \sqsubseteq y$; $y \sqsubseteq z$;
     $x \in$ *carrier L*; $y \in$ *carrier L*; $z \in$ *carrier L* |] ==> $x \sqsubseteq z$

constdefs (**structure** *L*)
  lless :: [-, *'a*, *'a*] => *bool* (**infixl** $\sqsubset_1$ *50*)
  $x \sqsubset y$ == $x \sqsubseteq y$ & $x \sim= y$

  — Upper and lower bounds of a set.
  Upper :: [-, *'a set*] => *'a set*
  Upper L A == $\{u.\ (ALL\ x.\ x \in A \cap carrier\ L \dashrightarrow x \sqsubseteq u)\} \cap$
    carrier L

  Lower :: [-, *'a set*] => *'a set*
  Lower L A == $\{l.\ (ALL\ x.\ x \in A \cap carrier\ L \dashrightarrow l \sqsubseteq x)\} \cap$
    carrier L

  — Least and greatest, as predicate.
  least :: [-, *'a*, *'a set*] => *bool*
  least L l A == $A \subseteq$ *carrier L* & $l \in A$ & $(ALL\ x : A.\ l \sqsubseteq x)$

  greatest :: [-, *'a*, *'a set*] => *bool*
  greatest L g A == $A \subseteq$ *carrier L* & $g \in A$ & $(ALL\ x : A.\ x \sqsubseteq g)$

  — Supremum and infimum
  sup :: [-, *'a set*] => *'a* ($\bigsqcup_1$- [*90*] *90*)
  $\bigsqcup A$ == *THE x. least L x* (*Upper L A*)

inf :: [-, *'a set*] => *'a* ($\bigsqcap_1$- [*90*] *90*)
$\bigsqcap A$ == *THE x. greatest L x* (*Lower L A*)

join :: [-, *'a*, *'a*] => *'a* (**infixl** $\sqcup_1$ *65*)
$x \sqcup y$ == *sup L* {x, y}

meet :: [-, *'a*, *'a*] => *'a* (**infixl** $\sqcap_1$ *70*)
$x \sqcap y$ == *inf L* {x, y}

1.1.1 Upper

lemma *Upper-closed* [*intro*, *simp*]:
Upper L A $\subseteq$ *carrier L*
⟨*proof*⟩

lemma *UpperD* [*dest*]:
fixes *L* (**structure**)
shows [| $u \in$ *Upper L A*; $x \in A$; $A \subseteq$ *carrier L* |] ==> $x \sqsubseteq u$
⟨*proof*⟩

lemma *Upper-memI*:
fixes *L* (**structure**)
shows [| !! *y*. $y \in A$ ==> $y \sqsubseteq x$; $x \in$ *carrier L* |] ==> $x \in$ *Upper L A*
⟨*proof*⟩

lemma *Upper-antimono*:
$A \subseteq B$ ==> *Upper L B* $\subseteq$ *Upper L A*
⟨*proof*⟩

1.1.2 Lower

lemma *Lower-closed* [*intro*, *simp*]:
Lower L A $\subseteq$ *carrier L*
⟨*proof*⟩

lemma *LowerD* [*dest*]:
fixes *L* (**structure**)
shows [| $l \in$ *Lower L A*; $x \in A$; $A \subseteq$ *carrier L* |] ==> $l \sqsubseteq x$
⟨*proof*⟩

lemma *Lower-memI*:
fixes *L* (**structure**)
shows [| !! *y*. $y \in A$ ==> $x \sqsubseteq y$; $x \in$ *carrier L* |] ==> $x \in$ *Lower L A*
⟨*proof*⟩

lemma *Lower-antimono*:
$A \subseteq B$ ==> *Lower L B* $\subseteq$ *Lower L A*
⟨*proof*⟩

1.1.3 least

lemma *least-carrier* [*intro*, *simp*]:
shows *least L l A* ==> *l* ∈ *carrier L*
⟨*proof*⟩

lemma *least-mem*:
least L l A ==> *l* ∈ *A*
⟨*proof*⟩

lemma (**in** *partial-order*) *least-unique*:
[| *least L x A*; *least L y A* |] ==> *x* = *y*
⟨*proof*⟩

lemma *least-le*:
fixes *L* (**structure**)
shows [| *least L x A*; *a* ∈ *A* |] ==> *x* ⊑ *a*
⟨*proof*⟩

lemma *least-UpperI*:
fixes *L* (**structure**)
assumes *above*: !! *x*. *x* ∈ *A* ==> *x* ⊑ *s*
and *below*: !! *y*. *y* ∈ *Upper L A* ==> *s* ⊑ *y*
and *L*: *A* ⊆ *carrier L* *s* ∈ *carrier L*
shows *least L s* (*Upper L A*)
⟨*proof*⟩

1.1.4 greatest

lemma *greatest-carrier* [*intro*, *simp*]:
shows *greatest L l A* ==> *l* ∈ *carrier L*
⟨*proof*⟩

lemma *greatest-mem*:
greatest L l A ==> *l* ∈ *A*
⟨*proof*⟩

lemma (**in** *partial-order*) *greatest-unique*:
[| *greatest L x A*; *greatest L y A* |] ==> *x* = *y*
⟨*proof*⟩

lemma *greatest-le*:
fixes *L* (**structure**)
shows [| *greatest L x A*; *a* ∈ *A* |] ==> *a* ⊑ *x*
⟨*proof*⟩

lemma *greatest-LowerI*:
fixes *L* (**structure**)
assumes *below*: !! *x*. *x* ∈ *A* ==> *i* ⊑ *x*
and *above*: !! *y*. *y* ∈ *Lower L A* ==> *y* ⊑ *i*

and L: $A \subseteq$ *carrier* L $i \in$ *carrier* L
shows *greatest* L i (*Lower* L A)
⟨*proof*⟩

1.2 Lattices

locale *lattice* = *partial-order* +
assumes *sup-of-two-exists*:
[| $x \in$ *carrier* L; $y \in$ *carrier* L |] ==> *EX* s. *least* L s (*Upper* L {x, y})
and *inf-of-two-exists*:
[| $x \in$ *carrier* L; $y \in$ *carrier* L |] ==> *EX* s. *greatest* L s (*Lower* L {x, y})

lemma *least-Upper-above*:
fixes L (**structure**)
shows [| *least* L s (*Upper* L A); $x \in A$; $A \subseteq$ *carrier* L |] ==> $x \sqsubseteq s$
⟨*proof*⟩

lemma *greatest-Lower-above*:
fixes L (**structure**)
shows [| *greatest* L i (*Lower* L A); $x \in A$; $A \subseteq$ *carrier* L |] ==> $i \sqsubseteq x$
⟨*proof*⟩

1.2.1 Supremum

lemma (**in** *lattice*) *joinI*:
[| !!l. *least* L l (*Upper* L {x, y}) ==> P l; $x \in$ *carrier* L; $y \in$ *carrier* L |]
==> P ($x \sqcup y$)
⟨*proof*⟩

lemma (**in** *lattice*) *join-closed* [*simp*]:
[| $x \in$ *carrier* L; $y \in$ *carrier* L |] ==> $x \sqcup y \in$ *carrier* L
⟨*proof*⟩

lemma (**in** *partial-order*) *sup-of-singletonI*:
$x \in$ *carrier* L ==> *least* L x (*Upper* L {x})
⟨*proof*⟩

lemma (**in** *partial-order*) *sup-of-singleton* [*simp*]:
$x \in$ *carrier* L ==> $\bigsqcup\{x\} = x$
⟨*proof*⟩

Condition on A: supremum exists.

lemma (**in** *lattice*) *sup-insertI*:
[| !!s. *least* L s (*Upper* L (*insert* x A)) ==> P s;
least L a (*Upper* L A); $x \in$ *carrier* L; $A \subseteq$ *carrier* L |]
==> P ($\bigsqcup$(*insert* x A))
⟨*proof*⟩

lemma (**in** *lattice*) *finite-sup-least*:

[| *finite A*; $A \subseteq$ *carrier L*; $A \sim= \{\}$ |] ==> *least L* ($\bigsqcup A$) (*Upper L A*)
⟨*proof*⟩

lemma (**in** *lattice*) *finite-sup-insertI*:
assumes *P*: !!*l*. *least L l* (*Upper L* (*insert x A*)) ==> *P l*
and *xA*: *finite A* $x \in$ *carrier L* $A \subseteq$ *carrier L*
shows *P* ($\bigsqcup$ (*insert x A*))
⟨*proof*⟩

lemma (**in** *lattice*) *finite-sup-closed*:
[| *finite A*; $A \subseteq$ *carrier L*; $A \sim= \{\}$ |] ==> $\bigsqcup A \in$ *carrier L*
⟨*proof*⟩

lemma (**in** *lattice*) *join-left*:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqsubseteq x \sqcup y$
⟨*proof*⟩

lemma (**in** *lattice*) *join-right*:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $y \sqsubseteq x \sqcup y$
⟨*proof*⟩

lemma (**in** *lattice*) *sup-of-two-least*:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> *least L* ($\bigsqcup \{x, y\}$) (*Upper L* $\{x, y\}$)
⟨*proof*⟩

lemma (**in** *lattice*) *join-le*:
assumes *sub*: $x \sqsubseteq z$ $y \sqsubseteq z$
and *x*: $x \in$ *carrier L* **and** *y*: $y \in$ *carrier L* **and** *z*: $z \in$ *carrier L*
shows $x \sqcup y \sqsubseteq z$
⟨*proof*⟩

lemma (**in** *lattice*) *join-assoc-lemma*:
assumes *L*: $x \in$ *carrier L* $y \in$ *carrier L* $z \in$ *carrier L*
shows $x \sqcup (y \sqcup z) = \bigsqcup \{x, y, z\}$
⟨*proof*⟩

lemma *join-comm*:
fixes *L* (**structure**)
shows $x \sqcup y = y \sqcup x$
⟨*proof*⟩

lemma (**in** *lattice*) *join-assoc*:
assumes *L*: $x \in$ *carrier L* $y \in$ *carrier L* $z \in$ *carrier L*
shows $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$
⟨*proof*⟩

1.2.2 Infimum

lemma (**in** *lattice*) *meetI*:

[| !!i. *greatest L i* (*Lower L* {x, y}) ==> P i;
$x \in$ *carrier L*; $y \in$ *carrier L* |]
==> P ($x \sqcap y$)
⟨*proof*⟩

lemma (**in** *lattice*) *meet-closed* [*simp*]:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqcap y \in$ *carrier L*
⟨*proof*⟩

lemma (**in** *partial-order*) *inf-of-singletonI*:
$x \in$ *carrier L* ==> *greatest L x* (*Lower L* {x})
⟨*proof*⟩

lemma (**in** *partial-order*) *inf-of-singleton* [*simp*]:
$x \in$ *carrier L* ==> $\bigsqcap$ {x} = x
⟨*proof*⟩

Condition on A: infimum exists.

lemma (**in** *lattice*) *inf-insertI*:
[| !!i. *greatest L i* (*Lower L* (*insert x A*)) ==> P i;
greatest L a (*Lower L A*); $x \in$ *carrier L*; $A \subseteq$ *carrier L* |]
==> P ($\bigsqcap$(*insert x A*))
⟨*proof*⟩

lemma (**in** *lattice*) *finite-inf-greatest*:
[| *finite A*; $A \subseteq$ *carrier L*; $A \sim= \{\}$ |] ==> *greatest L* ($\bigsqcap A$) (*Lower L A*)
⟨*proof*⟩

lemma (**in** *lattice*) *finite-inf-insertI*:
assumes P: !!i. *greatest L i* (*Lower L* (*insert x A*)) ==> P i
and xA: *finite A* $x \in$ *carrier L* $A \subseteq$ *carrier L*
shows P ($\bigsqcap$ (*insert x A*))
⟨*proof*⟩

lemma (**in** *lattice*) *finite-inf-closed*:
[| *finite A*; $A \subseteq$ *carrier L*; $A \sim= \{\}$ |] ==> $\bigsqcap A \in$ *carrier L*
⟨*proof*⟩

lemma (**in** *lattice*) *meet-left*:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqcap y \sqsubseteq x$
⟨*proof*⟩

lemma (**in** *lattice*) *meet-right*:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==> $x \sqcap y \sqsubseteq y$
⟨*proof*⟩

lemma (**in** *lattice*) *inf-of-two-greatest*:
[| $x \in$ *carrier L*; $y \in$ *carrier L* |] ==>
greatest L ($\bigsqcap$ {x, y}) (*Lower L* {x, y})

⟨*proof*⟩

lemma (**in** *lattice*) *meet-le*:
assumes *sub*: $z \sqsubseteq x$ $z \sqsubseteq y$
and x: $x \in$ *carrier* L **and** y: $y \in$ *carrier* L **and** z: $z \in$ *carrier* L
shows $z \sqsubseteq x \sqcap y$
⟨*proof*⟩

lemma (**in** *lattice*) *meet-assoc-lemma*:
assumes L: $x \in$ *carrier* L $y \in$ *carrier* L $z \in$ *carrier* L
shows $x \sqcap (y \sqcap z) = \bigsqcap \{x, y, z\}$
⟨*proof*⟩

lemma *meet-comm*:
fixes L (**structure**)
shows $x \sqcap y = y \sqcap x$
⟨*proof*⟩

lemma (**in** *lattice*) *meet-assoc*:
assumes L: $x \in$ *carrier* L $y \in$ *carrier* L $z \in$ *carrier* L
shows $(x \sqcap y) \sqcap z = x \sqcap (y \sqcap z)$
⟨*proof*⟩

1.3 Total Orders

locale *total-order* = *partial-order* +
assumes *total*: [| $x \in$ *carrier* L; $y \in$ *carrier* L |] ==> $x \sqsubseteq y \mid y \sqsubseteq x$

Introduction rule: the usual definition of total order

lemma (**in** *partial-order*) *total-orderI*:
assumes *total*: !!x y. [| $x \in$ *carrier* L; $y \in$ *carrier* L |] ==> $x \sqsubseteq y \mid y \sqsubseteq x$
shows *total-order* L
⟨*proof*⟩

Total orders are lattices.

interpretation *total-order* < *lattice*
⟨*proof*⟩

1.4 Complete lattices

locale *complete-lattice* = *lattice* +
assumes *sup-exists*:
[| $A \subseteq$ *carrier* L |] ==> *EX* s. *least* L s (*Upper* L A)
and *inf-exists*:
[| $A \subseteq$ *carrier* L |] ==> *EX* i. *greatest* L i (*Lower* L A)

Introduction rule: the usual definition of complete lattice

lemma (**in** *partial-order*) *complete-latticeI*:
assumes *sup-exists*:

!!A. [| $A \subseteq$ *carrier* L |] ==> *EX* s. *least* L s (*Upper* L A)
and *inf-exists*:
!!A. [| $A \subseteq$ *carrier* L |] ==> *EX* i. *greatest* L i (*Lower* L A)
shows *complete-lattice* L
⟨*proof*⟩

constdefs (**structure** L)
top :: - => 'a ($\top_1$)
$\top$ == *sup* L (*carrier* L)

bottom :: - => 'a ($\bot_1$)
$\bot$ == *inf* L (*carrier* L)

lemma (**in** *complete-lattice*) *supI*:
[| !!l. *least* L l (*Upper* L A) ==> P l; $A \subseteq$ *carrier* L |]
==> P ($\bigsqcup A$)
⟨*proof*⟩

lemma (**in** *complete-lattice*) *sup-closed* [*simp*]:
$A \subseteq$ *carrier* L ==> $\bigsqcup A \in$ *carrier* L
⟨*proof*⟩

lemma (**in** *complete-lattice*) *top-closed* [*simp*, *intro*]:
$\top \in$ *carrier* L
⟨*proof*⟩

lemma (**in** *complete-lattice*) *infI*:
[| !!i. *greatest* L i (*Lower* L A) ==> P i; $A \subseteq$ *carrier* L |]
==> P ($\bigsqcap A$)
⟨*proof*⟩

lemma (**in** *complete-lattice*) *inf-closed* [*simp*]:
$A \subseteq$ *carrier* L ==> $\bigsqcap A \in$ *carrier* L
⟨*proof*⟩

lemma (**in** *complete-lattice*) *bottom-closed* [*simp*, *intro*]:
$\bot \in$ *carrier* L
⟨*proof*⟩

Jacobson: Theorem 8.1

lemma *Lower-empty* [*simp*]:
Lower L {} = *carrier* L
⟨*proof*⟩

lemma *Upper-empty* [*simp*]:
Upper L {} = *carrier* L
⟨*proof*⟩

theorem (**in** *partial-order*) *complete-lattice-criterion1*:
assumes *top-exists*: *EX g. greatest L g* (*carrier L*)
and *inf-exists*:
!!*A*. [| *A* ⊆ *carrier L*; *A* ~= {} |] ==> *EX i. greatest L i* (*Lower L A*)
shows *complete-lattice L*
⟨*proof*⟩

1.5 Examples

1.5.1 Powerset of a Set is a Complete Lattice

theorem *powerset-is-complete-lattice*:
complete-lattice (| *carrier* = *Pow A*, *le* = *op* ⊆ |)
(**is** *complete-lattice ?L*)
⟨*proof*⟩

An other example, that of the lattice of subgroups of a group, can be found in Group theory (Section 2.7).

end

theory *Group* **imports** *FuncSet Lattice* **begin**

2 Monoids and Groups

2.1 Definitions

Definitions follow [2].

record *'a monoid* = *'a partial-object* +
mult :: ['a, 'a] ⇒ 'a (**infixl** ⊗$_1$ *70*)
one :: 'a (**1**$_1$)

constdefs (**structure** *G*)
m-inv :: ('a, 'b) *monoid-scheme* => 'a => 'a (*inv*$_1$ - [*81*] *80*)
inv x == (*THE y. y* ∈ *carrier G* & *x* ⊗ *y* = **1** & *y* ⊗ *x* = **1**)

Units :: - => 'a *set*
— The set of invertible elements
Units G == {*y. y* ∈ *carrier G* & (∃ *x* ∈ *carrier G. x* ⊗ *y* = **1** & *y* ⊗ *x* = **1**)}

consts
pow :: [('a, 'm) *monoid-scheme*, 'a, 'b::*number*] => 'a (**infixr** '(^')$_1$ *75*)

defs (**overloaded**)
nat-pow-def: *pow G a n* == *nat-rec* **1**$_G$ (%*u b. b* ⊗$_G$ *a*) *n*
int-pow-def: *pow G a z* ==
let p = *nat-rec* **1**$_G$ (%*u b. b* ⊗$_G$ *a*)

in if neg z then inv$_G$ *(p (nat (−z))) else p (nat z)*

locale *monoid* =
fixes *G* (**structure**)
assumes *m-closed* [*intro*, *simp*]:
⟦*x* ∈ *carrier G*; *y* ∈ *carrier G*⟧ ⟹ *x* ⊗ *y* ∈ *carrier G*
and *m-assoc*:
⟦*x* ∈ *carrier G*; *y* ∈ *carrier G*; *z* ∈ *carrier G*⟧
⟹ (*x* ⊗ *y*) ⊗ *z* = *x* ⊗ (*y* ⊗ *z*)
and *one-closed* [*intro*, *simp*]: **1** ∈ *carrier G*
and *l-one* [*simp*]: *x* ∈ *carrier G* ⟹ **1** ⊗ *x* = *x*
and *r-one* [*simp*]: *x* ∈ *carrier G* ⟹ *x* ⊗ **1** = *x*

lemma *monoidI*:
fixes *G* (**structure**)
assumes *m-closed*:
!!*x y*. [| *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==> *x* ⊗ *y* ∈ *carrier G*
and *one-closed*: **1** ∈ *carrier G*
and *m-assoc*:
!!*x y z*. [| *x* ∈ *carrier G*; *y* ∈ *carrier G*; *z* ∈ *carrier G* |] ==>
(*x* ⊗ *y*) ⊗ *z* = *x* ⊗ (*y* ⊗ *z*)
and *l-one*: !!*x*. *x* ∈ *carrier G* ==> **1** ⊗ *x* = *x*
and *r-one*: !!*x*. *x* ∈ *carrier G* ==> *x* ⊗ **1** = *x*
shows *monoid G*
⟨*proof*⟩

lemma (**in** *monoid*) *Units-closed* [*dest*]:
x ∈ *Units G* ==> *x* ∈ *carrier G*
⟨*proof*⟩

lemma (**in** *monoid*) *inv-unique*:
assumes *eq*: *y* ⊗ *x* = **1** *x* ⊗ *y*′ = **1**
and *G*: *x* ∈ *carrier G* *y* ∈ *carrier G* *y*′ ∈ *carrier G*
shows *y* = *y*′
⟨*proof*⟩

lemma (**in** *monoid*) *Units-one-closed* [*intro*, *simp*]:
1 ∈ *Units G*
⟨*proof*⟩

lemma (**in** *monoid*) *Units-inv-closed* [*intro*, *simp*]:
x ∈ *Units G* ==> *inv x* ∈ *carrier G*
⟨*proof*⟩

lemma (**in** *monoid*) *Units-l-inv-ex*:
x ∈ *Units G* ==> ∃ *y* ∈ *carrier G*. *y* ⊗ *x* = **1**
⟨*proof*⟩

lemma (**in** *monoid*) *Units-r-inv-ex*:

$x \in \mathit{Units}\ G \Longrightarrow \exists\, y \in \mathit{carrier}\ G.\ x \otimes y = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *monoid*) *Units-l-inv*:
$x \in \mathit{Units}\ G \Longrightarrow \mathit{inv}\ x \otimes x = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *monoid*) *Units-r-inv*:
$x \in \mathit{Units}\ G \Longrightarrow x \otimes \mathit{inv}\ x = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *monoid*) *Units-inv-Units* [*intro*, *simp*]:
$x \in \mathit{Units}\ G \Longrightarrow \mathit{inv}\ x \in \mathit{Units}\ G$
⟨*proof*⟩

lemma (**in** *monoid*) *Units-l-cancel* [*simp*]:
$[\!|\ x \in \mathit{Units}\ G;\ y \in \mathit{carrier}\ G;\ z \in \mathit{carrier}\ G\ |\!] \Longrightarrow$
$(x \otimes y = x \otimes z) = (y = z)$
⟨*proof*⟩

lemma (**in** *monoid*) *Units-inv-inv* [*simp*]:
$x \in \mathit{Units}\ G \Longrightarrow \mathit{inv}\ (\mathit{inv}\ x) = x$
⟨*proof*⟩

lemma (**in** *monoid*) *inv-inj-on-Units*:
$\mathit{inj\text{-}on}\ (\mathit{m\text{-}inv}\ G)\ (\mathit{Units}\ G)$
⟨*proof*⟩

lemma (**in** *monoid*) *Units-inv-comm*:
assumes *inv*: $x \otimes y = \mathbf{1}$
and *G*: $x \in \mathit{Units}\ G\ \ y \in \mathit{Units}\ G$
shows $y \otimes x = \mathbf{1}$
⟨*proof*⟩

Power

lemma (**in** *monoid*) *nat-pow-closed* [*intro*, *simp*]:
$x \in \mathit{carrier}\ G \Longrightarrow x\ (\hat{})\ (n{::}\mathit{nat}) \in \mathit{carrier}\ G$
⟨*proof*⟩

lemma (**in** *monoid*) *nat-pow-0* [*simp*]:
$x\ (\hat{})\ (0{::}\mathit{nat}) = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *monoid*) *nat-pow-Suc* [*simp*]:
$x\ (\hat{})\ (\mathit{Suc}\ n) = x\ (\hat{})\ n \otimes x$
⟨*proof*⟩

lemma (**in** *monoid*) *nat-pow-one* [*simp*]:
$\mathbf{1}\ (\hat{})\ (n{::}\mathit{nat}) = \mathbf{1}$

⟨*proof*⟩

lemma (**in** *monoid*) *nat-pow-mult*:
x ∈ *carrier G* ==> *x* (^) (*n*::*nat*) ⊗ *x* (^) *m* = *x* (^) (*n* + *m*)
⟨*proof*⟩

lemma (**in** *monoid*) *nat-pow-pow*:
x ∈ *carrier G* ==> (*x* (^) *n*) (^) *m* = *x* (^) (*n* ∗ *m*::*nat*)
⟨*proof*⟩

A group is a monoid all of whose elements are invertible.

locale *group* = *monoid* +
assumes *Units*: *carrier G* <= *Units G*

lemma (**in** *group*) *is-group*: *group G* ⟨*proof*⟩

theorem *groupI*:
fixes *G* (**structure**)
assumes *m-closed* [*simp*]:
!!*x y*. [| *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==> *x* ⊗ *y* ∈ *carrier G*
and *one-closed* [*simp*]: **1** ∈ *carrier G*
and *m-assoc*:
!!*x y z*. [| *x* ∈ *carrier G*; *y* ∈ *carrier G*; *z* ∈ *carrier G* |] ==>
(*x* ⊗ *y*) ⊗ *z* = *x* ⊗ (*y* ⊗ *z*)
and *l-one* [*simp*]: !!*x*. *x* ∈ *carrier G* ==> **1** ⊗ *x* = *x*
and *l-inv-ex*: !!*x*. *x* ∈ *carrier G* ==> ∃ *y* ∈ *carrier G*. *y* ⊗ *x* = **1**
shows *group G*
⟨*proof*⟩

lemma (**in** *monoid*) *monoid-groupI*:
assumes *l-inv-ex*:
!!*x*. *x* ∈ *carrier G* ==> ∃ *y* ∈ *carrier G*. *y* ⊗ *x* = **1**
shows *group G*
⟨*proof*⟩

lemma (**in** *group*) *Units-eq* [*simp*]:
Units G = *carrier G*
⟨*proof*⟩

lemma (**in** *group*) *inv-closed* [*intro*, *simp*]:
x ∈ *carrier G* ==> *inv x* ∈ *carrier G*
⟨*proof*⟩

lemma (**in** *group*) *l-inv-ex* [*simp*]:
x ∈ *carrier G* ==> ∃ *y* ∈ *carrier G*. *y* ⊗ *x* = **1**
⟨*proof*⟩

lemma (**in** *group*) *r-inv-ex* [*simp*]:

$x \in carrier\ G \Longrightarrow \exists\, y \in carrier\ G.\ x \otimes y = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *group*) *l-inv* [*simp*]:
$x \in carrier\ G \Longrightarrow inv\ x \otimes x = \mathbf{1}$
⟨*proof*⟩

2.2 Cancellation Laws and Basic Properties

lemma (**in** *group*) *l-cancel* [*simp*]:
$[|\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] \Longrightarrow$
$(x \otimes y = x \otimes z) = (y = z)$
⟨*proof*⟩

lemma (**in** *group*) *r-inv* [*simp*]:
$x \in carrier\ G \Longrightarrow x \otimes inv\ x = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *group*) *r-cancel* [*simp*]:
$[|\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] \Longrightarrow$
$(y \otimes x = z \otimes x) = (y = z)$
⟨*proof*⟩

lemma (**in** *group*) *inv-one* [*simp*]:
$inv\ \mathbf{1} = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *group*) *inv-inv* [*simp*]:
$x \in carrier\ G \Longrightarrow inv\ (inv\ x) = x$
⟨*proof*⟩

lemma (**in** *group*) *inv-inj*:
$inj\text{-}on\ (m\text{-}inv\ G)\ (carrier\ G)$
⟨*proof*⟩

lemma (**in** *group*) *inv-mult-group*:
$[|\ x \in carrier\ G;\ y \in carrier\ G\ |] \Longrightarrow inv\ (x \otimes y) = inv\ y \otimes inv\ x$
⟨*proof*⟩

lemma (**in** *group*) *inv-comm*:
$[|\ x \otimes y = \mathbf{1};\ x \in carrier\ G;\ y \in carrier\ G\ |] \Longrightarrow y \otimes x = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *group*) *inv-equality*:
$[|y \otimes x = \mathbf{1};\ x \in carrier\ G;\ y \in carrier\ G|] \Longrightarrow inv\ x = y$
⟨*proof*⟩

Power

lemma (**in** *group*) *int-pow-def2*:

$a\ (\hat{})\ (z{::}int) = (\mathit{if}\ \mathit{neg}\ z\ \mathit{then}\ \mathit{inv}\ (a\ (\hat{})\ (\mathit{nat}\ (-z)))\ \mathit{else}\ a\ (\hat{})\ (\mathit{nat}\ z))$
⟨*proof*⟩

lemma (**in** *group*) *int-pow-0* [*simp*]:
$x\ (\hat{})\ (0{::}int) = \mathbf{1}$
⟨*proof*⟩

lemma (**in** *group*) *int-pow-one* [*simp*]:
$\mathbf{1}\ (\hat{})\ (z{::}int) = \mathbf{1}$
⟨*proof*⟩

2.3 Subgroups

locale *subgroup* =
fixes H **and** G (**structure**)
assumes *subset*: $H \subseteq carrier\ G$
and *m-closed* [*intro*, *simp*]: $[\![x \in H;\ y \in H]\!] \Longrightarrow x \otimes y \in H$
and *one-closed* [*simp*]: $\mathbf{1} \in H$
and *m-inv-closed* [*intro*,*simp*]: $x \in H \Longrightarrow inv\ x \in H$

lemma (**in** *subgroup*) *is-subgroup*:
subgroup H G ⟨*proof*⟩

declare (**in** *subgroup*) *group.intro* [*intro*]

lemma (**in** *subgroup*) *mem-carrier* [*simp*]:
$x \in H \Longrightarrow x \in carrier\ G$
⟨*proof*⟩

lemma *subgroup-imp-subset*:
subgroup H $G \Longrightarrow H \subseteq carrier\ G$
⟨*proof*⟩

lemma (**in** *subgroup*) *subgroup-is-group* [*intro*]:
includes *group* G
shows *group* $(G(\!|carrier := H|\!))$
⟨*proof*⟩

Since H is nonempty, it contains some element x. Since it is closed under inverse, it contains $inv\ x$. Since it is closed under product, it contains $x \otimes inv\ x = \mathbf{1}$.

lemma (**in** *group*) *one-in-subset*:
$[|\ H \subseteq carrier\ G;\ H \neq \{\};\ \forall\, a \in H.\ inv\ a \in H;\ \forall\, a{\in}H.\ \forall\, b{\in}H.\ a \otimes b \in H\ |]$
$\Longrightarrow \mathbf{1} \in H$
⟨*proof*⟩

A characterization of subgroups: closed, non-empty subset.

lemma (**in** *group*) *subgroupI*:
assumes *subset*: $H \subseteq carrier\ G$ **and** *non-empty*: $H \neq \{\}$

and *inv*: !!a. $a \in H \Longrightarrow inv\ a \in H$
and *mult*: !!a b. $[\![a \in H;\ b \in H]\!] \Longrightarrow a \otimes b \in H$
shows *subgroup H G*
⟨*proof*⟩

declare *monoid.one-closed* [*iff*] *group.inv-closed* [*simp*]
monoid.l-one [*simp*] *monoid.r-one* [*simp*] *group.inv-inv* [*simp*]

lemma *subgroup-nonempty*:
~ *subgroup* {} *G*
⟨*proof*⟩

lemma (**in** *subgroup*) *finite-imp-card-positive*:
finite (*carrier G*) ==> 0 < *card H*
⟨*proof*⟩

2.4 Direct Products

constdefs
DirProd :: - ⇒ - ⇒ ('*a* × '*b*) *monoid* (**infixr** ×× 80)
$G \times\times H \equiv (\!|carrier = carrier\ G \times carrier\ H,$
mult = (λ(g, h) (g', h'). ($g \otimes_G g'$, $h \otimes_H h'$)),
one = ($\mathbf{1}_G$, $\mathbf{1}_H$)|)

lemma *DirProd-monoid*:
includes *monoid G* + *monoid H*
shows *monoid* (*G* ×× *H*)
⟨*proof*⟩

Does not use the previous result because it's easier just to use auto.

lemma *DirProd-group*:
includes *group G* + *group H*
shows *group* (*G* ×× *H*)
⟨*proof*⟩

lemma *carrier-DirProd* [*simp*]:
carrier (*G* ×× *H*) = *carrier G* × *carrier H*
⟨*proof*⟩

lemma *one-DirProd* [*simp*]:
$\mathbf{1}_{G \times\times H} = (\mathbf{1}_G, \mathbf{1}_H)$
⟨*proof*⟩

lemma *mult-DirProd* [*simp*]:
(g, h) $\otimes_{(G \times\times H)}$ (g', h') = ($g \otimes_G g'$, $h \otimes_H h'$)
⟨*proof*⟩

lemma *inv-DirProd* [*simp*]:
includes *group G* + *group H*

assumes g: $g \in carrier\ G$
and h: $h \in carrier\ H$
shows $m\text{-}inv\ (G \times\times H)\ (g, h) = (inv_G\ g, inv_H\ h)$
⟨*proof*⟩

This alternative proof of the previous result demonstrates interpret. It uses *Prod.inv-equality* (available after *interpret*) instead of *group.inv-equality* [*OF DirProd-group*].

lemma
includes $group\ G + group\ H$
assumes g: $g \in carrier\ G$
and h: $h \in carrier\ H$
shows $m\text{-}inv\ (G \times\times H)\ (g, h) = (inv_G\ g, inv_H\ h)$
⟨*proof*⟩

2.5 Homomorphisms and Isomorphisms

constdefs (**structure** G **and** H)
$hom :: - => - => ('a => 'b)\ set$
$hom\ G\ H ==$
$\{h.\ h \in carrier\ G -> carrier\ H\ \&$
$(\forall x \in carrier\ G.\ \forall y \in carrier\ G.\ h\ (x \otimes_G y) = h\ x \otimes_H h\ y)\}$

lemma *hom-mult*:
$[|\ h \in hom\ G\ H;\ x \in carrier\ G;\ y \in carrier\ G\ |]$
$==> h\ (x \otimes_G y) = h\ x \otimes_H h\ y$
⟨*proof*⟩

lemma *hom-closed*:
$[|\ h \in hom\ G\ H;\ x \in carrier\ G\ |] ==> h\ x \in carrier\ H$
⟨*proof*⟩

lemma (**in** *group*) *hom-compose*:
$[|h \in hom\ G\ H;\ i \in hom\ H\ I|] ==> compose\ (carrier\ G)\ i\ h \in hom\ G\ I$
⟨*proof*⟩

constdefs
$iso :: - => - => ('a => 'b)\ set$ (**infixr** $\cong$ *60*)
$G \cong H == \{h.\ h \in hom\ G\ H\ \&\ bij\text{-}betw\ h\ (carrier\ G)\ (carrier\ H)\}$

lemma *iso-refl*: $(\%x.\ x) \in G \cong G$
⟨*proof*⟩

lemma (**in** *group*) *iso-sym*:
$h \in G \cong H \Longrightarrow Inv\ (carrier\ G)\ h \in H \cong G$
⟨*proof*⟩

lemma (**in** *group*) *iso-trans*:
$[|h \in G \cong H;\ i \in H \cong I|] ==> (compose\ (carrier\ G)\ i\ h) \in G \cong I$

⟨*proof*⟩

lemma *DirProd-commute-iso*:
shows $(\lambda(x,y).\ (y,x)) \in (G \times\times H) \cong (H \times\times G)$
⟨*proof*⟩

lemma *DirProd-assoc-iso*:
shows $(\lambda(x,y,z).\ (x,(y,z))) \in (G \times\times H \times\times I) \cong (G \times\times (H \times\times I))$
⟨*proof*⟩

Basis for homomorphism proofs: we assume two groups G and H, with a homomorphism h between them

locale *group-hom* = *group G* + *group H* + *var h* +
assumes *homh*: $h \in hom\ G\ H$
notes *hom-mult* [*simp*] = *hom-mult* [*OF homh*]
and *hom-closed* [*simp*] = *hom-closed* [*OF homh*]

lemma (**in** *group-hom*) *one-closed* [*simp*]:
$h\ \mathbf{1} \in carrier\ H$
⟨*proof*⟩

lemma (**in** *group-hom*) *hom-one* [*simp*]:
$h\ \mathbf{1} = \mathbf{1}_H$
⟨*proof*⟩

lemma (**in** *group-hom*) *inv-closed* [*simp*]:
$x \in carrier\ G \Longrightarrow h\ (inv\ x) \in carrier\ H$
⟨*proof*⟩

lemma (**in** *group-hom*) *hom-inv* [*simp*]:
$x \in carrier\ G \Longrightarrow h\ (inv\ x) = inv_H\ (h\ x)$
⟨*proof*⟩

2.6 Commutative Structures

Naming convention: multiplicative structures that are commutative are called *commutative*, additive structures are called *Abelian*.

locale *comm-monoid* = *monoid* +
assumes *m-comm*: $[\![x \in carrier\ G;\ y \in carrier\ G]\!] \Longrightarrow x \otimes y = y \otimes x$

lemma (**in** *comm-monoid*) *m-lcomm*:
$[\![x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G]\!] \Longrightarrow$
$x \otimes (y \otimes z) = y \otimes (x \otimes z)$
⟨*proof*⟩

lemmas (**in** *comm-monoid*) *m-ac* = *m-assoc m-comm m-lcomm*

lemma *comm-monoidI*:

fixes G (**structure**)
assumes *m-closed*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y \in carrier\ G$
and *one-closed*: $\mathbf{1} \in carrier\ G$
and *m-assoc*:
$!!x\ y\ z.\ [|\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] ==>$
$(x \otimes y) \otimes z = x \otimes (y \otimes z)$
and *l-one*: $!!x.\ x \in carrier\ G ==> \mathbf{1} \otimes x = x$
and *m-comm*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y = y \otimes x$
shows *comm-monoid G*
⟨*proof*⟩

lemma (**in** *monoid*) *monoid-comm-monoidI*:
assumes *m-comm*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y = y \otimes x$
shows *comm-monoid G*
⟨*proof*⟩

lemma (**in** *comm-monoid*) *nat-pow-distr*:
$[|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==>$
$(x \otimes y)\ (\hat{})\ (n::nat) = x\ (\hat{})\ n \otimes y\ (\hat{})\ n$
⟨*proof*⟩

locale *comm-group* = *comm-monoid* + *group*

lemma (**in** *group*) *group-comm-groupI*:
assumes *m-comm*: $!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==>$
$x \otimes y = y \otimes x$
shows *comm-group G*
⟨*proof*⟩

lemma *comm-groupI*:
fixes G (**structure**)
assumes *m-closed*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y \in carrier\ G$
and *one-closed*: $\mathbf{1} \in carrier\ G$
and *m-assoc*:
$!!x\ y\ z.\ [|\ x \in carrier\ G;\ y \in carrier\ G;\ z \in carrier\ G\ |] ==>$
$(x \otimes y) \otimes z = x \otimes (y \otimes z)$
and *m-comm*:
$!!x\ y.\ [|\ x \in carrier\ G;\ y \in carrier\ G\ |] ==> x \otimes y = y \otimes x$
and *l-one*: $!!x.\ x \in carrier\ G ==> \mathbf{1} \otimes x = x$
and *l-inv-ex*: $!!x.\ x \in carrier\ G ==> \exists y \in carrier\ G.\ y \otimes x = \mathbf{1}$
shows *comm-group G*
⟨*proof*⟩

lemma (**in** *comm-group*) *inv-mult*:
[| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==> *inv* $(x \otimes y) =$ *inv* $x \otimes$ *inv* y
⟨*proof*⟩

2.7 The Lattice of Subgroups of a Group

theorem (**in** *group*) *subgroups-partial-order*:
partial-order (| *carrier* = {H. *subgroup* H G}, *le* = *op* ⊆ |)
⟨*proof*⟩

lemma (**in** *group*) *subgroup-self*:
subgroup (*carrier* G) G
⟨*proof*⟩

lemma (**in** *group*) *subgroup-imp-group*:
subgroup H G ==> *group* (G(| *carrier* := H |))
⟨*proof*⟩

lemma (**in** *group*) *is-monoid* [*intro*, *simp*]:
monoid G
⟨*proof*⟩

lemma (**in** *group*) *subgroup-inv-equality*:
[| *subgroup* H G; $x \in H$ |] ==> *m-inv* (G (| *carrier* := H |)) $x =$ *inv* x
⟨*proof*⟩

theorem (**in** *group*) *subgroups-Inter*:
assumes *subgr*: (!!H. $H \in A$ ==> *subgroup* H G)
and *not-empty*: $A \neq \{\}$
shows *subgroup* ($\bigcap A$) G
⟨*proof*⟩

theorem (**in** *group*) *subgroups-complete-lattice*:
complete-lattice (| *carrier* = {H. *subgroup* H G}, *le* = *op* ⊆ |)
(**is** *complete-lattice* ?L)
⟨*proof*⟩

end

theory *FiniteProduct* **imports** *Group* **begin**

3 Product Operator for Commutative Monoids

3.1 Inductive Definition of a Relation for Products over Sets

Instantiation of locale LC of theory *Finite-Set* is not possible, because here we have explicit typing rules like $x \in$ *carrier* G. We introduce an explicit

argument for the domain D.

inductive-set
  foldSetD :: [$'a$ *set*, $'b$ => $'a$ => $'a$, $'a$] => ($'b$ *set* * $'a$) *set*
  for D :: $'a$ *set* **and** f :: $'b$ => $'a$ => $'a$ **and** e :: $'a$
  where
    emptyI [*intro*]: $e \in D$ ==> ({}, e) $\in$ *foldSetD D f e*
  | *insertI* [*intro*]: [| x ~: A; $f\ x\ y \in D$; (A, y) $\in$ *foldSetD D f e* |] ==>
                  (*insert x A*, $f\ x\ y$) $\in$ *foldSetD D f e*

inductive-cases *empty-foldSetDE* [*elim*!]: ({}, x) $\in$ *foldSetD D f e*

constdefs
  foldD :: [$'a$ *set*, $'b$ => $'a$ => $'a$, $'a$, $'b$ *set*] => $'a$
  foldD D f e A == *THE* x. (A, x) $\in$ *foldSetD D f e*

lemma *foldSetD-closed*:
  [| (A, z) $\in$ *foldSetD D f e* ; $e \in D$; !!$x\ y$. [| $x \in A$; $y \in D$ |] ==> $f\ x\ y \in D$
     |] ==> $z \in D$
  ⟨*proof*⟩

lemma *Diff1-foldSetD*:
  [| (A − {x}, y) $\in$ *foldSetD D f e*; $x \in A$; $f\ x\ y \in D$ |] ==>
   (A, $f\ x\ y$) $\in$ *foldSetD D f e*
  ⟨*proof*⟩

lemma *foldSetD-imp-finite* [*simp*]: (A, x) $\in$ *foldSetD D f e* ==> *finite A*
  ⟨*proof*⟩

lemma *finite-imp-foldSetD*:
  [| *finite A*; $e \in D$; !!$x\ y$. [| $x \in A$; $y \in D$ |] ==> $f\ x\ y \in D$ |] ==>
   EX x. (A, x) $\in$ *foldSetD D f e*
⟨*proof*⟩

3.2 Left-Commutative Operations

locale *LCD* =
  fixes B :: $'b$ *set*
  and D :: $'a$ *set*
  and f :: $'b$ => $'a$ => $'a$ (**infixl** · *70*)
  assumes *left-commute*:
    [| $x \in B$; $y \in B$; $z \in D$ |] ==> $x \cdot (y \cdot z) = y \cdot (x \cdot z)$
  and *f-closed* [*simp*, *intro*!]: !!$x\ y$. [| $x \in B$; $y \in D$ |] ==> $f\ x\ y \in D$

lemma (**in** *LCD*) *foldSetD-closed* [*dest*]:
  (A, z) $\in$ *foldSetD D f e* ==> $z \in D$
  ⟨*proof*⟩

lemma (**in** *LCD*) *Diff1-foldSetD*:
  [| (A − {x}, y) $\in$ *foldSetD D f e*; $x \in A$; $A \subseteq B$ |] ==>

$(A, f\ x\ y) \in foldSetD\ D\ f\ e$
⟨*proof*⟩

lemma (**in** *LCD*) *foldSetD-imp-finite* [*simp*]:
$(A, x) \in foldSetD\ D\ f\ e \Longrightarrow finite\ A$
⟨*proof*⟩

lemma (**in** *LCD*) *finite-imp-foldSetD*:
$[|\ finite\ A;\ A \subseteq B;\ e \in D\ |] \Longrightarrow EX\ x.\ (A, x) \in foldSetD\ D\ f\ e$
⟨*proof*⟩

lemma (**in** *LCD*) *foldSetD-determ-aux*:
$e \in D \Longrightarrow \forall A\ x.\ A \subseteq B\ \&\ card\ A < n \longrightarrow (A, x) \in foldSetD\ D\ f\ e \longrightarrow$
$(\forall y.\ (A, y) \in foldSetD\ D\ f\ e \longrightarrow y = x)$
⟨*proof*⟩

lemma (**in** *LCD*) *foldSetD-determ*:
$[|\ (A, x) \in foldSetD\ D\ f\ e;\ (A, y) \in foldSetD\ D\ f\ e;\ e \in D;\ A \subseteq B\ |]$
$\Longrightarrow y = x$
⟨*proof*⟩

lemma (**in** *LCD*) *foldD-equality*:
$[|\ (A, y) \in foldSetD\ D\ f\ e;\ e \in D;\ A \subseteq B\ |] \Longrightarrow foldD\ D\ f\ e\ A = y$
⟨*proof*⟩

lemma *foldD-empty* [*simp*]:
$e \in D \Longrightarrow foldD\ D\ f\ e\ \{\} = e$
⟨*proof*⟩

lemma (**in** *LCD*) *foldD-insert-aux*:
$[|\ x \sim: A;\ x \in B;\ e \in D;\ A \subseteq B\ |] \Longrightarrow$
$((insert\ x\ A, v) \in foldSetD\ D\ f\ e) =$
$(EX\ y.\ (A, y) \in foldSetD\ D\ f\ e\ \&\ v = f\ x\ y)$
⟨*proof*⟩

lemma (**in** *LCD*) *foldD-insert*:
$[|\ finite\ A;\ x \sim: A;\ x \in B;\ e \in D;\ A \subseteq B\ |] \Longrightarrow$
$foldD\ D\ f\ e\ (insert\ x\ A) = f\ x\ (foldD\ D\ f\ e\ A)$
⟨*proof*⟩

lemma (**in** *LCD*) *foldD-closed* [*simp*]:
$[|\ finite\ A;\ e \in D;\ A \subseteq B\ |] \Longrightarrow foldD\ D\ f\ e\ A \in D$
⟨*proof*⟩

lemma (**in** *LCD*) *foldD-commute*:
$[|\ finite\ A;\ x \in B;\ e \in D;\ A \subseteq B\ |] \Longrightarrow$
$f\ x\ (foldD\ D\ f\ e\ A) = foldD\ D\ f\ (f\ x\ e)\ A$
⟨*proof*⟩

lemma *Int-mono2*:
[| $A \subseteq C$; $B \subseteq C$ |] ==> A *Int* $B \subseteq C$
⟨*proof*⟩

lemma (**in** *LCD*) *foldD-nest-Un-Int*:
[| *finite* A; *finite* C; $e \in D$; $A \subseteq B$; $C \subseteq B$ |] ==>
foldD D f (*foldD* D f e C) A = *foldD* D f (*foldD* D f e (A *Int* C)) (A *Un* C)
⟨*proof*⟩

lemma (**in** *LCD*) *foldD-nest-Un-disjoint*:
[| *finite* A; *finite* B; A *Int* B = {}; $e \in D$; $A \subseteq B$; $C \subseteq B$ |]
==> *foldD* D f e (A *Un* B) = *foldD* D f (*foldD* D f e B) A
⟨*proof*⟩

declare *foldSetD-imp-finite* [*simp del*]
empty-foldSetDE [*rule del*]
foldSetD.intros [*rule del*]
declare (**in** *LCD*)
foldSetD-closed [*rule del*]

3.3 Commutative Monoids

We enter a more restrictive context, with $f :: 'a => 'a => 'a$ instead of $'b => 'a => 'a$.

locale *ACeD* =
fixes $D :: 'a$ *set*
and $f :: 'a => 'a => 'a$ (**infixl** · 70)
and $e :: 'a$
assumes *ident* [*simp*]: $x \in D$ ==> $x \cdot e = x$
and *commute*: [| $x \in D$; $y \in D$ |] ==> $x \cdot y = y \cdot x$
and *assoc*: [| $x \in D$; $y \in D$; $z \in D$ |] ==> $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
and *e-closed* [*simp*]: $e \in D$
and *f-closed* [*simp*]: [| $x \in D$; $y \in D$ |] ==> $x \cdot y \in D$

lemma (**in** *ACeD*) *left-commute*:
[| $x \in D$; $y \in D$; $z \in D$ |] ==> $x \cdot (y \cdot z) = y \cdot (x \cdot z)$
⟨*proof*⟩

lemmas (**in** *ACeD*) *AC* = *assoc commute left-commute*

lemma (**in** *ACeD*) *left-ident* [*simp*]: $x \in D$ ==> $e \cdot x = x$
⟨*proof*⟩

lemma (**in** *ACeD*) *foldD-Un-Int*:
[| *finite* A; *finite* B; $A \subseteq D$; $B \subseteq D$ |] ==>
foldD D f e A · *foldD* D f e B =
foldD D f e (A *Un* B) · *foldD* D f e (A *Int* B)
⟨*proof*⟩

lemma (**in** *ACeD*) *foldD-Un-disjoint*:
[| *finite A*; *finite B*; *A Int B* = {}; *A* ⊆ *D*; *B* ⊆ *D* |] ==>
foldD D f e (*A Un B*) = *foldD D f e A* · *foldD D f e B*
⟨*proof*⟩

3.4 Products over Finite Sets

constdefs (**structure** *G*)
finprod :: [('*b*, '*m*) *monoid-scheme*, '*a* => '*b*, '*a set*] => '*b*
finprod G f A == *if finite A*
then foldD (*carrier G*) (*mult G o f*) **1** *A*
else arbitrary

syntax
-finprod :: *index* => *idt* => '*a set* => '*b* => '*b*
((*3*⨂ --:-. -) [*1000*, *0*, *51*, *10*] *10*)
syntax (*xsymbols*)
-finprod :: *index* => *idt* => '*a set* => '*b* => '*b*
((*3*⨂ --∈-. -) [*1000*, *0*, *51*, *10*] *10*)
syntax (*HTML* **output**)
-finprod :: *index* => *idt* => '*a set* => '*b* => '*b*
((*3*⨂ --∈-. -) [*1000*, *0*, *51*, *10*] *10*)
translations
⨂ι*i*:*A*. *b* == *finprod* ⋄ι (%*i*. *b*) *A*
— Beware of argument permutation!

lemma (**in** *comm-monoid*) *finprod-empty* [*simp*]:
finprod G f {} = **1**
⟨*proof*⟩

declare *funcsetI* [*intro*]
funcset-mem [*dest*]

lemma (**in** *comm-monoid*) *finprod-insert* [*simp*]:
[| *finite F*; *a* ∉ *F*; *f* ∈ *F* -> *carrier G*; *f a* ∈ *carrier G* |] ==>
finprod G f (*insert a F*) = *f a* ⊗ *finprod G f F*
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-one* [*simp*]:
finite A ==> (⨂ *i*:*A*. **1**) = **1**
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-closed* [*simp*]:
fixes *A*
assumes *fin*: *finite A* **and** *f*: *f* ∈ *A* -> *carrier G*
shows *finprod G f A* ∈ *carrier G*
⟨*proof*⟩

lemma *funcset-Int-left* [*simp*, *intro*]:

[| $f \in A \to C$; $f \in B \to C$ |] ==> $f \in A$ *Int* $B \to C$
⟨*proof*⟩

lemma *funcset-Un-left* [*iff*]:
$(f \in A$ *Un* $B \to C) = (f \in A \to C \,\&\, f \in B \to C)$
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-Un-Int*:
[| *finite* A; *finite* B; $g \in A \to$ *carrier* G; $g \in B \to$ *carrier* G |] ==>
finprod G g $(A$ *Un* $B) \otimes$ *finprod* G g $(A$ *Int* $B) =$
finprod G g $A \otimes$ *finprod* G g B
— The reversed orientation looks more natural, but LOOPS as a simprule!
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-Un-disjoint*:
[| *finite* A; *finite* B; A *Int* $B = \{\}$;
$g \in A \to$ *carrier* G; $g \in B \to$ *carrier* G |]
==> *finprod* G g $(A$ *Un* $B) =$ *finprod* G g $A \otimes$ *finprod* G g B
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-multf*:
[| *finite* A; $f \in A \to$ *carrier* G; $g \in A \to$ *carrier* G |] ==>
finprod G $(\%x.\ f\ x \otimes g\ x)$ $A =$ (*finprod* G f $A \otimes$ *finprod* G g A)
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-cong'*:
[| $A = B$; $g \in B \to$ *carrier* G;
!!i. $i \in B$ ==> $f\ i = g\ i$ |] ==> *finprod* G f $A =$ *finprod* G g B
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-cong*:
[| $A = B$; $f \in B \to$ *carrier* $G =$ *True*;
!!i. $i \in B$ ==> $f\ i = g\ i$ |] ==> *finprod* G f $A =$ *finprod* G g B

⟨*proof*⟩

Usually, if this rule causes a failed congruence proof error, the reason is that the premise $g \in B \to$ *carrier* G cannot be shown. Adding *Pi-def* to the simpset is often useful. For this reason, *comm-monoid.finprod-cong* is not added to the simpset by default.

declare *funcsetI* [*rule del*]
funcset-mem [*rule del*]

lemma (**in** *comm-monoid*) *finprod-0* [*simp*]:
$f \in \{0$::*nat*$\} \to$ *carrier* G ==> *finprod* G f $\{..0\} = f\ 0$
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-Suc* [*simp*]:
$f \in \{..$*Suc* $n\} \to$ *carrier* G ==>

finprod G f {..Suc n} = (f (Suc n) ⊗ finprod G f {..n})
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-Suc2*:
f ∈ {..Suc n} −> carrier G ==>
finprod G f {..Suc n} = (finprod G (%i. f (Suc i)) {..n} ⊗ f 0)
⟨*proof*⟩

lemma (**in** *comm-monoid*) *finprod-mult* [*simp*]:
[| f ∈ {..n} −> carrier G; g ∈ {..n} −> carrier G |] ==>
finprod G (%i. f i ⊗ g i) {..n::nat} =
finprod G f {..n} ⊗ finprod G g {..n}
⟨*proof*⟩

end

theory *Exponent* **imports** *Main Primes Binomial* **begin**

4 The Combinatorial Argument Underlying the First Sylow Theorem

definition *exponent* :: *nat => nat => nat* **where**
exponent p s == if prime p then (GREATEST r. p^r dvd s) else 0

4.1 Prime Theorems

lemma *prime-imp-one-less*: *prime p ==> Suc 0 < p*
⟨*proof*⟩

lemma *prime-iff*:
*(prime p) = (Suc 0 < p & (∀ a b. p dvd a*b −−> (p dvd a) | (p dvd b)))*
⟨*proof*⟩

lemma *zero-less-prime-power*: *prime p ==> 0 < p^a*
⟨*proof*⟩

lemma *zero-less-card-empty*: *[| finite S; S ≠ {} |] ==> 0 < card(S)*
⟨*proof*⟩

lemma *prime-dvd-cases*:
*[| p*k dvd m*n; prime p |]*
*==> (∃ x. k dvd x*n & m = p*x) | (∃ y. k dvd m*y & n = p*y)*
⟨*proof*⟩

lemma *prime-power-dvd-cases* [*rule-format* (*no-asm*)]: *prime p*
*==> ∀ m n. p^c dvd m*n −−>*
(∀ a b. a+b = Suc c −−> p^a dvd m | p^b dvd n)
⟨*proof*⟩

lemma *div-combine*:
*[| prime p; ~ (p ^ (Suc r) dvd n); p^(a+r) dvd n*k |]*
==> p ^ a dvd k
⟨*proof*⟩

lemma *Suc-le-power*: *Suc 0 < p ==> Suc n <= p^n*
⟨*proof*⟩

lemma *power-dvd-bound*: *[|p^n dvd a; Suc 0 < p; a > 0|] ==> n < a*
⟨*proof*⟩

4.2 Exponent Theorems

lemma *exponent-ge* [*rule-format*]:
[|p^k dvd n; prime p; 0<n|] ==> k <= exponent p n
⟨*proof*⟩

lemma *power-exponent-dvd*: *s>0 ==> (p ^ exponent p s) dvd s*
⟨*proof*⟩

lemma *power-Suc-exponent-Not-dvd*:
*[|(p * p ^ exponent p s) dvd s; prime p |] ==> s=0*
⟨*proof*⟩

lemma *exponent-power-eq* [*simp*]: *prime p ==> exponent p (p^a) = a*
⟨*proof*⟩

lemma *exponent-equalityI*:
!r::nat. (p^r dvd a) = (p^r dvd b) ==> exponent p a = exponent p b
⟨*proof*⟩

lemma *exponent-eq-0* [*simp*]: *¬ prime p ==> exponent p s = 0*
⟨*proof*⟩

lemma *exponent-mult-add1*: *[| a > 0; b > 0 |]*
*==> (exponent p a) + (exponent p b) <= exponent p (a * b)*
⟨*proof*⟩

lemma *exponent-mult-add2*: [| $a > 0$; $b > 0$ |]
  ==> *exponent* p $(a * b)$ <= $($*exponent* p $a)$ + $($*exponent* p $b)$
⟨*proof*⟩

lemma *exponent-mult-add*: [| $a > 0$; $b > 0$ |]
  ==> *exponent* p $(a * b)$ = $($*exponent* p $a)$ + $($*exponent* p $b)$
⟨*proof*⟩

lemma *not-divides-exponent-0*: ~ $(p$ *dvd* $n)$ ==> *exponent* p $n = 0$
⟨*proof*⟩

lemma *exponent-1-eq-0* [*simp*]: *exponent* p $($*Suc* $0) = 0$
⟨*proof*⟩

4.3 Main Combinatorial Argument

lemma *le-extend-mult*: [| $c > 0$; a <= b |] ==> a <= $b * (c$::*nat*$)$
⟨*proof*⟩

lemma *p-fac-forw-lemma*:
  [| $(m$::*nat*$) > 0$; $k > 0$; $k < p$ ^ a; $(p$ ^ $r)$ *dvd* $(p$ ^ $a)* m - k$ |] ==> r <= a
⟨*proof*⟩

lemma *p-fac-forw*: [| $(m$::*nat*$) > 0$; $k>0$; $k < p$ ^ a; $(p$ ^ $r)$ *dvd* $(p$ ^ $a)* m - k$ |]
  ==> $(p$ ^ $r)$ *dvd* $(p$ ^ $a) - k$
⟨*proof*⟩

lemma *r-le-a-forw*:
  [| $(k$::*nat*$) > 0$; $k < p$ ^ a; $p>0$; $(p$ ^ $r)$ *dvd* $(p$ ^ $a) - k$ |] ==> r <= a
⟨*proof*⟩

lemma *p-fac-backw*: [| $m>0$; $k>0$; $(p$::*nat*$)\neq 0$; $k < p$ ^ a; $(p$ ^ $r)$ *dvd* p ^ $a - k$ |]
  ==> $(p$ ^ $r)$ *dvd* $(p$ ^ $a)* m - k$
⟨*proof*⟩

lemma *exponent-p-a-m-k-equation*: [| $m>0$; $k>0$; $(p$::*nat*$)\neq 0$; $k < p$ ^ a |]
  ==> *exponent* p $(p$ ^ $a * m - k) =$ *exponent* p $(p$ ^ $a - k)$
⟨*proof*⟩

Suc rules that we have to delete from the simpset

lemmas *bad-Sucs* = *binomial-Suc-Suc mult-Suc mult-Suc-right*

lemma *p-not-div-choose-lemma* [*rule-format*]:
  [| $\forall i$. *Suc* $i < K$ −−> *exponent* p $($*Suc* $i)$ = *exponent* p $(Suc(j+i))$|]

$==>$ $k<K$ $-->$ *exponent p* $((j+k)$ *choose* $k) = 0$
⟨*proof*⟩

lemma *p-not-div-choose*:
$[|\ k<K;\ k<=n;$
$\forall j.\ 0<j\ \&\ j<K$ $-->$ *exponent p* $(n - k + (K - j)) =$ *exponent p* $(K - j)|]$
$==>$ *exponent p* $(n$ *choose* $k) = 0$
⟨*proof*⟩

lemma *const-p-fac-right*:
$m>0$ $==>$ *exponent p* $((p\hat{}a * m -$ *Suc* $0)$ *choose* $(p\hat{}a -$ *Suc* $0)) = 0$
⟨*proof*⟩

lemma *const-p-fac*:
$m>0$ $==>$ *exponent p* $(((p\hat{}a) * m)$ *choose* $p\hat{}a) =$ *exponent p* m
⟨*proof*⟩

end

theory *Coset* **imports** *Group Exponent* **begin**

5 Cosets and Quotient Groups

constdefs (**structure** G)
r-coset :: [-, *'a set*, *'a*] ⇒ *'a set* (**infixl** $\#>_1$ *60*)
$H \#> a \equiv \bigcup h{\in}H.\ \{h \otimes a\}$

l-coset :: [-, *'a*, *'a set*] ⇒ *'a set* (**infixl** $<\#_1$ *60*)
$a <\# H \equiv \bigcup h{\in}H.\ \{a \otimes h\}$

RCOSETS :: [-, *'a set*] ⇒ (*'a set*)*set* (*rcosets*$_1$ - [*81*] *80*)
rcosets $H \equiv \bigcup a{\in}$*carrier* $G.\ \{H \#> a\}$

set-mult :: [-, *'a set* ,*'a set*] ⇒ *'a set* (**infixl** $<\#>_1$ *60*)
$H <\#> K \equiv \bigcup h{\in}H.\ \bigcup k{\in}K.\ \{h \otimes k\}$

SET-INV :: [-,*'a set*] ⇒ *'a set* (*set'-inv*$_1$ - [*81*] *80*)
set-inv $H \equiv \bigcup h{\in}H.\ \{$*inv* $h\}$

locale *normal* = *subgroup* + *group* +
assumes *coset-eq*: $(\forall x \in$ *carrier* $G.\ H \#> x = x <\# H)$

abbreviation
normal-rel :: [*'a set*, (*'a*, *'b*) *monoid-scheme*] ⇒ *bool* (**infixl** ◁ *60*) **where**
$H \lhd G \equiv$ *normal H G*

5.1 Basic Properties of Cosets

lemma (**in** *group*) *coset-mult-assoc*:
[| $M \subseteq$ *carrier G*; $g \in$ *carrier G*; $h \in$ *carrier G* |]
==> $(M \#> g) \#> h = M \#> (g \otimes h)$
⟨*proof*⟩

lemma (**in** *group*) *coset-mult-one* [*simp*]: $M \subseteq$ *carrier G* ==> $M \#> \mathbf{1} = M$
⟨*proof*⟩

lemma (**in** *group*) *coset-mult-inv1*:
[| $M \#> (x \otimes (inv\ y)) = M$; $x \in$ *carrier G* ; $y \in$ *carrier G*;
$M \subseteq$ *carrier G* |] ==> $M \#> x = M \#> y$
⟨*proof*⟩

lemma (**in** *group*) *coset-mult-inv2*:
[| $M \#> x = M \#> y$; $x \in$ *carrier G*; $y \in$ *carrier G*; $M \subseteq$ *carrier G* |]
==> $M \#> (x \otimes (inv\ y)) = M$
⟨*proof*⟩

lemma (**in** *group*) *coset-join1*:
[| $H \#> x = H$; $x \in$ *carrier G*; *subgroup H G* |] ==> $x \in H$
⟨*proof*⟩

lemma (**in** *group*) *solve-equation*:
⟦*subgroup H G*; $x \in H$; $y \in H$⟧ ⟹ $\exists h \in H.\ y = h \otimes x$
⟨*proof*⟩

lemma (**in** *group*) *repr-independence*:
⟦$y \in H \#> x$; $x \in$ *carrier G*; *subgroup H G*⟧ ⟹ $H \#> x = H \#> y$
⟨*proof*⟩

lemma (**in** *group*) *coset-join2*:
⟦$x \in$ *carrier G*; *subgroup H G*; $x \in H$⟧ ⟹ $H \#> x = H$
— Alternative proof is to put $x = \mathbf{1}$ in *repr-independence*.
⟨*proof*⟩

lemma (**in** *monoid*) *r-coset-subset-G*:
[| $H \subseteq$ *carrier G*; $x \in$ *carrier G* |] ==> $H \#> x \subseteq$ *carrier G*
⟨*proof*⟩

lemma (**in** *group*) *rcosI*:
[| $h \in H$; $H \subseteq$ *carrier G*; $x \in$ *carrier G*|] ==> $h \otimes x \in H \#> x$
⟨*proof*⟩

lemma (**in** *group*) *rcosetsI*:
$[\![H \subseteq carrier\ G;\ x \in carrier\ G]\!] \Longrightarrow H\ \#>\ x \in rcosets\ H$
⟨*proof*⟩

Really needed?

lemma (**in** *group*) *transpose-inv*:
$[|\ x \otimes y = z;\ \ x \in carrier\ G;\ \ y \in carrier\ G;\ \ z \in carrier\ G\ |]$
$==>\ (inv\ x) \otimes z = y$
⟨*proof*⟩

lemma (**in** *group*) *rcos-self*: $[|\ x \in carrier\ G;\ subgroup\ H\ G\ |] ==> x \in H\ \#>\ x$
⟨*proof*⟩

Opposite of *repr-independence*

lemma (**in** *group*) *repr-independenceD*:
includes *subgroup H G*
assumes *ycarr*: $y \in carrier\ G$
and *repr*: $H\ \#>\ x = H\ \#>\ y$
shows $y \in H\ \#>\ x$
⟨*proof*⟩

Elements of a right coset are in the carrier

lemma (**in** *subgroup*) *elemrcos-carrier*:
includes *group*
assumes *acarr*: $a \in carrier\ G$
and *a'*: $a' \in H\ \#>\ a$
shows $a' \in carrier\ G$
⟨*proof*⟩

lemma (**in** *subgroup*) *rcos-const*:
includes *group*
assumes *hH*: $h \in H$
shows $H\ \#>\ h = H$
⟨*proof*⟩

Step one for lemma *rcos-module*

lemma (**in** *subgroup*) *rcos-module-imp*:
includes *group*
assumes *xcarr*: $x \in carrier\ G$
and *x'cos*: $x' \in H\ \#>\ x$
shows $(x' \otimes inv\ x) \in H$
⟨*proof*⟩

Step two for lemma *rcos-module*

lemma (**in** *subgroup*) *rcos-module-rev*:
includes *group*
assumes *carr*: $x \in carrier\ G\ x' \in carrier\ G$
and *xixH*: $(x' \otimes inv\ x) \in H$

shows $x' \in H \mathbin{\#>} x$
⟨*proof*⟩

Module property of right cosets

lemma (**in** *subgroup*) *rcos-module*:
includes *group*
assumes *carr*: $x \in carrier\ G\ x' \in carrier\ G$
shows $(x' \in H \mathbin{\#>} x) = (x' \otimes inv\ x \in H)$
⟨*proof*⟩

Right cosets are subsets of the carrier.

lemma (**in** *subgroup*) *rcosets-carrier*:
includes *group*
assumes *XH*: $X \in rcosets\ H$
shows $X \subseteq carrier\ G$
⟨*proof*⟩

Multiplication of general subsets

lemma (**in** *monoid*) *set-mult-closed*:
assumes *Acarr*: $A \subseteq carrier\ G$
and *Bcarr*: $B \subseteq carrier\ G$
shows $A \mathbin{<\#>} B \subseteq carrier\ G$
⟨*proof*⟩

lemma (**in** *comm-group*) *mult-subgroups*:
assumes *subH*: *subgroup H G*
and *subK*: *subgroup K G*
shows *subgroup* $(H \mathbin{<\#>} K)\ G$
⟨*proof*⟩

lemma (**in** *subgroup*) *lcos-module-rev*:
includes *group*
assumes *carr*: $x \in carrier\ G\ x' \in carrier\ G$
and *xixH*: $(inv\ x \otimes x') \in H$
shows $x' \in x \mathbin{<\#} H$
⟨*proof*⟩

5.2 Normal subgroups

lemma *normal-imp-subgroup*: $H \lhd G \Longrightarrow subgroup\ H\ G$
⟨*proof*⟩

lemma (**in** *group*) *normalI*:
subgroup H G $\Longrightarrow (\forall x \in carrier\ G.\ H \mathbin{\#>} x = x \mathbin{<\#} H) \Longrightarrow H \lhd G$
⟨*proof*⟩

lemma (**in** *normal*) *inv-op-closed1*:
$[\![x \in carrier\ G; h \in H]\!] \Longrightarrow (inv\ x) \otimes h \otimes x \in H$
⟨*proof*⟩

lemma (**in** *normal*) *inv-op-closed2*:
$[\![x \in carrier\ G;\ h \in H]\!] \Longrightarrow x \otimes h \otimes (inv\ x) \in H$
⟨*proof*⟩

Alternative characterization of normal subgroups

lemma (**in** *group*) *normal-inv-iff*:
$(N \lhd G) =$
$(subgroup\ N\ G\ \&\ (\forall x \in carrier\ G.\ \forall h \in N.\ x \otimes h \otimes (inv\ x) \in N))$
(**is** - = *?rhs*)
⟨*proof*⟩

5.3 More Properties of Cosets

lemma (**in** *group*) *lcos-m-assoc*:
$[|\ M \subseteq carrier\ G;\ g \in carrier\ G;\ h \in carrier\ G\ |]$
$==>\ g <\#\ (h <\#\ M) = (g \otimes h) <\#\ M$
⟨*proof*⟩

lemma (**in** *group*) *lcos-mult-one*: $M \subseteq carrier\ G ==> \mathbf{1} <\#\ M = M$
⟨*proof*⟩

lemma (**in** *group*) *l-coset-subset-G*:
$[|\ H \subseteq carrier\ G;\ x \in carrier\ G\ |] ==> x <\#\ H \subseteq carrier\ G$
⟨*proof*⟩

lemma (**in** *group*) *l-coset-swap*:
$[\![y \in x <\#\ H;\ \ x \in carrier\ G;\ \ subgroup\ H\ G]\!] \Longrightarrow x \in y <\#\ H$
⟨*proof*⟩

lemma (**in** *group*) *l-coset-carrier*:
$[|\ y \in x <\#\ H;\ \ x \in carrier\ G;\ \ subgroup\ H\ G\ |] ==> y \in carrier\ G$
⟨*proof*⟩

lemma (**in** *group*) *l-repr-imp-subset*:
assumes *y*: $y \in x <\#\ H$ **and** *x*: $x \in carrier\ G$ **and** *sb*: *subgroup H G*
shows $y <\#\ H \subseteq x <\#\ H$
⟨*proof*⟩

lemma (**in** *group*) *l-repr-independence*:
assumes *y*: $y \in x <\#\ H$ **and** *x*: $x \in carrier\ G$ **and** *sb*: *subgroup H G*
shows $x <\#\ H = y <\#\ H$
⟨*proof*⟩

lemma (**in** *group*) *setmult-subset-G*:
$[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G]\!] \Longrightarrow H <\#> K \subseteq carrier\ G$
⟨*proof*⟩

lemma (**in** *group*) *subgroup-mult-id*: $subgroup\ H\ G \Longrightarrow H <\#> H = H$

⟨*proof*⟩

5.3.1 Set of Inverses of an *r-coset*.

lemma (**in** *normal*) *rcos-inv*:
assumes *x*: $x \in carrier\ G$
shows *set-inv* $(H\ \#>\ x) = H\ \#>\ (inv\ x)$
⟨*proof*⟩

5.3.2 Theorems for <#> with #> or <#.

lemma (**in** *group*) *setmult-rcos-assoc*:
$[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G;\ x \in carrier\ G]\!]$
$\Longrightarrow H\ <\#>\ (K\ \#>\ x) = (H\ <\#>\ K)\ \#>\ x$
⟨*proof*⟩

lemma (**in** *group*) *rcos-assoc-lcos*:
$[\![H \subseteq carrier\ G;\ K \subseteq carrier\ G;\ x \in carrier\ G]\!]$
$\Longrightarrow (H\ \#>\ x)\ <\#>\ K = H\ <\#>\ (x\ <\#\ K)$
⟨*proof*⟩

lemma (**in** *normal*) *rcos-mult-step1*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H\ \#>\ x)\ <\#>\ (H\ \#>\ y) = (H\ <\#>\ (x\ <\#\ H))\ \#>\ y$
⟨*proof*⟩

lemma (**in** *normal*) *rcos-mult-step2*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H\ <\#>\ (x\ <\#\ H))\ \#>\ y = (H\ <\#>\ (H\ \#>\ x))\ \#>\ y$
⟨*proof*⟩

lemma (**in** *normal*) *rcos-mult-step3*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H\ <\#>\ (H\ \#>\ x))\ \#>\ y = H\ \#>\ (x \otimes y)$
⟨*proof*⟩

lemma (**in** *normal*) *rcos-sum*:
$[\![x \in carrier\ G;\ y \in carrier\ G]\!]$
$\Longrightarrow (H\ \#>\ x)\ <\#>\ (H\ \#>\ y) = H\ \#>\ (x \otimes y)$
⟨*proof*⟩

lemma (**in** *normal*) *rcosets-mult-eq*: $M \in rcosets\ H \Longrightarrow H\ <\#>\ M = M$
— generalizes *subgroup-mult-id*
⟨*proof*⟩

5.3.3 An Equivalence Relation

constdefs (**structure** *G*)
r-congruent :: $[('a, 'b)monoid\text{-}scheme,\ 'a\ set] \Rightarrow ('a * 'a)set$
$(rcong_1$ -)

rcong H ≡ {(*x*,*y*). *x* ∈ *carrier G* & *y* ∈ *carrier G* & *inv x* ⊗ *y* ∈ *H*}

lemma (**in** *subgroup*) *equiv-rcong*:
includes *group G*
shows *equiv* (*carrier G*) (*rcong H*)
⟨*proof*⟩

Equivalence classes of *rcong* correspond to left cosets. Was there a mistake in the definitions? I'd have expected them to correspond to right cosets.

lemma (**in** *subgroup*) *l-coset-eq-rcong*:
includes *group G*
assumes *a*: *a* ∈ *carrier G*
shows *a* <# *H* = *rcong H* ‘‘ {*a*}
⟨*proof*⟩

5.3.4 Two Distinct Right Cosets are Disjoint

lemma (**in** *group*) *rcos-equation*:
includes *subgroup H G*
shows
⟦*ha* ⊗ *a* = *h* ⊗ *b*; *a* ∈ *carrier G*; *b* ∈ *carrier G*;
h ∈ *H*; *ha* ∈ *H*; *hb* ∈ *H*⟧
⟹ *hb* ⊗ *a* ∈ (⋃*h*∈*H*. {*h* ⊗ *b*})
⟨*proof*⟩

lemma (**in** *group*) *rcos-disjoint*:
includes *subgroup H G*
shows ⟦*a* ∈ *rcosets H*; *b* ∈ *rcosets H*; *a*≠*b*⟧ ⟹ *a* ∩ *b* = {}
⟨*proof*⟩

5.4 Further lemmas for *r-congruent*

The relation is a congruence

lemma (**in** *normal*) *congruent-rcong*:
shows *congruent2* (*rcong H*) (*rcong H*) (λ*a b*. *a* ⊗ *b* <# *H*)
⟨*proof*⟩

5.5 Order of a Group and Lagrange's Theorem

constdefs
order :: ('*a*, '*b*) *monoid-scheme* ⇒ *nat*
order S ≡ *card* (*carrier S*)

lemma (**in** *group*) *rcosets-part-G*:
includes *subgroup*
shows ⋃(*rcosets H*) = *carrier G*
⟨*proof*⟩

lemma (**in** *group*) *cosets-finite*:
$[\![c \in rcosets\ H;\ H \subseteq carrier\ G;\ finite\ (carrier\ G)]\!] \Longrightarrow finite\ c$
⟨*proof*⟩

The next two lemmas support the proof of *card-cosets-equal*.

lemma (**in** *group*) *inj-on-f*:
$[\![H \subseteq carrier\ G;\ a \in carrier\ G]\!] \Longrightarrow inj\text{-}on\ (\lambda y.\ y \otimes inv\ a)\ (H\ \#>\ a)$
⟨*proof*⟩

lemma (**in** *group*) *inj-on-g*:
$[\![H \subseteq carrier\ G;\ a \in carrier\ G]\!] \Longrightarrow inj\text{-}on\ (\lambda y.\ y \otimes a)\ H$
⟨*proof*⟩

lemma (**in** *group*) *card-cosets-equal*:
$[\![c \in rcosets\ H;\ H \subseteq carrier\ G; finite(carrier\ G)]\!]$
$\Longrightarrow card\ c = card\ H$
⟨*proof*⟩

lemma (**in** *group*) *rcosets-subset-PowG*:
$subgroup\ H\ G \Longrightarrow rcosets\ H \subseteq Pow(carrier\ G)$
⟨*proof*⟩

theorem (**in** *group*) *lagrange*:
$[\![finite(carrier\ G); subgroup\ H\ G]\!]$
$\Longrightarrow card(rcosets\ H) * card(H) = order(G)$
⟨*proof*⟩

5.6 Quotient Groups: Factorization of a Group

constdefs
FactGroup :: [('*a*,'*b*) *monoid-scheme*, '*a set*] ⇒ ('*a set*) *monoid*
(**infixl** *Mod 65*)
— Actually defined for groups rather than monoids
$FactGroup\ G\ H \equiv$
$(\!|carrier = rcosets_G\ H,\ mult = set\text{-}mult\ G,\ one = H|\!)$

lemma (**in** *normal*) *setmult-closed*:
$[\![K1 \in rcosets\ H;\ K2 \in rcosets\ H]\!] \Longrightarrow K1 <\#> K2 \in rcosets\ H$
⟨*proof*⟩

lemma (**in** *normal*) *setinv-closed*:
$K \in rcosets\ H \Longrightarrow set\text{-}inv\ K \in rcosets\ H$
⟨*proof*⟩

lemma (**in** *normal*) *rcosets-assoc*:
$[\![M1 \in rcosets\ H;\ M2 \in rcosets\ H;\ M3 \in rcosets\ H]\!]$
$\Longrightarrow M1 <\#> M2 <\#> M3 = M1 <\#> (M2 <\#> M3)$
⟨*proof*⟩

lemma (**in** *subgroup*) *subgroup-in-rcosets*:
includes *group G*
shows $H \in$ *rcosets H*
⟨*proof*⟩

lemma (**in** *normal*) *rcosets-inv-mult-group-eq*:
$M \in$ *rcosets H* $\Longrightarrow$ *set-inv M <#> M* $= H$
⟨*proof*⟩

theorem (**in** *normal*) *factorgroup-is-group*:
group (*G Mod H*)
⟨*proof*⟩

lemma *mult-FactGroup* [*simp*]: $X \otimes_{(G\ Mod\ H)} X' = X <\#>_G X'$
⟨*proof*⟩

lemma (**in** *normal*) *inv-FactGroup*:
$X \in$ *carrier* (*G Mod H*) $\Longrightarrow$ $inv_{G\ Mod\ H}\ X =$ *set-inv X*
⟨*proof*⟩

The coset map is a homomorphism from G to the quotient group $G\ Mod\ H$

lemma (**in** *normal*) *r-coset-hom-Mod*:
($\lambda a.\ H\ \#>\ a$) $\in$ *hom G* (*G Mod H*)
⟨*proof*⟩

5.7 The First Isomorphism Theorem

The quotient by the kernel of a homomorphism is isomorphic to the range of that homomorphism.

constdefs
kernel :: ('*a*, '*m*) *monoid-scheme* ⇒ ('*b*, '*n*) *monoid-scheme* ⇒
('*a* ⇒ '*b*) ⇒ '*a set*
— the kernel of a homomorphism
kernel G H h ≡ {*x*. *x* ∈ *carrier G* & *h x* = $\mathbf{1}_H$}

lemma (**in** *group-hom*) *subgroup-kernel*: *subgroup* (*kernel G H h*) *G*
⟨*proof*⟩

The kernel of a homomorphism is a normal subgroup

lemma (**in** *group-hom*) *normal-kernel*: (*kernel G H h*) $\lhd$ *G*
⟨*proof*⟩

lemma (**in** *group-hom*) *FactGroup-nonempty*:
assumes *X*: *X* ∈ *carrier* (*G Mod kernel G H h*)
shows *X* ≠ {}
⟨*proof*⟩

lemma (**in** *group-hom*) *FactGroup-contents-mem*:
assumes *X*: *X* ∈ *carrier* (*G Mod* (*kernel G H h*))
shows *contents* (*h*‘*X*) ∈ *carrier H*
⟨*proof*⟩

lemma (**in** *group-hom*) *FactGroup-hom*:
(λ*X*. *contents* (*h*‘*X*)) ∈ *hom* (*G Mod* (*kernel G H h*)) *H*
⟨*proof*⟩

Lemma for the following injectivity result

lemma (**in** *group-hom*) *FactGroup-subset*:
⟦*g* ∈ *carrier G*; *g*′ ∈ *carrier G*; *h g* = *h g*′⟧
⟹ *kernel G H h* #> *g* ⊆ *kernel G H h* #> *g*′
⟨*proof*⟩

lemma (**in** *group-hom*) *FactGroup-inj-on*:
inj-on (λ*X*. *contents* (*h* ‘ *X*)) (*carrier* (*G Mod kernel G H h*))
⟨*proof*⟩

If the homomorphism *h* is onto *H*, then so is the homomorphism from the quotient group

lemma (**in** *group-hom*) *FactGroup-onto*:
assumes *h*: *h* ‘ *carrier G* = *carrier H*
shows (λ*X*. *contents* (*h* ‘ *X*)) ‘ *carrier* (*G Mod kernel G H h*) = *carrier H*
⟨*proof*⟩

If *h* is a homomorphism from *G* onto *H*, then the quotient group *G Mod kernel G H h* is isomorphic to *H*.

theorem (**in** *group-hom*) *FactGroup-iso*:
h ‘ *carrier G* = *carrier H*
⟹ (λ*X*. *contents* (*h*‘*X*)) ∈ (*G Mod* (*kernel G H h*)) ≅ *H*
⟨*proof*⟩

end

theory *Sylow* **imports** *Coset* **begin**

6 Sylow's Theorem

See also [3].

The combinatorial argument is in theory Exponent

locale *sylow* = *group* +

fixes p **and** a **and** m **and** *calM* **and** *RelM*
assumes *prime-p*: $prime\ p$
and *order-G*: $order(G) = (p \hat{} a) * m$
and *finite-G* [*iff*]: $finite\ (carrier\ G)$
defines $calM == \{s.\ s \subseteq carrier(G)\ \&\ card(s) = p \hat{} a\}$
and $RelM == \{(N1,N2).\ N1 \in calM\ \&\ N2 \in calM\ \&$
$(\exists g \in carrier(G).\ N1 = (N2\ \#>\ g)\ )\}$

lemma (**in** *sylow*) *RelM-refl*: *refl calM RelM*
⟨*proof*⟩

lemma (**in** *sylow*) *RelM-sym*: *sym RelM*
⟨*proof*⟩

lemma (**in** *sylow*) *RelM-trans*: *trans RelM*
⟨*proof*⟩

lemma (**in** *sylow*) *RelM-equiv*: *equiv calM RelM*
⟨*proof*⟩

lemma (**in** *sylow*) *M-subset-calM-prep*: $M' \in calM\ //\ RelM\ ==>\ M' \subseteq calM$
⟨*proof*⟩

6.1 Main Part of the Proof

locale *sylow-central* = *sylow* +
fixes H **and** *M1* **and** M
assumes *M-in-quot*: $M \in calM\ //\ RelM$
and *not-dvd-M*: $\sim(p\ \hat{}\ Suc(exponent\ p\ m)\ dvd\ card(M))$
and *M1-in-M*: $M1 \in M$
defines $H == \{g.\ g \in carrier\ G\ \&\ M1\ \#>\ g = M1\}$

lemma (**in** *sylow-central*) *M-subset-calM*: $M \subseteq calM$
⟨*proof*⟩

lemma (**in** *sylow-central*) *card-M1*: $card(M1) = p \hat{} a$
⟨*proof*⟩

lemma *card-nonempty*: $0 < card(S) ==> S \neq \{\}$
⟨*proof*⟩

lemma (**in** *sylow-central*) *exists-x-in-M1*: $\exists x.\ x \in M1$
⟨*proof*⟩

lemma (**in** *sylow-central*) *M1-subset-G* [*simp*]: $M1 \subseteq carrier\ G$
⟨*proof*⟩

lemma (**in** *sylow-central*) *M1-inj-H*: $\exists f \in H \rightarrow M1.\ inj\text{-}on\ f\ H$
⟨*proof*⟩

6.2 Discharging the Assumptions of *sylow-central*

lemma (**in** *sylow*) *EmptyNotInEquivSet*: $\{\} \notin calM\ //\ RelM$
⟨*proof*⟩

lemma (**in** *sylow*) *existsM1inM*: $M \in calM\ //\ RelM \Longrightarrow \exists M1.\ M1 \in M$
⟨*proof*⟩

lemma (**in** *sylow*) *zero-less-o-G*: $0 < order(G)$
⟨*proof*⟩

lemma (**in** *sylow*) *zero-less-m*: $m > 0$
⟨*proof*⟩

lemma (**in** *sylow*) *card-calM*: $card(calM) = (p \hat{} a) * m\ choose\ p \hat{} a$
⟨*proof*⟩

lemma (**in** *sylow*) *zero-less-card-calM*: $card\ calM > 0$
⟨*proof*⟩

lemma (**in** *sylow*) *max-p-div-calM*:
$\sim (p \hat{} Suc(exponent\ p\ m)\ dvd\ card(calM))$
⟨*proof*⟩

lemma (**in** *sylow*) *finite-calM*: *finite calM*
⟨*proof*⟩

lemma (**in** *sylow*) *lemma-A1*:
$\exists M \in calM\ //\ RelM.\ \sim (p \hat{} Suc(exponent\ p\ m)\ dvd\ card(M))$
⟨*proof*⟩

6.2.1 Introduction and Destruct Rules for H

lemma (**in** *sylow-central*) *H-I*: $[|g \in carrier\ G;\ M1\ \#>\ g = M1|] \Longrightarrow g \in H$
⟨*proof*⟩

lemma (**in** *sylow-central*) *H-into-carrier-G*: $x \in H \Longrightarrow x \in carrier\ G$
⟨*proof*⟩

lemma (**in** *sylow-central*) *in-H-imp-eq*: $g : H \Longrightarrow M1\ \#>\ g = M1$
⟨*proof*⟩

lemma (**in** *sylow-central*) *H-m-closed*: $[|\ x{\in}H;\ y{\in}H|] \Longrightarrow x \otimes y \in H$
⟨*proof*⟩

lemma (**in** *sylow-central*) *H-not-empty*: $H \neq \{\}$
⟨*proof*⟩

lemma (**in** *sylow-central*) *H-is-subgroup*: *subgroup H G*
⟨*proof*⟩

lemma (**in** *sylow-central*) *rcosetGM1g-subset-G*:
[| $g \in carrier\ G$; $x \in M1\ \#>\ g$ |] ==> $x \in carrier\ G$
⟨*proof*⟩

lemma (**in** *sylow-central*) *finite-M1*: *finite M1*
⟨*proof*⟩

lemma (**in** *sylow-central*) *finite-rcosetGM1g*: $g \in carrier\ G$ ==> *finite* $(M1\ \#>\ g)$
⟨*proof*⟩

lemma (**in** *sylow-central*) *M1-cardeq-rcosetGM1g*:
$g \in carrier\ G$ ==> $card(M1\ \#>\ g) = card(M1)$
⟨*proof*⟩

lemma (**in** *sylow-central*) *M1-RelM-rcosetGM1g*:
$g \in carrier\ G$ ==> $(M1, M1\ \#>\ g) \in RelM$
⟨*proof*⟩

6.3 Equal Cardinalities of M and the Set of Cosets

Injections between M and $rcosets_G\ H$ show that their cardinalities are equal.

lemma *ElemClassEquiv*:
[| *equiv A r*; $C \in A\ //\ r$ |] ==> $\forall x \in C.\ \forall y \in C.\ (x,y) \in r$
⟨*proof*⟩

lemma (**in** *sylow-central*) *M-elem-map*:
$M2 \in M$ ==> $\exists g.\ g \in carrier\ G\ \&\ M1\ \#>\ g = M2$
⟨*proof*⟩

lemmas (**in** *sylow-central*) *M-elem-map-carrier* =
M-elem-map [*THEN someI-ex*, *THEN conjunct1*]

lemmas (**in** *sylow-central*) *M-elem-map-eq* =
M-elem-map [*THEN someI-ex*, *THEN conjunct2*]

lemma (**in** *sylow-central*) *M-funcset-rcosets-H*:
$(\%x{:}M.\ H\ \#>\ (SOME\ g.\ g \in carrier\ G\ \&\ M1\ \#>\ g = x)) \in M \to rcosets\ H$
⟨*proof*⟩

lemma (**in** *sylow-central*) *inj-M-GmodH*: $\exists f \in M \to rcosets\ H.\ inj\text{-}on\ f\ M$
⟨*proof*⟩

6.3.1 The Opposite Injection

lemma (**in** *sylow-central*) *H-elem-map*:
$H1 \in rcosets\ H$ ==> $\exists g.\ g \in carrier\ G\ \&\ H\ \#>\ g = H1$
⟨*proof*⟩

lemmas (**in** *sylow-central*) *H-elem-map-carrier* =
H-elem-map [*THEN someI-ex*, *THEN conjunct1*]

lemmas (**in** *sylow-central*) *H-elem-map-eq* =
H-elem-map [*THEN someI-ex*, *THEN conjunct2*]

lemma *EquivElemClass*:
[|*equiv A r*; $M \in A//r$; *M1*$\in$*M*; (*M1*,*M2*) $\in r$ |] ==> *M2* $\in$ *M*
⟨*proof*⟩

lemma (**in** *sylow-central*) *rcosets-H-funcset-M*:
$(\lambda C \in rcosets\ H.\ M1\ \#>\ (@g.\ g \in carrier\ G \wedge H\ \#>\ g = C)) \in rcosets\ H \rightarrow M$
⟨*proof*⟩

close to a duplicate of *inj-M-GmodH*

lemma (**in** *sylow-central*) *inj-GmodH-M*:
$\exists\, g \in rcosets\ H \rightarrow M.\ inj\text{-}on\ g\ (rcosets\ H)$
⟨*proof*⟩

lemma (**in** *sylow-central*) *calM-subset-PowG*: $calM \subseteq Pow(carrier\ G)$
⟨*proof*⟩

lemma (**in** *sylow-central*) *finite-M*: *finite M*
⟨*proof*⟩

lemma (**in** *sylow-central*) *cardMeqIndexH*: $card(M) = card(rcosets\ H)$
⟨*proof*⟩

lemma (**in** *sylow-central*) *index-lem*: $card(M) * card(H) = order(G)$
⟨*proof*⟩

lemma (**in** *sylow-central*) *lemma-leq1*: $p\hat{}a \leq card(H)$
⟨*proof*⟩

lemma (**in** *sylow-central*) *lemma-leq2*: $card(H) \leq p\hat{}a$
⟨*proof*⟩

lemma (**in** *sylow-central*) *card-H-eq*: $card(H) = p\hat{}a$
⟨*proof*⟩

lemma (**in** *sylow*) *sylow-thm*: $\exists H.\ subgroup\ H\ G\ \&\ card(H) = p\hat{}a$
⟨*proof*⟩

Needed because the locale's automatic definition refers to *semigroup G* and

group-axioms G rather than simply to *group G*.

lemma *sylow-eq*: *sylow G p a m* = (*group G* & *sylow-axioms G p a m*)
⟨*proof*⟩

6.4 Sylow's Theorem

theorem *sylow-thm*:
[| *prime p*; *group*(*G*); *order*(*G*) = (*p* ^ *a*) * *m*; *finite* (*carrier G*)|]
==> ∃ *H*. *subgroup H G* & *card*(*H*) = *p* ^ *a*
⟨*proof*⟩

end

theory *Bij* **imports** *Group* **begin**

7 Bijections of a Set, Permutation Groups and Automorphism Groups

constdefs
Bij :: 'a *set* ⇒ ('a ⇒ 'a) *set*
— Only extensional functions, since otherwise we get too many.
Bij S ≡ *extensional S* ∩ {*f*. *bij-betw f S S*}

BijGroup :: 'a *set* ⇒ ('a ⇒ 'a) *monoid*
BijGroup S ≡
(|*carrier* = *Bij S*,
mult = λ*g* ∈ *Bij S*. λ*f* ∈ *Bij S*. *compose S g f*,
one = λ*x* ∈ *S*. *x*|)

declare *Id-compose* [*simp*] *compose-Id* [*simp*]

lemma *Bij-imp-extensional*: *f* ∈ *Bij S* ⟹ *f* ∈ *extensional S*
⟨*proof*⟩

lemma *Bij-imp-funcset*: *f* ∈ *Bij S* ⟹ *f* ∈ *S* → *S*
⟨*proof*⟩

7.1 Bijections Form a Group

lemma *restrict-Inv-Bij*: *f* ∈ *Bij S* ⟹ (λ*x* ∈ *S*. (*Inv S f*) *x*) ∈ *Bij S*
⟨*proof*⟩

lemma *id-Bij*: (λ*x*∈*S*. *x*) ∈ *Bij S*
⟨*proof*⟩

lemma *compose-Bij*: $[\![x \in Bij\ S;\ y \in Bij\ S]\!] \Longrightarrow compose\ S\ x\ y \in Bij\ S$
⟨*proof*⟩

lemma *Bij-compose-restrict-eq*:
$f \in Bij\ S \Longrightarrow compose\ S\ (restrict\ (Inv\ S\ f)\ S)\ f = (\lambda x{\in}S.\ x)$
⟨*proof*⟩

theorem *group-BijGroup*: *group* (*BijGroup S*)
⟨*proof*⟩

7.2 Automorphisms Form a Group

lemma *Bij-Inv-mem*: $[\![\ f \in Bij\ S;\ \ x \in S]\!] \Longrightarrow Inv\ S\ f\ x \in S$
⟨*proof*⟩

lemma *Bij-Inv-lemma*:
assumes *eq*: $\bigwedge x\ y.\ [\![x \in S;\ y \in S]\!] \Longrightarrow h(g\ x\ y) = g\ (h\ x)\ (h\ y)$
shows $[\![h \in Bij\ S;\ \ g \in S \to S \to S;\ \ x \in S;\ \ y \in S]\!]$
$\Longrightarrow Inv\ S\ h\ (g\ x\ y) = g\ (Inv\ S\ h\ x)\ (Inv\ S\ h\ y)$
⟨*proof*⟩

constdefs
auto :: ('*a*, '*b*) *monoid-scheme* ⇒ ('*a* ⇒ '*a*) *set*
auto G ≡ *hom G G* ∩ *Bij* (*carrier G*)

AutoGroup :: ('*a*, '*c*) *monoid-scheme* ⇒ ('*a* ⇒ '*a*) *monoid*
AutoGroup G ≡ *BijGroup* (*carrier G*) (|*carrier* := *auto G*|)

lemma (**in** *group*) *id-in-auto*: $(\lambda x \in carrier\ G.\ x) \in auto\ G$
⟨*proof*⟩

lemma (**in** *group*) *mult-funcset*: *mult G* ∈ *carrier G* → *carrier G* → *carrier G*
⟨*proof*⟩

lemma (**in** *group*) *restrict-Inv-hom*:
$[\![h \in hom\ G\ G;\ h \in Bij\ (carrier\ G)]\!]$
$\Longrightarrow restrict\ (Inv\ (carrier\ G)\ h)\ (carrier\ G) \in hom\ G\ G$
⟨*proof*⟩

lemma *inv-BijGroup*:
$f \in Bij\ S \Longrightarrow m\text{-}inv\ (BijGroup\ S)\ f = (\lambda x \in S.\ (Inv\ S\ f)\ x)$
⟨*proof*⟩

lemma (**in** *group*) *subgroup-auto*:
subgroup (*auto G*) (*BijGroup* (*carrier G*))
⟨*proof*⟩

theorem (**in** *group*) *AutoGroup*: *group* (*AutoGroup G*)
⟨*proof*⟩

end

theory *Ring* **imports** *FiniteProduct*
uses (*ringsimp.ML*) **begin**

8 Abelian Groups

record *'a ring* = *'a monoid* +
  zero :: *'a* (**0**$_1$)
  add :: [*'a*, *'a*] => *'a* (**infixl** $\oplus_1$ *65*)

Derived operations.

constdefs (**structure** *R*)
  a-inv :: [(*'a*, *'m*) *ring-scheme*, *'a*] => *'a* ($\ominus_1$ - [*81*] *80*)
  a-inv R == *m-inv* (| *carrier* = *carrier R*, *mult* = *add R*, *one* = *zero R* |)

  a-minus :: [(*'a*, *'m*) *ring-scheme*, *'a*, *'a*] => *'a* (**infixl** $\ominus_1$ *65*)
  [| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> *x* ⊖ *y* == *x* ⊕ (⊖ *y*)

locale *abelian-monoid* =
  fixes *G* (**structure**)
  assumes *a-comm-monoid*:
    comm-monoid (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)

The following definition is redundant but simple to use.

locale *abelian-group* = *abelian-monoid* +
  assumes *a-comm-group*:
    comm-group (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)

8.1 Basic Properties

lemma *abelian-monoidI*:
  fixes *R* (**structure**)
  assumes *a-closed*:
    !!*x y*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> *x* ⊕ *y* ∈ *carrier R*
   and *zero-closed*: **0** ∈ *carrier R*
   and *a-assoc*:
    !!*x y z*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R*; *z* ∈ *carrier R* |] ==>
    (*x* ⊕ *y*) ⊕ *z* = *x* ⊕ (*y* ⊕ *z*)
   and *l-zero*: !!*x*. *x* ∈ *carrier R* ==> **0** ⊕ *x* = *x*
   and *a-comm*:
    !!*x y*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> *x* ⊕ *y* = *y* ⊕ *x*
  shows *abelian-monoid R*

⟨*proof*⟩

lemma *abelian-groupI*:
fixes R (**structure**)
assumes *a-closed*:
!!x y. [| $x \in$ *carrier* R; $y \in$ *carrier* R |] ==> $x \oplus y \in$ *carrier* R
and *zero-closed*: *zero* $R \in$ *carrier* R
and *a-assoc*:
!!x y z. [| $x \in$ *carrier* R; $y \in$ *carrier* R; $z \in$ *carrier* R |] ==>
$(x \oplus y) \oplus z = x \oplus (y \oplus z)$
and *a-comm*:
!!x y. [| $x \in$ *carrier* R; $y \in$ *carrier* R |] ==> $x \oplus y = y \oplus x$
and *l-zero*: !!x. $x \in$ *carrier* R ==> $\mathbf{0} \oplus x = x$
and *l-inv-ex*: !!x. $x \in$ *carrier* R ==> *EX* y : *carrier* R. $y \oplus x = \mathbf{0}$
shows *abelian-group* R
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *a-monoid*:
monoid (| *carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G |)
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-group*:
group (| *carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G |)
⟨*proof*⟩

lemmas *monoid-record-simps* = *partial-object.simps monoid.simps*

lemma (**in** *abelian-monoid*) *a-closed* [*intro*, *simp*]:
⟦ $x \in$ *carrier* G; $y \in$ *carrier* G ⟧ ⟹ $x \oplus y \in$ *carrier* G
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *zero-closed* [*intro*, *simp*]:
$\mathbf{0} \in$ *carrier* G
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-inv-closed* [*intro*, *simp*]:
$x \in$ *carrier* G ==> $\ominus x \in$ *carrier* G
⟨*proof*⟩

lemma (**in** *abelian-group*) *minus-closed* [*intro*, *simp*]:
[| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==> $x \ominus y \in$ *carrier* G
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-l-cancel* [*simp*]:
[| $x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* G |] ==>
$(x \oplus y = x \oplus z) = (y = z)$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-r-cancel* [*simp*]:

[| $x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* G |] ==>
($y \oplus x = z \oplus x$) = ($y = z$)
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *a-assoc*:
⟦$x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* G⟧ ⟹
($x \oplus y$) $\oplus z = x \oplus$ ($y \oplus z$)
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *l-zero* [*simp*]:
$x \in$ *carrier* G ==> $\mathbf{0} \oplus x = x$
⟨*proof*⟩

lemma (**in** *abelian-group*) *l-neg*:
$x \in$ *carrier* G ==> $\ominus x \oplus x = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *a-comm*:
⟦$x \in$ *carrier* G; $y \in$ *carrier* G⟧ ⟹ $x \oplus y = y \oplus x$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *a-lcomm*:
⟦$x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* G⟧ ⟹
$x \oplus$ ($y \oplus z$) = $y \oplus$ ($x \oplus z$)
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *r-zero* [*simp*]:
$x \in$ *carrier* G ==> $x \oplus \mathbf{0} = x$
⟨*proof*⟩

lemma (**in** *abelian-group*) *r-neg*:
$x \in$ *carrier* G ==> $x \oplus (\ominus x) = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *abelian-group*) *minus-zero* [*simp*]:
$\ominus \mathbf{0} = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *abelian-group*) *minus-minus* [*simp*]:
$x \in$ *carrier* G ==> $\ominus$ ($\ominus x$) = x
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-inv-inj*:
inj-on (*a-inv* G) (*carrier* G)
⟨*proof*⟩

lemma (**in** *abelian-group*) *minus-add*:
[| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==> $\ominus$ ($x \oplus y$) = $\ominus x \oplus \ominus y$
⟨*proof*⟩

lemma (**in** *abelian-group*) *minus-equality*:
[| $x \in$ *carrier* G; $y \in$ *carrier* G; $y \oplus x = \mathbf{0}$ |] ==> $\ominus x = y$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *minus-unique*:
[| $x \in$ *carrier* G; $y \in$ *carrier* G; $y' \in$ *carrier* G;
$y \oplus x = \mathbf{0}$; $x \oplus y' = \mathbf{0}$ |] ==> $y = y'$
⟨*proof*⟩

lemmas (**in** *abelian-monoid*) *a-ac* = *a-assoc a-comm a-lcomm*

Derive an *abelian-group* from a *comm-group*

lemma *comm-group-abelian-groupI*:
fixes G (**structure**)
assumes *cg*: *comm-group* (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|)
shows *abelian-group* G
⟨*proof*⟩

8.2 Sums over Finite Sets

This definition makes it easy to lift lemmas from *finprod*.

constdefs
finsum :: [('*b*, '*m*) *ring-scheme*, '*a* => '*b*, '*a set*] => '*b*
finsum G f A == *finprod* (| *carrier* = *carrier* G,
mult = *add* G, *one* = *zero* G |) f A

syntax
-*finsum* :: *index* => *idt* => '*a set* => '*b* => '*b*
((3⨁ --:-. -) [1000, 0, 51, 10] 10)
syntax (*xsymbols*)
-*finsum* :: *index* => *idt* => '*a set* => '*b* => '*b*
((3⨁ --∈-. -) [1000, 0, 51, 10] 10)
syntax (*HTML* **output**)
-*finsum* :: *index* => *idt* => '*a set* => '*b* => '*b*
((3⨁ --∈-. -) [1000, 0, 51, 10] 10)
translations
⨁1*i*:*A*. *b* == *finsum* ⋄1 (%*i*. *b*) *A*
— Beware of argument permutation!

lemma (**in** *abelian-monoid*) *finsum-empty* [*simp*]:
finsum G f {} = $\mathbf{0}$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-insert* [*simp*]:
[| *finite* F; $a \notin F$; $f \in F$ -> *carrier* G; $f\ a \in$ *carrier* G |]
==> *finsum* G f (*insert* a F) = $f\ a \oplus$ *finsum* G f F

⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-zero* [*simp*]:
finite A ==> $(\bigoplus i{\in}A.\ \mathbf{0}) = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-closed* [*simp*]:
fixes *A*
assumes *fin*: *finite A* **and** *f*: *f* ∈ *A* -> *carrier G*
shows *finsum G f A* ∈ *carrier G*
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-Un-Int*:
[| *finite A*; *finite B*; *g* ∈ *A* -> *carrier G*; *g* ∈ *B* -> *carrier G* |] ==>
finsum G g (*A Un B*) ⊕ *finsum G g* (*A Int B*) =
finsum G g A ⊕ *finsum G g B*
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-Un-disjoint*:
[| *finite A*; *finite B*; *A Int B* = {};
g ∈ *A* -> *carrier G*; *g* ∈ *B* -> *carrier G* |]
==> *finsum G g* (*A Un B*) = *finsum G g A* ⊕ *finsum G g B*
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-addf*:
[| *finite A*; *f* ∈ *A* -> *carrier G*; *g* ∈ *A* -> *carrier G* |] ==>
finsum G (%*x*. *f x* ⊕ *g x*) *A* = (*finsum G f A* ⊕ *finsum G g A*)
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-cong'*:
[| *A* = *B*; *g* : *B* -> *carrier G*;
!!*i*. *i* : *B* ==> *f i* = *g i* |] ==> *finsum G f A* = *finsum G g B*
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-0* [*simp*]:
f : {*0*::*nat*} -> *carrier G* ==> *finsum G f* {..*0*} = *f 0*
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-Suc* [*simp*]:
f : {..*Suc n*} -> *carrier G* ==>
finsum G f {..*Suc n*} = (*f* (*Suc n*) ⊕ *finsum G f* {..*n*})
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-Suc2*:
f : {..*Suc n*} -> *carrier G* ==>
finsum G f {..*Suc n*} = (*finsum G* (%*i*. *f* (*Suc i*)) {..*n*} ⊕ *f 0*)
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-add* [*simp*]:

[| $f : \{..n\}$ -> *carrier* G; $g : \{..n\}$ -> *carrier* G |] ==>
finsum G (%i. $f\ i \oplus g\ i$) $\{..n::nat\}$ =
finsum $G\ f\ \{..n\} \oplus$ *finsum* $G\ g\ \{..n\}$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *finsum-cong*:
[| $A = B$; $f : B$ -> *carrier* G;
!!i. $i : B$ =*simp*=> $f\ i = g\ i$ |] ==> *finsum* $G\ f\ A$ = *finsum* $G\ g\ B$
⟨*proof*⟩

Usually, if this rule causes a failed congruence proof error, the reason is that the premise $g \in B$ -> *carrier* G cannot be shown. Adding *Pi-def* to the simpset is often useful.

9 The Algebraic Hierarchy of Rings

9.1 Basic Definitions

locale *ring* = *abelian-group* R + *monoid* R +
assumes *l-distr*: [| $x \in$ *carrier* R; $y \in$ *carrier* R; $z \in$ *carrier* R |]
==> $(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z$
and *r-distr*: [| $x \in$ *carrier* R; $y \in$ *carrier* R; $z \in$ *carrier* R |]
==> $z \otimes (x \oplus y) = z \otimes x \oplus z \otimes y$

locale *cring* = *ring* + *comm-monoid* R

locale *domain* = *cring* +
assumes *one-not-zero* [*simp*]: $\mathbf{1} \sim= \mathbf{0}$
and *integral*: [| $a \otimes b = \mathbf{0}$; $a \in$ *carrier* R; $b \in$ *carrier* R |] ==>
$a = \mathbf{0} \mid b = \mathbf{0}$

locale *field* = *domain* +
assumes *field-Units*: *Units* R = *carrier* $R - \{\mathbf{0}\}$

9.2 Rings

lemma *ringI*:
fixes R (**structure**)
assumes *abelian-group*: *abelian-group* R
and *monoid*: *monoid* R
and *l-distr*: !!$x\ y\ z$. [| $x \in$ *carrier* R; $y \in$ *carrier* R; $z \in$ *carrier* R |]
==> $(x \oplus y) \otimes z = x \otimes z \oplus y \otimes z$
and *r-distr*: !!$x\ y\ z$. [| $x \in$ *carrier* R; $y \in$ *carrier* R; $z \in$ *carrier* R |]
==> $z \otimes (x \oplus y) = z \otimes x \oplus z \otimes y$
shows *ring* R
⟨*proof*⟩

lemma (**in** *ring*) *is-abelian-group*:
abelian-group R

⟨*proof*⟩

lemma (**in** *ring*) *is-monoid*:
monoid R
⟨*proof*⟩

lemma (**in** *ring*) *is-ring*:
ring R
⟨*proof*⟩

lemmas *ring-record-simps* = *monoid-record-simps ring.simps*

lemma *cringI*:
fixes *R* (**structure**)
assumes *abelian-group*: *abelian-group R*
and *comm-monoid*: *comm-monoid R*
and *l-distr*: !!*x y z*. [| *x* ∈ *carrier R*; *y* ∈ *carrier R*; *z* ∈ *carrier R* |]
==> (*x* ⊕ *y*) ⊗ *z* = *x* ⊗ *z* ⊕ *y* ⊗ *z*
shows *cring R*
⟨*proof*⟩

lemma (**in** *cring*) *is-comm-monoid*:
comm-monoid R
⟨*proof*⟩

lemma (**in** *cring*) *is-cring*:
cring R ⟨*proof*⟩

9.2.1 Normaliser for Rings

lemma (**in** *abelian-group*) *r-neg2*:
[| *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==> *x* ⊕ (⊖ *x* ⊕ *y*) = *y*
⟨*proof*⟩

lemma (**in** *abelian-group*) *r-neg1*:
[| *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==> ⊖ *x* ⊕ (*x* ⊕ *y*) = *y*
⟨*proof*⟩

The following proofs are from Jacobson, Basic Algebra I, pp. 88–89

lemma (**in** *ring*) *l-null* [*simp*]:
x ∈ *carrier R* ==> **0** ⊗ *x* = **0**
⟨*proof*⟩

lemma (**in** *ring*) *r-null* [*simp*]:
x ∈ *carrier R* ==> *x* ⊗ **0** = **0**
⟨*proof*⟩

lemma (**in** *ring*) *l-minus*:
[| *x* ∈ *carrier R*; *y* ∈ *carrier R* |] ==> ⊖ *x* ⊗ *y* = ⊖ (*x* ⊗ *y*)

⟨*proof*⟩

lemma (**in** *ring*) *r-minus*:
[| $x \in$ *carrier* R; $y \in$ *carrier* R |] ==> $x \otimes \ominus y = \ominus (x \otimes y)$
⟨*proof*⟩

lemma (**in** *abelian-group*) *minus-eq*:
[| $x \in$ *carrier* G; $y \in$ *carrier* G |] ==> $x \ominus y = x \oplus \ominus y$
⟨*proof*⟩

Setup algebra method: compute distributive normal form in locale contexts

⟨*ML*⟩

lemmas (**in** *ring*) *ring-simprules*
[*algebra ring zero R add R a-inv R a-minus R one R mult R*] =
a-closed zero-closed a-inv-closed minus-closed m-closed one-closed
a-assoc l-zero l-neg a-comm m-assoc l-one l-distr minus-eq
r-zero r-neg r-neg2 r-neg1 minus-add minus-minus minus-zero
a-lcomm r-distr l-null r-null l-minus r-minus

lemmas (**in** *cring*)
[*algebra del*: *ring zero R add R a-inv R a-minus R one R mult R*] =
-

lemmas (**in** *cring*) *cring-simprules*
[*algebra add*: *cring zero R add R a-inv R a-minus R one R mult R*] =
a-closed zero-closed a-inv-closed minus-closed m-closed one-closed
a-assoc l-zero l-neg a-comm m-assoc l-one l-distr m-comm minus-eq
r-zero r-neg r-neg2 r-neg1 minus-add minus-minus minus-zero
a-lcomm m-lcomm r-distr l-null r-null l-minus r-minus

lemma (**in** *cring*) *nat-pow-zero*:
(n::*nat*) ~= *0* ==> $\mathbf{0}$ (^) $n = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *ring*) *one-zeroD*:
assumes *onezero*: $\mathbf{1} = \mathbf{0}$
shows *carrier* $R = \{\mathbf{0}\}$
⟨*proof*⟩

lemma (**in** *ring*) *one-zeroI*:
assumes *carrzero*: *carrier* $R = \{\mathbf{0}\}$
shows $\mathbf{1} = \mathbf{0}$
⟨*proof*⟩

lemma (**in** *ring*) *one-zero*:
shows (*carrier* $R = \{\mathbf{0}\}$) = ($\mathbf{1} = \mathbf{0}$)
⟨*proof*⟩

lemma (**in** *ring*) *one-not-zero*:
shows (*carrier R* ≠ {**0**}) = (**1** ≠ **0**)
⟨*proof*⟩

Two examples for use of method algebra

lemma
includes *ring R* + *cring S*
shows [| *a* ∈ *carrier R*; *b* ∈ *carrier R*; *c* ∈ *carrier S*; *d* ∈ *carrier S* |] ==>
a ⊕ ⊖ (*a* ⊕ ⊖ *b*) = *b* & *c* ⊗$_S$ *d* = *d* ⊗$_S$ *c*
⟨*proof*⟩

lemma
includes *cring*
shows [| *a* ∈ *carrier R*; *b* ∈ *carrier R* |] ==> *a* ⊖ (*a* ⊖ *b*) = *b*
⟨*proof*⟩

9.2.2 Sums over Finite Sets

lemma (**in** *cring*) *finsum-ldistr*:
[| *finite A*; *a* ∈ *carrier R*; *f* ∈ *A* -> *carrier R* |] ==>
finsum R f A ⊗ *a* = *finsum R* (%*i*. *f i* ⊗ *a*) *A*
⟨*proof*⟩

lemma (**in** *cring*) *finsum-rdistr*:
[| *finite A*; *a* ∈ *carrier R*; *f* ∈ *A* -> *carrier R* |] ==>
a ⊗ *finsum R f A* = *finsum R* (%*i*. *a* ⊗ *f i*) *A*
⟨*proof*⟩

9.3 Integral Domains

lemma (**in** *domain*) *zero-not-one* [*simp*]:
0 ~= **1**
⟨*proof*⟩

lemma (**in** *domain*) *integral-iff*:
[| *a* ∈ *carrier R*; *b* ∈ *carrier R* |] ==> (*a* ⊗ *b* = **0**) = (*a* = **0** | *b* = **0**)
⟨*proof*⟩

lemma (**in** *domain*) *m-lcancel*:
assumes *prem*: *a* ~= **0**
and *R*: *a* ∈ *carrier R b* ∈ *carrier R c* ∈ *carrier R*
shows (*a* ⊗ *b* = *a* ⊗ *c*) = (*b* = *c*)
⟨*proof*⟩

lemma (**in** *domain*) *m-rcancel*:
assumes *prem*: *a* ~= **0**
and *R*: *a* ∈ *carrier R b* ∈ *carrier R c* ∈ *carrier R*
shows *conc*: (*b* ⊗ *a* = *c* ⊗ *a*) = (*b* = *c*)
⟨*proof*⟩

9.4 Fields

Field would not need to be derived from domain, the properties for domain follow from the assumptions of field

lemma (**in** *cring*) *cring-fieldI*:
assumes *field-Units*: *Units* R = *carrier* R − {**0**}
shows *field* R
⟨*proof*⟩

Another variant to show that something is a field

lemma (**in** *cring*) *cring-fieldI2*:
assumes *notzero*: $\mathbf{0} \neq \mathbf{1}$
and *invex*: $\bigwedge a.\ [\![a \in carrier\ R;\ a \neq \mathbf{0}]\!] \Longrightarrow \exists\, b{\in}carrier\ R.\ a \otimes b = \mathbf{1}$
shows *field* R
⟨*proof*⟩

9.5 Morphisms

constdefs (**structure** R S)
ring-hom :: [('*a*, '*m*) *ring-scheme*, ('*b*, '*n*) *ring-scheme*] => ('*a* => '*b*) *set*
ring-hom R S == {*h*. *h* ∈ *carrier* R −> *carrier* S &
(*ALL x y*. *x* ∈ *carrier* R & *y* ∈ *carrier* R −−>
h (*x* ⊗ *y*) = *h x* ⊗$_S$ *h y* & *h* (*x* ⊕ *y*) = *h x* ⊕$_S$ *h y*) &
h **1** = **1**$_S$}

lemma *ring-hom-memI*:
fixes R (**structure**) **and** S (**structure**)
assumes *hom-closed*: !!*x*. *x* ∈ *carrier* R ==> *h x* ∈ *carrier* S
and *hom-mult*: !!*x y*. [| *x* ∈ *carrier* R; *y* ∈ *carrier* R |] ==>
h (*x* ⊗ *y*) = *h x* ⊗$_S$ *h y*
and *hom-add*: !!*x y*. [| *x* ∈ *carrier* R; *y* ∈ *carrier* R |] ==>
h (*x* ⊕ *y*) = *h x* ⊕$_S$ *h y*
and *hom-one*: *h* **1** = **1**$_S$
shows *h* ∈ *ring-hom* R S
⟨*proof*⟩

lemma *ring-hom-closed*:
[| *h* ∈ *ring-hom* R S; *x* ∈ *carrier* R |] ==> *h x* ∈ *carrier* S
⟨*proof*⟩

lemma *ring-hom-mult*:
fixes R (**structure**) **and** S (**structure**)
shows
[| *h* ∈ *ring-hom* R S; *x* ∈ *carrier* R; *y* ∈ *carrier* R |] ==>
h (*x* ⊗ *y*) = *h x* ⊗$_S$ *h y*
⟨*proof*⟩

lemma *ring-hom-add*:
fixes R (**structure**) **and** S (**structure**)

shows
[| $h \in$ *ring-hom* $R\ S$; $x \in$ *carrier* R; $y \in$ *carrier* R |] ==>
$h\ (x \oplus y) = h\ x \oplus_S h\ y$
⟨*proof*⟩

lemma *ring-hom-one*:
fixes R (**structure**) **and** S (**structure**)
shows $h \in$ *ring-hom* $R\ S$ ==> $h\ \mathbf{1} = \mathbf{1}_S$
⟨*proof*⟩

locale *ring-hom-cring* = *cring* R + *cring* S +
fixes h
assumes *homh* [*simp*, *intro*]: $h \in$ *ring-hom* $R\ S$
notes *hom-closed* [*simp*, *intro*] = *ring-hom-closed* [*OF homh*]
and *hom-mult* [*simp*] = *ring-hom-mult* [*OF homh*]
and *hom-add* [*simp*] = *ring-hom-add* [*OF homh*]
and *hom-one* [*simp*] = *ring-hom-one* [*OF homh*]

lemma (**in** *ring-hom-cring*) *hom-zero* [*simp*]:
$h\ \mathbf{0} = \mathbf{0}_S$
⟨*proof*⟩

lemma (**in** *ring-hom-cring*) *hom-a-inv* [*simp*]:
$x \in$ *carrier* R ==> $h\ (\ominus x) = \ominus_S h\ x$
⟨*proof*⟩

lemma (**in** *ring-hom-cring*) *hom-finsum* [*simp*]:
[| *finite* A; $f \in A$ -> *carrier* R |] ==>
h (*finsum* $R\ f\ A$) = *finsum* S (h *o* f) A
⟨*proof*⟩

lemma (**in** *ring-hom-cring*) *hom-finprod*:
[| *finite* A; $f \in A$ -> *carrier* R |] ==>
h (*finprod* $R\ f\ A$) = *finprod* S (h *o* f) A
⟨*proof*⟩

declare *ring-hom-cring.hom-finprod* [*simp*]

lemma *id-ring-hom* [*simp*]:
id $\in$ *ring-hom* $R\ R$
⟨*proof*⟩

end

theory *Module* **imports** *Ring* **begin**

10 Modules over an Abelian Group

10.1 Definitions

record $('a, 'b)$ *module* $=$ $'b$ *ring* $+$
  smult :: $['a, 'b] \Rightarrow 'b$ (**infixl** $\odot_1$ *70*)

locale *module* $=$ *cring* R $+$ *abelian-group* M $+$
  assumes *smult-closed* [*simp*, *intro*]:
    $[\![\ a \in \text{carrier } R;\ x \in \text{carrier } M\ ]\!] \Longrightarrow a \odot_M x \in \text{carrier } M$
  and *smult-l-distr*:
    $[\![\ a \in \text{carrier } R;\ b \in \text{carrier } R;\ x \in \text{carrier } M\ ]\!] \Longrightarrow$
    $(a \oplus b) \odot_M x = a \odot_M x \oplus_M b \odot_M x$
  and *smult-r-distr*:
    $[\![\ a \in \text{carrier } R;\ x \in \text{carrier } M;\ y \in \text{carrier } M\ ]\!] \Longrightarrow$
    $a \odot_M (x \oplus_M y) = a \odot_M x \oplus_M a \odot_M y$
  and *smult-assoc1*:
    $[\![\ a \in \text{carrier } R;\ b \in \text{carrier } R;\ x \in \text{carrier } M\ ]\!] \Longrightarrow$
    $(a \otimes b) \odot_M x = a \odot_M (b \odot_M x)$
  and *smult-one* [*simp*]:
    $x \in \text{carrier } M \Longrightarrow \mathbf{1} \odot_M x = x$

locale *algebra* $=$ *module* R M $+$ *cring* M $+$
  assumes *smult-assoc2*:
    $[\![\ a \in \text{carrier } R;\ x \in \text{carrier } M;\ y \in \text{carrier } M\ ]\!] \Longrightarrow$
    $(a \odot_M x) \otimes_M y = a \odot_M (x \otimes_M y)$

lemma *moduleI*:
  fixes R (**structure**) **and** M (**structure**)
  assumes *cring*: *cring* R
    and *abelian-group*: *abelian-group* M
    and *smult-closed*:
      $!!a\ x.\ [\![\ a \in \text{carrier } R;\ x \in \text{carrier } M\ ]\!] \Longrightarrow a \odot_M x \in \text{carrier } M$
    and *smult-l-distr*:
      $!!a\ b\ x.\ [\![\ a \in \text{carrier } R;\ b \in \text{carrier } R;\ x \in \text{carrier } M\ ]\!] \Longrightarrow$
      $(a \oplus b) \odot_M x = (a \odot_M x) \oplus_M (b \odot_M x)$
    and *smult-r-distr*:
      $!!a\ x\ y.\ [\![\ a \in \text{carrier } R;\ x \in \text{carrier } M;\ y \in \text{carrier } M\ ]\!] \Longrightarrow$
      $a \odot_M (x \oplus_M y) = (a \odot_M x) \oplus_M (a \odot_M y)$
    and *smult-assoc1*:
      $!!a\ b\ x.\ [\![\ a \in \text{carrier } R;\ b \in \text{carrier } R;\ x \in \text{carrier } M\ ]\!] \Longrightarrow$
      $(a \otimes b) \odot_M x = a \odot_M (b \odot_M x)$
    and *smult-one*:
      $!!x.\ x \in \text{carrier } M \Longrightarrow \mathbf{1} \odot_M x = x$
  shows *module* R M
  ⟨*proof*⟩

lemma *algebraI*:
  fixes R (**structure**) **and** M (**structure**)
  assumes *R-cring*: *cring* R

and *M-cring*: *cring M*
and *smult-closed*:
!!*a x*. [| *a* ∈ *carrier R*; *x* ∈ *carrier M* |] ==> *a* ⊙$_M$ *x* ∈ *carrier M*
and *smult-l-distr*:
!!*a b x*. [| *a* ∈ *carrier R*; *b* ∈ *carrier R*; *x* ∈ *carrier M* |] ==>
(*a* ⊕ *b*) ⊙$_M$ *x* = (*a* ⊙$_M$ *x*) ⊕$_M$ (*b* ⊙$_M$ *x*)
and *smult-r-distr*:
!!*a x y*. [| *a* ∈ *carrier R*; *x* ∈ *carrier M*; *y* ∈ *carrier M* |] ==>
a ⊙$_M$ (*x* ⊕$_M$ *y*) = (*a* ⊙$_M$ *x*) ⊕$_M$ (*a* ⊙$_M$ *y*)
and *smult-assoc1*:
!!*a b x*. [| *a* ∈ *carrier R*; *b* ∈ *carrier R*; *x* ∈ *carrier M* |] ==>
(*a* ⊗ *b*) ⊙$_M$ *x* = *a* ⊙$_M$ (*b* ⊙$_M$ *x*)
and *smult-one*:
!!*x*. *x* ∈ *carrier M* ==> (*one R*) ⊙$_M$ *x* = *x*
and *smult-assoc2*:
!!*a x y*. [| *a* ∈ *carrier R*; *x* ∈ *carrier M*; *y* ∈ *carrier M* |] ==>
(*a* ⊙$_M$ *x*) ⊗$_M$ *y* = *a* ⊙$_M$ (*x* ⊗$_M$ *y*)
shows *algebra R M*
⟨*proof*⟩

lemma (**in** *algebra*) *R-cring*:
cring R
⟨*proof*⟩

lemma (**in** *algebra*) *M-cring*:
cring M
⟨*proof*⟩

lemma (**in** *algebra*) *module*:
module R M
⟨*proof*⟩

10.2 Basic Properties of Algebras

lemma (**in** *algebra*) *smult-l-null* [*simp*]:
x ∈ *carrier M* ==> **0** ⊙$_M$ *x* = **0**$_M$
⟨*proof*⟩

lemma (**in** *algebra*) *smult-r-null* [*simp*]:
a ∈ *carrier R* ==> *a* ⊙$_M$ **0**$_M$ = **0**$_M$
⟨*proof*⟩

lemma (**in** *algebra*) *smult-l-minus*:
[| *a* ∈ *carrier R*; *x* ∈ *carrier M* |] ==> (⊖*a*) ⊙$_M$ *x* = ⊖$_M$ (*a* ⊙$_M$ *x*)
⟨*proof*⟩

lemma (**in** *algebra*) *smult-r-minus*:
[| *a* ∈ *carrier R*; *x* ∈ *carrier M* |] ==> *a* ⊙$_M$ (⊖$_M$*x*) = ⊖$_M$ (*a* ⊙$_M$ *x*)
⟨*proof*⟩

end

theory *UnivPoly* **imports** *Module* **begin**

11 Univariate Polynomials

Polynomials are formalised as modules with additional operations for extracting coefficients from polynomials and for obtaining monomials from coefficients and exponents (record *up-ring*). The carrier set is a set of bounded functions from Nat to the coefficient domain. Bounded means that these functions return zero above a certain bound (the degree). There is a chapter on the formalisation of polynomials in the PhD thesis [1], which was implemented with axiomatic type classes. This was later ported to Locales.

11.1 The Constructor for Univariate Polynomials

Functions with finite support.

locale *bound* =
  fixes $z :: 'a$
    and $n :: nat$
    and $f :: nat \Rightarrow 'a$
  assumes *bound*: $!!m.\ n < m \Longrightarrow f\ m = z$

declare *bound.intro* [*intro*!]
  and *bound.bound* [*dest*]

lemma *bound-below*:
  assumes *bound*: *bound z m f* **and** *nonzero*: $f\ n \neq z$ **shows** $n \leq m$
⟨*proof*⟩

record $('a, 'p)$ *up-ring* = $('a, 'p)$ *module* +
  monom :: $['a, nat] \Rightarrow 'p$
  coeff :: $['p, nat] \Rightarrow 'a$

constdefs (**structure** *R*)
  up :: $('a, 'm)$ *ring-scheme* $\Rightarrow$ $(nat \Rightarrow 'a)$ *set*
  up R == $\{f.\ f \in UNIV \rightarrow carrier\ R\ \&\ (EX\ n.\ bound\ \mathbf{0}\ n\ f)\}$
  UP :: $('a, 'm)$ *ring-scheme* $\Rightarrow$ $('a, nat \Rightarrow 'a)$ *up-ring*
  UP R == (|
    carrier = *up R*,
    mult = $(\%p{:}up\ R.\ \%q{:}up\ R.\ \%n.\ \bigoplus i \in \{..n\}.\ p\ i \otimes q\ (n-i))$,
    one = $(\%i.$ *if* $i=0$ *then* $\mathbf{1}$ *else* $\mathbf{0})$,
    zero = $(\%i.\ \mathbf{0})$,
    add = $(\%p{:}up\ R.\ \%q{:}up\ R.\ \%i.\ p\ i \oplus q\ i)$,

smult = (%*a*:*carrier R*. %*p*:*up R*. %*i*. *a* ⊗ *p i*),
monom = (%*a*:*carrier R*. %*n i*. *if i*=*n then a else* **0**),
coeff = (%*p*:*up R*. %*n*. *p n*) |)

Properties of the set of polynomials *up*.

lemma *mem-upI* [*intro*]:
[| !!*n*. *f n* ∈ *carrier R*; *EX n*. *bound* (*zero R*) *n f* |] ==> *f* ∈ *up R*
⟨*proof*⟩

lemma *mem-upD* [*dest*]:
f ∈ *up R* ==> *f n* ∈ *carrier R*
⟨*proof*⟩

lemma (**in** *cring*) *bound-upD* [*dest*]:
f ∈ *up R* ==> *EX n*. *bound* **0** *n f*
⟨*proof*⟩

lemma (**in** *cring*) *up-one-closed*:
(%*n*. *if n* = *0 then* **1** *else* **0**) ∈ *up R*
⟨*proof*⟩

lemma (**in** *cring*) *up-smult-closed*:
[| *a* ∈ *carrier R*; *p* ∈ *up R* |] ==> (%*i*. *a* ⊗ *p i*) ∈ *up R*
⟨*proof*⟩

lemma (**in** *cring*) *up-add-closed*:
[| *p* ∈ *up R*; *q* ∈ *up R* |] ==> (%*i*. *p i* ⊕ *q i*) ∈ *up R*
⟨*proof*⟩

lemma (**in** *cring*) *up-a-inv-closed*:
p ∈ *up R* ==> (%*i*. ⊖ (*p i*)) ∈ *up R*
⟨*proof*⟩

lemma (**in** *cring*) *up-mult-closed*:
[| *p* ∈ *up R*; *q* ∈ *up R* |] ==>
(%*n*. ⨁ *i* ∈ {..*n*}. *p i* ⊗ *q* (*n*−*i*)) ∈ *up R*
⟨*proof*⟩

11.2 Effect of Operations on Coefficients

locale *UP* =
fixes *R* (**structure**) **and** *P* (**structure**)
defines *P-def*: *P* == *UP R*

locale *UP-cring* = *UP* + *cring R*

locale *UP-domain* = *UP-cring* + *domain R*

Temporarily declare *P* ≡ *UP R* as simp rule.

declare (**in** *UP*) *P-def* [*simp*]

lemma (**in** *UP-cring*) *coeff-monom* [*simp*]:
a ∈ *carrier R* ==>
coeff P (*monom P a m*) *n* = (*if m*=*n then a else* **0**)
⟨*proof*⟩

lemma (**in** *UP-cring*) *coeff-zero* [*simp*]:
coeff P $\mathbf{0}_P$ *n* = **0**
⟨*proof*⟩

lemma (**in** *UP-cring*) *coeff-one* [*simp*]:
coeff P $\mathbf{1}_P$ *n* = (*if n*=*0 then* **1** *else* **0**)
⟨*proof*⟩

lemma (**in** *UP-cring*) *coeff-smult* [*simp*]:
[| *a* ∈ *carrier R*; *p* ∈ *carrier P* |] ==>
coeff P (*a* $\odot_P$ *p*) *n* = *a* ⊗ *coeff P p n*
⟨*proof*⟩

lemma (**in** *UP-cring*) *coeff-add* [*simp*]:
[| *p* ∈ *carrier P*; *q* ∈ *carrier P* |] ==>
coeff P (*p* $\oplus_P$ *q*) *n* = *coeff P p n* ⊕ *coeff P q n*
⟨*proof*⟩

lemma (**in** *UP-cring*) *coeff-mult* [*simp*]:
[| *p* ∈ *carrier P*; *q* ∈ *carrier P* |] ==>
coeff P (*p* $\otimes_P$ *q*) *n* = ($\bigoplus$ *i* ∈ {..*n*}. *coeff P p i* ⊗ *coeff P q* (*n*−*i*))
⟨*proof*⟩

lemma (**in** *UP*) *up-eqI*:
assumes *prem*: !!*n*. *coeff P p n* = *coeff P q n*
and *R*: *p* ∈ *carrier P q* ∈ *carrier P*
shows *p* = *q*
⟨*proof*⟩

11.3 Polynomials Form a Commutative Ring.

Operations are closed over *P*.

lemma (**in** *UP-cring*) *UP-mult-closed* [*simp*]:
[| *p* ∈ *carrier P*; *q* ∈ *carrier P* |] ==> *p* $\otimes_P$ *q* ∈ *carrier P*
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-one-closed* [*simp*]:
$\mathbf{1}_P$ ∈ *carrier P*
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-zero-closed* [*intro*, *simp*]:
$\mathbf{0}_P$ ∈ *carrier P*

⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-a-closed* [*intro*, *simp*]:
[| $p \in$ *carrier* P; $q \in$ *carrier* P |] ==> $p \oplus_P q \in$ *carrier* P
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-closed* [*simp*]:
$a \in$ *carrier* R ==> *monom* P a $n \in$ *carrier* P
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-smult-closed* [*simp*]:
[| $a \in$ *carrier* R; $p \in$ *carrier* P |] ==> $a \odot_P p \in$ *carrier* P
⟨*proof*⟩

lemma (**in** *UP*) *coeff-closed* [*simp*]:
$p \in$ *carrier* P ==> *coeff* P p $n \in$ *carrier* R
⟨*proof*⟩

declare (**in** *UP*) *P-def* [*simp del*]

Algebraic ring properties

lemma (**in** *UP-cring*) *UP-a-assoc*:
assumes R: $p \in$ *carrier* P $q \in$ *carrier* P $r \in$ *carrier* P
shows $(p \oplus_P q) \oplus_P r = p \oplus_P (q \oplus_P r)$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-l-zero* [*simp*]:
assumes R: $p \in$ *carrier* P
shows $\mathbf{0}_P \oplus_P p = p$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-l-neg-ex*:
assumes R: $p \in$ *carrier* P
shows *EX* q : *carrier* P. $q \oplus_P p = \mathbf{0}_P$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-a-comm*:
assumes R: $p \in$ *carrier* P $q \in$ *carrier* P
shows $p \oplus_P q = q \oplus_P p$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-m-assoc*:
assumes R: $p \in$ *carrier* P $q \in$ *carrier* P $r \in$ *carrier* P
shows $(p \otimes_P q) \otimes_P r = p \otimes_P (q \otimes_P r)$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-l-one* [*simp*]:
assumes R: $p \in$ *carrier* P
shows $\mathbf{1}_P \otimes_P p = p$

⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-l-distr*:
assumes *R*: $p \in carrier\ P\ q \in carrier\ P\ r \in carrier\ P$
shows $(p \oplus_P q) \otimes_P r = (p \otimes_P r) \oplus_P (q \otimes_P r)$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-m-comm*:
assumes *R*: $p \in carrier\ P\ q \in carrier\ P$
shows $p \otimes_P q = q \otimes_P p$
⟨*proof*⟩

theorem (**in** *UP-cring*) *UP-cring*:
cring P
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-ring*:

ring P
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-a-inv-closed* [*intro*, *simp*]:
$p \in carrier\ P \Longrightarrow \ominus_P p \in carrier\ P$
⟨*proof*⟩

lemma (**in** *UP-cring*) *coeff-a-inv* [*simp*]:
assumes *R*: $p \in carrier\ P$
shows $coeff\ P\ (\ominus_P p)\ n = \ominus\ (coeff\ P\ p\ n)$
⟨*proof*⟩

Interpretation of lemmas from *cring*. Saves lifting 43 lemmas manually.

interpretation *UP-cring* < *cring P*
⟨*proof*⟩

11.4 Polynomials Form an Algebra

lemma (**in** *UP-cring*) *UP-smult-l-distr*:
$[|\ a \in carrier\ R;\ b \in carrier\ R;\ p \in carrier\ P\ |] \Longrightarrow$
$(a \oplus b) \odot_P p = a \odot_P p \oplus_P b \odot_P p$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-smult-r-distr*:
$[|\ a \in carrier\ R;\ p \in carrier\ P;\ q \in carrier\ P\ |] \Longrightarrow$
$a \odot_P (p \oplus_P q) = a \odot_P p \oplus_P a \odot_P q$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-smult-assoc1*:
$[|\ a \in carrier\ R;\ b \in carrier\ R;\ p \in carrier\ P\ |] \Longrightarrow$
$(a \otimes b) \odot_P p = a \odot_P (b \odot_P p)$

⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-smult-one* [*simp*]:
  $p \in carrier\ P \Longrightarrow \mathbf{1} \odot_P p = p$
⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-smult-assoc2*:
  [| $a \in carrier\ R$; $p \in carrier\ P$; $q \in carrier\ P$ |] ==>
  $(a \odot_P p) \otimes_P q = a \odot_P (p \otimes_P q)$
  ⟨*proof*⟩

Interpretation of lemmas from *algebra*.

lemma (**in** *cring*) *cring*:
  cring R
  ⟨*proof*⟩

lemma (**in** *UP-cring*) *UP-algebra*:
  algebra R P
  ⟨*proof*⟩

interpretation *UP-cring* < *algebra R P*
  ⟨*proof*⟩

11.5 Further Lemmas Involving Monomials

lemma (**in** *UP-cring*) *monom-zero* [*simp*]:
  $monom\ P\ \mathbf{0}\ n = \mathbf{0}_P$
  ⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-mult-is-smult*:
  assumes *R*: $a \in carrier\ R\ p \in carrier\ P$
  shows $monom\ P\ a\ 0 \otimes_P p = a \odot_P p$
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-add* [*simp*]:
  [| $a \in carrier\ R$; $b \in carrier\ R$ |] ==>
  $monom\ P\ (a \oplus b)\ n = monom\ P\ a\ n \oplus_P monom\ P\ b\ n$
  ⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-one-Suc*:
  $monom\ P\ \mathbf{1}\ (Suc\ n) = monom\ P\ \mathbf{1}\ n \otimes_P monom\ P\ \mathbf{1}\ 1$
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-mult-smult*:
  [| $a \in carrier\ R$; $b \in carrier\ R$ |] ==> $monom\ P\ (a \otimes b)\ n = a \odot_P monom\ P\ b\ n$
  ⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-one* [*simp*]:

monom P **1** *0* = $\mathbf{1}_P$
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-one-mult*:
monom P **1** (*n* + *m*) = *monom P* **1** *n* $\otimes_P$ *monom P* **1** *m*
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-mult* [*simp*]:
assumes *R*: *a* ∈ *carrier R b* ∈ *carrier R*
shows *monom P* (*a* ⊗ *b*) (*n* + *m*) = *monom P a n* $\otimes_P$ *monom P b m*
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-a-inv* [*simp*]:
a ∈ *carrier R* ==> *monom P* (⊖ *a*) *n* = $\ominus_P$ *monom P a n*
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-inj*:
inj-on (%*a*. *monom P a n*) (*carrier R*)
⟨*proof*⟩

11.6 The Degree Function

constdefs (**structure** *R*)
deg :: [('*a*, '*m*) *ring-scheme*, *nat* => '*a*] => *nat*
deg R p == *LEAST n*. *bound* **0** *n* (*coeff* (*UP R*) *p*)

lemma (**in** *UP-cring*) *deg-aboveI*:
[| (!!*m*. *n* < *m* ==> *coeff P p m* = **0**); *p* ∈ *carrier P* |] ==> *deg R p* <= *n*
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-aboveD*:
assumes *deg R p* < *m* **and** *p* ∈ *carrier P*
shows *coeff P p m* = **0**
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-belowI*:
assumes *non-zero*: *n* ~= *0* ==> *coeff P p n* ~= **0**
and *R*: *p* ∈ *carrier P*
shows *n* <= *deg R p*
— Logically, this is a slightly stronger version of *deg-aboveD*
⟨*proof*⟩

lemma (**in** *UP-cring*) *lcoeff-nonzero-deg*:
assumes *deg*: *deg R p* ~= *0* **and** *R*: *p* ∈ *carrier P*
shows *coeff P p* (*deg R p*) ~= **0**
⟨*proof*⟩

lemma (**in** *UP-cring*) *lcoeff-nonzero-nonzero*:
assumes *deg*: *deg R p = 0* **and** *nonzero*: $p \neq \mathbf{0}_P$ **and** *R*: $p \in$ *carrier P*
shows *coeff P p 0* $\neq \mathbf{0}$
⟨*proof*⟩

lemma (**in** *UP-cring*) *lcoeff-nonzero*:
assumes *neq*: $p \neq \mathbf{0}_P$ **and** *R*: $p \in$ *carrier P*
shows *coeff P p* (*deg R p*) $\neq \mathbf{0}$
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-eqI*:
[| !!*m*. *n < m ==> coeff P p m* = $\mathbf{0}$;
!!*n*. *n* $\neq$ *0 ==> coeff P p n* $\neq \mathbf{0}$; $p \in$ *carrier P* |] *==> deg R p = n*
⟨*proof*⟩

Degree and polynomial operations

lemma (**in** *UP-cring*) *deg-add* [*simp*]:
assumes *R*: $p \in$ *carrier P* $q \in$ *carrier P*
shows *deg R* ($p \oplus_P q$) *<= max* (*deg R p*) (*deg R q*)
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-monom-le*:
$a \in$ *carrier R ==> deg R* (*monom P a n*) *<= n*
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-monom* [*simp*]:
[| $a \neq \mathbf{0}$; $a \in$ *carrier R* |] *==> deg R* (*monom P a n*) = *n*
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-const* [*simp*]:
assumes *R*: $a \in$ *carrier R* **shows** *deg R* (*monom P a 0*) = *0*
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-zero* [*simp*]:
deg R $\mathbf{0}_P$ = *0*
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-one* [*simp*]:
deg R $\mathbf{1}_P$ = *0*
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-uminus* [*simp*]:
assumes *R*: $p \in$ *carrier P* **shows** *deg R* ($\ominus_P p$) = *deg R p*
⟨*proof*⟩

lemma (**in** *UP-domain*) *deg-smult-ring*:
[| $a \in$ *carrier R*; $p \in$ *carrier P* |] *==>*
deg R ($a \odot_P p$) *<=* (*if a* = $\mathbf{0}$ *then 0 else deg R p*)
⟨*proof*⟩

lemma (**in** *UP-domain*) *deg-smult* [*simp*]:
assumes *R*: $a \in carrier\ R\ p \in carrier\ P$
shows $deg\ R\ (a \odot_P p) = (if\ a = \mathbf{0}\ then\ 0\ else\ deg\ R\ p)$
⟨*proof*⟩

lemma (**in** *UP-cring*) *deg-mult-cring*:
assumes *R*: $p \in carrier\ P\ q \in carrier\ P$
shows $deg\ R\ (p \otimes_P q) <= deg\ R\ p + deg\ R\ q$
⟨*proof*⟩

lemma (**in** *UP-domain*) *deg-mult* [*simp*]:
$[|\ p \sim= \mathbf{0}_P;\ q \sim= \mathbf{0}_P;\ p \in carrier\ P;\ q \in carrier\ P\ |] ==>$
$deg\ R\ (p \otimes_P q) = deg\ R\ p + deg\ R\ q$
⟨*proof*⟩

lemma (**in** *UP-cring*) *coeff-finsum*:
assumes *fin*: *finite A*
shows $p \in A -> carrier\ P ==>$
$coeff\ P\ (finsum\ P\ p\ A)\ k = (\bigoplus i \in A.\ coeff\ P\ (p\ i)\ k)$
⟨*proof*⟩

lemma (**in** *UP-cring*) *up-repr*:
assumes *R*: $p \in carrier\ P$
shows $(\bigoplus_P i \in \{..deg\ R\ p\}.\ monom\ P\ (coeff\ P\ p\ i)\ i) = p$
⟨*proof*⟩

lemma (**in** *UP-cring*) *up-repr-le*:
$[|\ deg\ R\ p <= n;\ p \in carrier\ P\ |] ==>$
$(\bigoplus_P i \in \{..n\}.\ monom\ P\ (coeff\ P\ p\ i)\ i) = p$
⟨*proof*⟩

11.7 Polynomials over Integral Domains

lemma *domainI*:
assumes *cring*: *cring R*
and *one-not-zero*: $one\ R \sim= zero\ R$
and *integral*: $!!a\ b.\ [|\ mult\ R\ a\ b = zero\ R;\ a \in carrier\ R;$
$b \in carrier\ R\ |] ==> a = zero\ R \mid b = zero\ R$
shows *domain R*
⟨*proof*⟩

lemma (**in** *UP-domain*) *UP-one-not-zero*:
$\mathbf{1}_P \sim= \mathbf{0}_P$
⟨*proof*⟩

lemma (**in** *UP-domain*) *UP-integral*:
$[|\ p \otimes_P q = \mathbf{0}_P;\ p \in carrier\ P;\ q \in carrier\ P\ |] ==> p = \mathbf{0}_P \mid q = \mathbf{0}_P$
⟨*proof*⟩

theorem (**in** *UP-domain*) *UP-domain*:
domain P
⟨*proof*⟩

Interpretation of theorems from *domain*.

interpretation *UP-domain* < *domain P*
⟨*proof*⟩

11.8 The Evaluation Homomorphism and Universal Property

theorem (**in** *cring*) *diagonal-sum*:
[| $f \in \{..n + m::nat\} \to carrier\ R$; $g \in \{..n + m\} \to carrier\ R$ |] ==>
$(\bigoplus k \in \{..n + m\}. \bigoplus i \in \{..k\}. f\ i \otimes g\ (k - i)) =$
$(\bigoplus k \in \{..n + m\}. \bigoplus i \in \{..n + m - k\}. f\ k \otimes g\ i)$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *boundD-carrier*:
[| *bound* **0** *n f*; $n < m$ |] ==> $f\ m \in carrier\ G$
⟨*proof*⟩

theorem (**in** *cring*) *cauchy-product*:
assumes *bf*: *bound* **0** *n f* **and** *bg*: *bound* **0** *m g*
and *Rf*: $f \in \{..n\} \to carrier\ R$ **and** *Rg*: $g \in \{..m\} \to carrier\ R$
shows $(\bigoplus k \in \{..n + m\}. \bigoplus i \in \{..k\}. f\ i \otimes g\ (k - i)) =$
$(\bigoplus i \in \{..n\}. f\ i) \otimes (\bigoplus i \in \{..m\}. g\ i)$
⟨*proof*⟩

lemma (**in** *UP-cring*) *const-ring-hom*:
$(\%a.\ monom\ P\ a\ 0) \in ring\text{-}hom\ R\ P$
⟨*proof*⟩

constdefs (**structure** *S*)
eval :: [('*a*, '*m*) *ring-scheme*, ('*b*, '*n*) *ring-scheme*,
'*a* => '*b*, '*b*, *nat* => '*a*] => '*b*
eval R S phi s == $\lambda p \in carrier\ (UP\ R).$
$\bigoplus i \in \{..deg\ R\ p\}.\ phi\ (coeff\ (UP\ R)\ p\ i) \otimes s\ (\hat{}\ )\ i$

lemma (**in** *UP*) *eval-on-carrier*:
fixes *S* (**structure**)
shows $p \in carrier\ P$ ==>
eval R S phi s p = $(\bigoplus_S i \in \{..deg\ R\ p\}.\ phi\ (coeff\ P\ p\ i) \otimes_S s\ (\hat{}\ )_S\ i)$
⟨*proof*⟩

lemma (**in** *UP*) *eval-extensional*:
eval R S phi p $\in$ *extensional* (*carrier P*)
⟨*proof*⟩

The universal property of the polynomial ring

locale *UP-pre-univ-prop* = *ring-hom-cring R S h* + *UP-cring R P*

locale *UP-univ-prop* = *UP-pre-univ-prop* +
  fixes *s* **and** *Eval*
  assumes *indet-img-carrier* [*simp*, *intro*]: $s \in carrier\ S$
  defines *Eval-def*: *Eval* == *eval R S h s*

theorem (**in** *UP-pre-univ-prop*) *eval-ring-hom*:
  assumes *S*: $s \in carrier\ S$
  shows *eval R S h s* $\in$ *ring-hom P S*
⟨*proof*⟩

Interpretation of ring homomorphism lemmas.

interpretation *UP-univ-prop* < *ring-hom-cring P S Eval*
  ⟨*proof*⟩

Further properties of the evaluation homomorphism.

The following lemma could be proved in *UP-cring* with the additional assumption that h is closed.

lemma (**in** *UP-pre-univ-prop*) *eval-const*:
  [| $s \in carrier\ S$; $r \in carrier\ R$ |] ==> *eval R S h s* (*monom P r 0*) = *h r*
  ⟨*proof*⟩

The following proof is complicated by the fact that in arbitrary rings one might have $\mathbf{1}_R = \mathbf{0}_R$.

lemma (**in** *UP-pre-univ-prop*) *eval-monom1*:
  assumes *S*: $s \in carrier\ S$
  shows *eval R S h s* (*monom P* **1** *1*) = *s*
⟨*proof*⟩

lemma (**in** *UP-cring*) *monom-pow*:
  assumes *R*: $a \in carrier\ R$
  shows (*monom P a n*) (^)$_P$ *m* = *monom P* (*a* (^) *m*) (*n* * *m*)
⟨*proof*⟩

lemma (**in** *ring-hom-cring*) *hom-pow* [*simp*]:
  $x \in carrier\ R$ ==> *h* (*x* (^) *n*) = *h x* (^)$_S$ (*n*::*nat*)
  ⟨*proof*⟩

lemma (**in** *UP-univ-prop*) *Eval-monom*:
  $r \in carrier\ R$ ==> *Eval* (*monom P r n*) = *h r* $\otimes_S$ *s* (^)$_S$ *n*
⟨*proof*⟩

lemma (**in** *UP-pre-univ-prop*) *eval-monom*:
  assumes *R*: $r \in carrier\ R$ **and** *S*: $s \in carrier\ S$
  shows *eval R S h s* (*monom P r n*) = *h r* $\otimes_S$ *s* (^)$_S$ *n*

⟨*proof*⟩

lemma (**in** *UP-univ-prop*) *Eval-smult*:
[| $r \in$ *carrier* R; $p \in$ *carrier* P |] ==> *Eval* $(r \odot_P p) = h\ r \otimes_S$ *Eval* p
⟨*proof*⟩

lemma *ring-hom-cringI*:
assumes *cring* R
and *cring* S
and $h \in$ *ring-hom* $R\ S$
shows *ring-hom-cring* $R\ S\ h$
⟨*proof*⟩

lemma (**in** *UP-pre-univ-prop*) *UP-hom-unique*:
includes *ring-hom-cring* $P\ S$ *Phi*
assumes *Phi*: *Phi* (*monom* P **1** (*Suc* 0)) $= s$
!!r. $r \in$ *carrier* R ==> *Phi* (*monom* $P\ r$ 0) $= h\ r$
includes *ring-hom-cring* $P\ S$ *Psi*
assumes *Psi*: *Psi* (*monom* P **1** (*Suc* 0)) $= s$
!!r. $r \in$ *carrier* R ==> *Psi* (*monom* $P\ r$ 0) $= h\ r$
and P: $p \in$ *carrier* P **and** S: $s \in$ *carrier* S
shows *Phi* $p =$ *Psi* p
⟨*proof*⟩

lemma (**in** *UP-pre-univ-prop*) *ring-homD*:
assumes *Phi*: *Phi* $\in$ *ring-hom* $P\ S$
shows *ring-hom-cring* $P\ S$ *Phi*
⟨*proof*⟩

theorem (**in** *UP-pre-univ-prop*) *UP-universal-property*:
assumes S: $s \in$ *carrier* S
shows *EX*! *Phi*. *Phi* $\in$ *ring-hom* $P\ S \cap$ *extensional* (*carrier* P) &
Phi (*monom* P **1** *1*) $= s$ &
(*ALL* r : *carrier* R. *Phi* (*monom* $P\ r$ 0) $= h\ r$)
⟨*proof*⟩

11.9 Sample Application of Evaluation Homomorphism

lemma *UP-pre-univ-propI*:
assumes *cring* R
and *cring* S
and $h \in$ *ring-hom* $R\ S$
shows *UP-pre-univ-prop* $R\ S\ h$
⟨*proof*⟩

constdefs
INTEG :: *int ring*
INTEG == (| *carrier* = *UNIV*, *mult* = *op* $*$, *one* = *1*, *zero* = *0*, *add* = *op* + |)

lemma *INTEG-cring*:
cring INTEG
⟨*proof*⟩

lemma *INTEG-id-eval*:
UP-pre-univ-prop INTEG INTEG id
⟨*proof*⟩

Interpretation now enables to import all theorems and lemmas valid in the context of homomorphisms between *INTEG* and *UP INTEG* globally.

interpretation *INTEG*: *UP-pre-univ-prop* [*INTEG INTEG id*]
⟨*proof*⟩

lemma *INTEG-closed* [*intro*, *simp*]:
$z \in$ *carrier INTEG*
⟨*proof*⟩

lemma *INTEG-mult* [*simp*]:
mult INTEG z w = $z * w$
⟨*proof*⟩

lemma *INTEG-pow* [*simp*]:
pow INTEG z n = $z \hat{} n$
⟨*proof*⟩

lemma *eval INTEG INTEG id 10* (*monom* (*UP INTEG*) *5 2*) = *500*
⟨*proof*⟩

end

theory *AbelCoset*
imports *Coset Ring*
begin

12 More Lifting from Groups to Abelian Groups

12.1 Definitions

Hiding <+> from *Sum-Type* until I come up with better syntax here

hide *const Plus*

constdefs (**structure** *G*)
a-r-coset :: [-, 'a *set*, 'a] ⇒ 'a *set* (**infixl** +>$_1$ *60*)
a-r-coset G ≡ *r-coset* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

a-l-coset :: [-, *'a*, *'a set*] ⇒ *'a set* (**infixl** <+₁ *60*)
a-l-coset G ≡ *l-coset* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

A-RCOSETS :: [-, *'a set*] ⇒ (*'a set*)*set* (*a'-rcosets*₁ - [*81*] *80*)
A-RCOSETS G H ≡ *RCOSETS* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|) *H*

set-add :: [-, *'a set* ,*'a set*] ⇒ *'a set* (**infixl** <+>₁ *60*)
set-add G ≡ *set-mult* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

A-SET-INV :: [-,*'a set*] ⇒ *'a set* (*a'-set'-inv*₁ - [*81*] *80*)
A-SET-INV G H ≡ *SET-INV* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|) *H*

constdefs (**structure** *G*)
a-r-congruent :: [(*'a*,*'b*)*ring-scheme*, *'a set*] ⇒ (*'a*∗*'a*)*set*
(*racong*₁ -)
a-r-congruent G ≡ *r-congruent* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

constdefs
A-FactGroup :: [(*'a*,*'b*) *ring-scheme*, *'a set*] ⇒ (*'a set*) *monoid*
(**infixl** *A'-Mod 65*)
— Actually defined for groups rather than monoids
A-FactGroup G H ≡ *FactGroup* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|) *H*

constdefs
a-kernel :: (*'a*, *'m*) *ring-scheme* ⇒ (*'b*, *'n*) *ring-scheme* ⇒
(*'a* ⇒ *'b*) ⇒ *'a set*
— the kernel of a homomorphism (additive)
a-kernel G H h ≡ *kernel* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)
(|*carrier* = *carrier H*, *mult* = *add H*, *one* = *zero H*|) *h*

locale *abelian-group-hom* = *abelian-group G* + *abelian-group H* + *var h* +
assumes *a-group-hom*: *group-hom* (| *carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G* |)
(| *carrier* = *carrier H*, *mult* = *add H*, *one* = *zero H* |) *h*

lemmas *a-r-coset-defs* =
a-r-coset-def r-coset-def

lemma *a-r-coset-def'*:
includes *struct G*
shows *H* +> *a* ≡ ⋃*h*∈*H*. {*h* ⊕ *a*}
⟨*proof*⟩

lemmas *a-l-coset-defs* =
a-l-coset-def l-coset-def

lemma *a-l-coset-def'*:
includes *struct G*
shows $a <+ H \equiv \bigcup h \in H.\ \{a \oplus h\}$
⟨*proof*⟩

lemmas *A-RCOSETS-defs* =
A-RCOSETS-def RCOSETS-def

lemma *A-RCOSETS-def'*:
includes *struct G*
shows *a-rcosets H* $\equiv \bigcup a \in$ *carrier G*. $\{H +> a\}$
⟨*proof*⟩

lemmas *set-add-defs* =
set-add-def set-mult-def

lemma *set-add-def'*:
includes *struct G*
shows $H <+> K \equiv \bigcup h \in H.\ \bigcup k \in K.\ \{h \oplus k\}$
⟨*proof*⟩

lemmas *A-SET-INV-defs* =
A-SET-INV-def SET-INV-def

lemma *A-SET-INV-def'*:
includes *struct G*
shows *a-set-inv H* $\equiv \bigcup h \in H.\ \{\ominus h\}$
⟨*proof*⟩

12.2 Cosets

lemma (**in** *abelian-group*) *a-coset-add-assoc*:
$[|\ M \subseteq$ *carrier G*; $g \in$ *carrier G*; $h \in$ *carrier G* $|]$
$==> (M +> g) +> h = M +> (g \oplus h)$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-coset-add-zero* [*simp*]:
$M \subseteq$ *carrier G* $==> M +> \mathbf{0} = M$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-coset-add-inv1*:
$[|\ M +> (x \oplus (\ominus y)) = M;\ x \in$ *carrier G* ; $y \in$ *carrier G*;
$M \subseteq$ *carrier G* $|] ==> M +> x = M +> y$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-coset-add-inv2*:
$[|\ M +> x = M +> y;\ x \in$ *carrier G*; $y \in$ *carrier G*; $M \subseteq$ *carrier G* $|]$
$==> M +> (x \oplus (\ominus y)) = M$

⟨*proof*⟩

lemma (**in** *abelian-group*) *a-coset-join1*:
[| H +> $x = H$; $x \in$ *carrier* G; *subgroup* H (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|) |] ==> $x \in H$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-solve-equation*:
⟦*subgroup* H (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|); $x \in H$; $y \in H$⟧ ⟹ ∃h∈H. $y = h \oplus x$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-repr-independence*:
⟦$y \in H$ +> x; $x \in$ *carrier* G; *subgroup* H (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|) ⟧ ⟹ H +> $x = H$ +> y
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-coset-join2*:
⟦$x \in$ *carrier* G; *subgroup* H (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|); x∈H⟧ ⟹ H +> $x = H$
⟨*proof*⟩

lemma (**in** *abelian-monoid*) *a-r-coset-subset-G*:
[| $H \subseteq$ *carrier* G; $x \in$ *carrier* G |] ==> H +> $x \subseteq$ *carrier* G
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-rcosI*:
[| $h \in H$; $H \subseteq$ *carrier* G; $x \in$ *carrier* G|] ==> $h \oplus x \in H$ +> x
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-rcosetsI*:
⟦$H \subseteq$ *carrier* G; $x \in$ *carrier* G⟧ ⟹ H +> $x \in$ *a-rcosets* H
⟨*proof*⟩

Really needed?

lemma (**in** *abelian-group*) *a-transpose-inv*:
[| $x \oplus y = z$; $x \in$ *carrier* G; $y \in$ *carrier* G; $z \in$ *carrier* G |]
==> (⊖ x) ⊕ $z = y$
⟨*proof*⟩

12.3 Subgroups

locale *additive-subgroup* = *var* H + *struct* G +
assumes *a-subgroup*: *subgroup* H (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|)

lemma (**in** *additive-subgroup*) *is-additive-subgroup*:
shows *additive-subgroup* H G
⟨*proof*⟩

lemma *additive-subgroupI*:
includes *struct G*
assumes *a-subgroup*: *subgroup H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)
shows *additive-subgroup H G*
⟨*proof*⟩

lemma (**in** *additive-subgroup*) *a-subset*:
$H \subseteq$ *carrier G*
⟨*proof*⟩

lemma (**in** *additive-subgroup*) *a-closed* [*intro*, *simp*]:
$[\![x \in H; y \in H]\!] \Longrightarrow x \oplus y \in H$
⟨*proof*⟩

lemma (**in** *additive-subgroup*) *zero-closed* [*simp*]:
$\mathbf{0} \in H$
⟨*proof*⟩

lemma (**in** *additive-subgroup*) *a-inv-closed* [*intro*,*simp*]:
$x \in H \Longrightarrow \ominus x \in H$
⟨*proof*⟩

12.4 Normal additive subgroups

12.4.1 Definition of *abelian-subgroup*

Every subgroup of an *abelian-group* is normal

locale *abelian-subgroup* = *additive-subgroup H G* + *abelian-group G* +
assumes *a-normal*: *normal H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

lemma (**in** *abelian-subgroup*) *is-abelian-subgroup*:
shows *abelian-subgroup H G*
⟨*proof*⟩

lemma *abelian-subgroupI*:
assumes *a-normal*: *normal H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)
and *a-comm*: !!*x y*. [| *x* ∈ *carrier G*; *y* ∈ *carrier G* |] ==> $x \oplus_G y = y \oplus_G x$
shows *abelian-subgroup H G*
⟨*proof*⟩

lemma *abelian-subgroupI2*:
includes *struct G*
assumes *a-comm-group*: *comm-group* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)

and *a-subgroup*: *subgroup H* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)
shows *abelian-subgroup H G*
⟨*proof*⟩

lemma *abelian-subgroupI3*:
includes *struct G*
assumes *asg*: *additive-subgroup H G*
and *ag*: *abelian-group G*
shows *abelian-subgroup H G*
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-coset-eq*:
(∀ *x* ∈ *carrier G*. *H* +> *x* = *x* <+ *H*)
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-inv-op-closed1*:
shows ⟦*x* ∈ *carrier G*; *h* ∈ *H*⟧ ⟹ (⊖ *x*) ⊕ *h* ⊕ *x* ∈ *H*
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-inv-op-closed2*:
shows ⟦*x* ∈ *carrier G*; *h* ∈ *H*⟧ ⟹ *x* ⊕ *h* ⊕ (⊖ *x*) ∈ *H*
⟨*proof*⟩

Alternative characterization of normal subgroups

lemma (**in** *abelian-group*) *a-normal-inv-iff*:
(*N* ◁ (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|)) =
(*subgroup N* (|*carrier* = *carrier G*, *mult* = *add G*, *one* = *zero G*|) & (∀ *x* ∈ *carrier G*. ∀ *h* ∈ *N*. *x* ⊕ *h* ⊕ (⊖ *x*) ∈ *N*))
(**is** - = *?rhs*)
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-lcos-m-assoc*:
[| *M* ⊆ *carrier G*; *g* ∈ *carrier G*; *h* ∈ *carrier G* |]
==> *g* <+ (*h* <+ *M*) = (*g* ⊕ *h*) <+ *M*
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-lcos-mult-one*:
M ⊆ *carrier G* ==> **0** <+ *M* = *M*
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-l-coset-subset-G*:
[| *H* ⊆ *carrier G*; *x* ∈ *carrier G* |] ==> *x* <+ *H* ⊆ *carrier G*
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-l-coset-swap*:
⟦*y* ∈ *x* <+ *H*; *x* ∈ *carrier G*; *subgroup H* (|*carrier* = *carrier G*, *mult* = *add*

G, *one* = *zero* G|)]] $\Longrightarrow$ $x \in y <+ H$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-l-coset-carrier*:
[| $y \in x <+ H$; $x \in$ *carrier* G; *subgroup* H (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|) |] ==> $y \in$ *carrier* G
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-l-repr-imp-subset*:
assumes *y*: $y \in x <+ H$ **and** *x*: $x \in$ *carrier* G **and** *sb*: *subgroup* H (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|)
shows $y <+ H \subseteq x <+ H$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-l-repr-independence*:
assumes *y*: $y \in x <+ H$ **and** *x*: $x \in$ *carrier* G **and** *sb*: *subgroup* H (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|)
shows $x <+ H = y <+ H$
⟨*proof*⟩

lemma (**in** *abelian-group*) *setadd-subset-G*:
[|$H \subseteq$ *carrier* G; $K \subseteq$ *carrier* G|] $\Longrightarrow$ $H <+> K \subseteq$ *carrier* G
⟨*proof*⟩

lemma (**in** *abelian-group*) *subgroup-add-id*: *subgroup* H (|*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G|) $\Longrightarrow$ $H <+> H = H$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-inv*:
assumes *x*: $x \in$ *carrier* G
shows *a-set-inv* $(H +> x) = H +> (\ominus x)$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-setmult-rcos-assoc*:
[|$H \subseteq$ *carrier* G; $K \subseteq$ *carrier* G; $x \in$ *carrier* G|]
$\Longrightarrow$ $H <+> (K +> x) = (H <+> K) +> x$
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-rcos-assoc-lcos*:
[|$H \subseteq$ *carrier* G; $K \subseteq$ *carrier* G; $x \in$ *carrier* G|]
$\Longrightarrow$ $(H +> x) <+> K = H <+> (x <+ K)$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-sum*:
[|$x \in$ *carrier* G; $y \in$ *carrier* G|]
$\Longrightarrow$ $(H +> x) <+> (H +> y) = H +> (x \oplus y)$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *rcosets-add-eq*:

$M \in$ *a-rcosets* $H \Longrightarrow H <+> M = M$
— generalizes *subgroup-mult-id*
⟨*proof*⟩

12.5 Congruence Relation

lemma (**in** *abelian-subgroup*) *a-equiv-rcong*:
shows *equiv* (*carrier* G) (*racong* H)
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-l-coset-eq-rcong*:
assumes *a*: $a \in$ *carrier* G
shows $a <+ H =$ *racong* H `` $\{a\}$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-equation*:
shows
$[\![ha \oplus a = h \oplus b;\ a \in$ *carrier* $G;\ b \in$ *carrier* $G;$
$h \in H;\ ha \in H;\ hb \in H]\!]$
$\Longrightarrow hb \oplus a \in (\bigcup h \in H.\ \{h \oplus b\})$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-disjoint*:
shows $[\![a \in$ *a-rcosets* $H;\ b \in$ *a-rcosets* $H;\ a \neq b]\!] \Longrightarrow a \cap b = \{\}$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-self*:
shows $x \in$ *carrier* $G \Longrightarrow x \in H +> x$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcosets-part-G*:
shows $\bigcup$(*a-rcosets* H) = *carrier* G
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-cosets-finite*:
$[\![c \in$ *a-rcosets* $H;\ H \subseteq$ *carrier* $G;$ *finite* (*carrier* G)$]\!] \Longrightarrow$ *finite* c
⟨*proof*⟩

lemma (**in** *abelian-group*) *a-card-cosets-equal*:
$[\![c \in$ *a-rcosets* $H;\ H \subseteq$ *carrier* G; *finite*(*carrier* G)$]\!]$
$\Longrightarrow$ *card* c = *card* H
⟨*proof*⟩

lemma (**in** *abelian-group*) *rcosets-subset-PowG*:
additive-subgroup H $G \Longrightarrow$ *a-rcosets* $H \subseteq$ *Pow*(*carrier* G)
⟨*proof*⟩

theorem (**in** *abelian-group*) *a-lagrange*:
$[\![$*finite*(*carrier* G); *additive-subgroup* H $G]\!]$

$\Longrightarrow$ *card*(*a-rcosets H*) $*$ *card*(*H*) = *order*(*G*)
⟨*proof*⟩

12.6 Factorization

lemmas *A-FactGroup-defs* = *A-FactGroup-def FactGroup-def*

lemma *A-FactGroup-def′*:
includes *struct G*
shows *G A-Mod H* ≡ (|*carrier* = *a-rcosets*$_G$ *H*, *mult* = *set-add G*, *one* = *H*|)
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-setmult-closed*:
⟦*K1* ∈ *a-rcosets H*; *K2* ∈ *a-rcosets H*⟧ $\Longrightarrow$ *K1* <+> *K2* ∈ *a-rcosets H*
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-setinv-closed*:
K ∈ *a-rcosets H* $\Longrightarrow$ *a-set-inv K* ∈ *a-rcosets H*
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcosets-assoc*:
⟦*M1* ∈ *a-rcosets H*; *M2* ∈ *a-rcosets H*; *M3* ∈ *a-rcosets H*⟧
$\Longrightarrow$ *M1* <+> *M2* <+> *M3* = *M1* <+> (*M2* <+> *M3*)
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-subgroup-in-rcosets*:
H ∈ *a-rcosets H*
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcosets-inv-mult-group-eq*:
M ∈ *a-rcosets H* $\Longrightarrow$ *a-set-inv M* <+> *M* = *H*
⟨*proof*⟩

theorem (**in** *abelian-subgroup*) *a-factorgroup-is-group*:
group (*G A-Mod H*)
⟨*proof*⟩

Since the Factorization is based on an *abelian* subgroup, is results in a commutative group

theorem (**in** *abelian-subgroup*) *a-factorgroup-is-comm-group*:
comm-group (*G A-Mod H*)
⟨*proof*⟩

lemma *add-A-FactGroup* [*simp*]: *X* $\otimes_{(G\ A\text{-}Mod\ H)}$ *X′* = *X* <+>$_G$ *X′*
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-inv-FactGroup*:
X ∈ *carrier* (*G A-Mod H*) $\Longrightarrow$ *inv*$_{G\ A\text{-}Mod\ H}$ *X* = *a-set-inv X*

⟨*proof*⟩

The coset map is a homomorphism from G to the quotient group G *Mod* H

lemma (**in** *abelian-subgroup*) *a-r-coset-hom-A-Mod*:
(λ*a*. H +> a) ∈ *hom* (⦇*carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G⦈) (G *A-Mod* H)
⟨*proof*⟩

The isomorphism theorems have been omitted from lifting, at least for now

12.7 The First Isomorphism Theorem

The quotient by the kernel of a homomorphism is isomorphic to the range of that homomorphism.

lemmas *a-kernel-defs* =
a-kernel-def kernel-def

lemma *a-kernel-def'*:
a-kernel R S h ≡ {x ∈ *carrier* R. h x = $\mathbf{0}_S$}
⟨*proof*⟩

12.8 Homomorphisms

lemma *abelian-group-homI*:
includes *abelian-group* G
includes *abelian-group* H
assumes *a-group-hom*: *group-hom* (| *carrier* = *carrier* G, *mult* = *add* G, *one* = *zero* G |)
(| *carrier* = *carrier* H, *mult* = *add* H, *one* = *zero* H |) h
shows *abelian-group-hom* G H h
⟨*proof*⟩

lemma (**in** *abelian-group-hom*) *is-abelian-group-hom*:
abelian-group-hom G H h
⟨*proof*⟩

lemma (**in** *abelian-group-hom*) *hom-add* [*simp*]:
[| x : *carrier* G; y : *carrier* G |]
==> h (x ⊕$_G$ y) = h x ⊕$_H$ h y
⟨*proof*⟩

lemma (**in** *abelian-group-hom*) *hom-closed* [*simp*]:
x ∈ *carrier* G ⟹ h x ∈ *carrier* H
⟨*proof*⟩

lemma (**in** *abelian-group-hom*) *zero-closed* [*simp*]:
h **0** ∈ *carrier* H
⟨*proof*⟩

lemma (**in** *abelian-group-hom*) *hom-zero* [*simp*]:
  $h\ \mathbf{0} = \mathbf{0}_H$
⟨*proof*⟩

lemma (**in** *abelian-group-hom*) *a-inv-closed* [*simp*]:
  $x \in carrier\ G \Longrightarrow h\ (\ominus x) \in carrier\ H$
⟨*proof*⟩

lemma (**in** *abelian-group-hom*) *hom-a-inv* [*simp*]:
  $x \in carrier\ G \Longrightarrow h\ (\ominus x) = \ominus_H\ (h\ x)$
⟨*proof*⟩

lemma (**in** *abelian-group-hom*) *additive-subgroup-a-kernel*:
  additive-subgroup (*a-kernel G H h*) *G*
⟨*proof*⟩

The kernel of a homomorphism is an abelian subgroup

lemma (**in** *abelian-group-hom*) *abelian-subgroup-a-kernel*:
  abelian-subgroup (*a-kernel G H h*) *G*
⟨*proof*⟩

lemma (**in** *abelian-group-hom*) *A-FactGroup-nonempty*:
  assumes *X*: $X \in carrier\ (G\ A\text{-}Mod\ a\text{-}kernel\ G\ H\ h)$
  shows $X \neq \{\}$
⟨*proof*⟩

lemma (**in** *abelian-group-hom*) *FactGroup-contents-mem*:
  assumes *X*: $X \in carrier\ (G\ A\text{-}Mod\ (a\text{-}kernel\ G\ H\ h))$
  shows $contents\ (h`X) \in carrier\ H$
⟨*proof*⟩

lemma (**in** *abelian-group-hom*) *A-FactGroup-hom*:
    $(\lambda X.\ contents\ (h`X)) \in hom\ (G\ A\text{-}Mod\ (a\text{-}kernel\ G\ H\ h))$
        $(\!|carrier = carrier\ H,\ mult = add\ H,\ one = zero\ H|\!)$
⟨*proof*⟩

lemma (**in** *abelian-group-hom*) *A-FactGroup-inj-on*:
    inj-on $(\lambda X.\ contents\ (h\ `\ X))\ (carrier\ (G\ A\text{-}Mod\ a\text{-}kernel\ G\ H\ h))$
⟨*proof*⟩

If the homomorphism h is onto H, then so is the homomorphism from the quotient group

lemma (**in** *abelian-group-hom*) *A-FactGroup-onto*:
  assumes *h*: $h\ `\ carrier\ G = carrier\ H$
  shows $(\lambda X.\ contents\ (h\ `\ X))\ `\ carrier\ (G\ A\text{-}Mod\ a\text{-}kernel\ G\ H\ h) = carrier\ H$
⟨*proof*⟩

If h is a homomorphism from G onto H, then the quotient group $G\ Mod$

kernel G H h is isomorphic to *H*.

theorem (**in** *abelian-group-hom*) *A-FactGroup-iso*:
  h ‘ carrier G = carrier H
  $\Longrightarrow$ $(\lambda X.\ contents\ (h`X)) \in (G\ A\text{-}Mod\ (a\text{-}kernel\ G\ H\ h)) \cong$
      $(|\ carrier = carrier\ H,\ mult = add\ H,\ one = zero\ H\ |)$
⟨*proof*⟩

13 Lemmas Lifted from CosetExt.thy

Not eveything from `CosetExt.thy` is lifted here.

13.1 General Lemmas from `AlgebraExt.thy`

lemma (**in** *additive-subgroup*) *a-Hcarr* [*simp*]:
  assumes *hH*: $h \in H$
  shows $h \in carrier\ G$
⟨*proof*⟩

13.2 Lemmas for Right Cosets

lemma (**in** *abelian-subgroup*) *a-elemrcos-carrier*:
  assumes *acarr*: $a \in carrier\ G$
    and *a'*: $a' \in H +> a$
  shows $a' \in carrier\ G$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-const*:
  assumes *hH*: $h \in H$
  shows $H +> h = H$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-module-imp*:
  assumes *xcarr*: $x \in carrier\ G$
    and *x'cos*: $x' \in H +> x$
  shows $(x' \oplus \ominus x) \in H$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-module-rev*:
  assumes $x \in carrier\ G\ x' \in carrier\ G$
    and $(x' \oplus \ominus x) \in H$
  shows $x' \in H +> x$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-rcos-module*:
  assumes $x \in carrier\ G\ x' \in carrier\ G$
  shows $(x' \in H +> x) = (x' \oplus \ominus x \in H)$
⟨*proof*⟩
lemma (**in** *abelian-subgroup*) *a-rcos-module-minus*:

includes *ring G*
assumes *carr*: $x \in$ *carrier G* $x' \in$ *carrier G*
shows $(x' \in H +> x) = (x' \ominus x \in H)$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-repr-independence'*:
assumes *y*: $y \in H +> x$
and *xcarr*: $x \in$ *carrier G*
shows $H +> x = H +> y$
⟨*proof*⟩

lemma (**in** *abelian-subgroup*) *a-repr-independenceD*:
assumes *ycarr*: $y \in$ *carrier G*
and *repr*: $H +> x = H +> y$
shows $y \in H +> x$
⟨*proof*⟩

13.3 Lemmas for the Set of Right Cosets

lemma (**in** *abelian-subgroup*) *a-rcosets-carrier*:
$X \in$ *a-rcosets* $H \Longrightarrow X \subseteq$ *carrier G*
⟨*proof*⟩

13.4 Addition of Subgroups

lemma (**in** *abelian-monoid*) *set-add-closed*:
assumes *Acarr*: $A \subseteq$ *carrier G*
and *Bcarr*: $B \subseteq$ *carrier G*
shows $A <+> B \subseteq$ *carrier G*
⟨*proof*⟩

lemma (**in** *abelian-group*) *add-additive-subgroups*:
assumes *subH*: *additive-subgroup H G*
and *subK*: *additive-subgroup K G*
shows *additive-subgroup* $(H <+> K)$ *G*
⟨*proof*⟩

end

theory *Ideal*
imports *Ring AbelCoset*
begin

14 Ideals

14.1 General definition

locale *ideal* = *additive-subgroup I R* + *ring R* +
  assumes *I-l-closed*: $[\![a \in I;\ x \in carrier\ R]\!] \Longrightarrow x \otimes a \in I$
    and *I-r-closed*: $[\![a \in I;\ x \in carrier\ R]\!] \Longrightarrow a \otimes x \in I$

interpretation *ideal* $\subseteq$ *abelian-subgroup I R*
⟨*proof*⟩

lemma (**in** *ideal*) *is-ideal*:
  ideal I R
⟨*proof*⟩

lemma *idealI*:
  includes *ring*
  assumes *a-subgroup*: *subgroup I* (|*carrier* = *carrier R*, *mult* = *add R*, *one* = *zero R*|)
    and *I-l-closed*: $\bigwedge a\ x.\ [\![a \in I;\ x \in carrier\ R]\!] \Longrightarrow x \otimes a \in I$
    and *I-r-closed*: $\bigwedge a\ x.\ [\![a \in I;\ x \in carrier\ R]\!] \Longrightarrow a \otimes x \in I$
  shows *ideal I R*
  ⟨*proof*⟩

14.2 Ideals Generated by a Subset of *carrier R*

constdefs (**structure** *R*)
  genideal :: ('*a*, '*b*) *ring-scheme* ⇒ '*a set* ⇒ '*a set* (*Idl*₁ - [*80*] *79*)
  genideal R S ≡ *Inter* {*I*. *ideal I R* ∧ *S* ⊆ *I*}

14.3 Principal Ideals

locale *principalideal* = *ideal* +
  assumes *generate*: $\exists i \in carrier\ R.\ I = Idl\ \{i\}$

lemma (**in** *principalideal*) *is-principalideal*:
  shows *principalideal I R*
⟨*proof*⟩

lemma *principalidealI*:
  includes *ideal*
  assumes *generate*: $\exists i \in carrier\ R.\ I = Idl\ \{i\}$
  shows *principalideal I R*
  ⟨*proof*⟩

14.4 Maximal Ideals

locale *maximalideal* = *ideal* +
  assumes *I-notcarr*: $carrier\ R \neq I$

and *I-maximal*: $[\![ideal\ J\ R;\ I \subseteq J;\ J \subseteq carrier\ R]\!] \Longrightarrow J = I \vee J = carrier\ R$

lemma (**in** *maximalideal*) *is-maximalideal*:
shows *maximalideal I R*
⟨*proof*⟩

lemma *maximalidealI*:
includes *ideal*
assumes *I-notcarr*: $carrier\ R \neq I$
and *I-maximal*: $\bigwedge J.\ [\![ideal\ J\ R;\ I \subseteq J;\ J \subseteq carrier\ R]\!] \Longrightarrow J = I \vee J = carrier\ R$
shows *maximalideal I R*
⟨*proof*⟩

14.5 Prime Ideals

locale *primeideal* = *ideal* + *cring* +
assumes *I-notcarr*: $carrier\ R \neq I$
and *I-prime*: $[\![a \in carrier\ R;\ b \in carrier\ R;\ a \otimes b \in I]\!] \Longrightarrow a \in I \vee b \in I$

lemma (**in** *primeideal*) *is-primeideal*:
shows *primeideal I R*
⟨*proof*⟩

lemma *primeidealI*:
includes *ideal*
includes *cring*
assumes *I-notcarr*: $carrier\ R \neq I$
and *I-prime*: $\bigwedge a\ b.\ [\![a \in carrier\ R;\ b \in carrier\ R;\ a \otimes b \in I]\!] \Longrightarrow a \in I \vee b \in I$
shows *primeideal I R*
⟨*proof*⟩

lemma *primeidealI2*:
includes *additive-subgroup I R*
includes *cring*
assumes *I-l-closed*: $\bigwedge a\ x.\ [\![a \in I;\ x \in carrier\ R]\!] \Longrightarrow x \otimes a \in I$
and *I-r-closed*: $\bigwedge a\ x.\ [\![a \in I;\ x \in carrier\ R]\!] \Longrightarrow a \otimes x \in I$
and *I-notcarr*: $carrier\ R \neq I$
and *I-prime*: $\bigwedge a\ b.\ [\![a \in carrier\ R;\ b \in carrier\ R;\ a \otimes b \in I]\!] \Longrightarrow a \in I \vee b \in I$
shows *primeideal I R*
⟨*proof*⟩

15 Properties of Ideals

15.1 Special Ideals

lemma (**in** *ring*) *zeroideal*:
shows *ideal* {**0**} *R*
⟨*proof*⟩

lemma (**in** *ring*) *oneideal*:
shows *ideal* (*carrier R*) *R*
⟨*proof*⟩

lemma (**in** *domain*) *zeroprimeideal*:
shows *primeideal* {**0**} *R*
⟨*proof*⟩

15.2 General Ideal Properies

lemma (**in** *ideal*) *one-imp-carrier*:
assumes *I-one-closed*: **1** ∈ *I*
shows *I* = *carrier R*
⟨*proof*⟩

lemma (**in** *ideal*) *Icarr*:
assumes *iI*: *i* ∈ *I*
shows *i* ∈ *carrier R*
⟨*proof*⟩

15.3 Intersection of Ideals

Intersection of two ideals The intersection of any two ideals is again an ideal in *R*

lemma (**in** *ring*) *i-intersect*:
includes *ideal I R*
includes *ideal J R*
shows *ideal* (*I* ∩ *J*) *R*
⟨*proof*⟩

15.3.1 Intersection of a Set of Ideals

The intersection of any Number of Ideals is again an Ideal in *R*

lemma (**in** *ring*) *i-Intersect*:
assumes *Sideals*: ⋀*I*. *I* ∈ *S* ⟹ *ideal I R*
and *notempty*: *S* ≠ {}
shows *ideal* (*Inter S*) *R*
⟨*proof*⟩

15.4 Addition of Ideals

lemma (**in** *ring*) *add-ideals*:
assumes *idealI*: *ideal I R*
and *idealJ*: *ideal J R*
shows *ideal* (*I* <+> *J*) *R*
⟨*proof*⟩

15.5 Ideals generated by a subset of *carrier R*

15.5.1 Generation of Ideals in General Rings

genideal generates an ideal

lemma (**in** *ring*) *genideal-ideal*:
assumes *Scarr*: *S* ⊆ *carrier R*
shows *ideal* (*Idl S*) *R*
⟨*proof*⟩

lemma (**in** *ring*) *genideal-self*:
assumes *S* ⊆ *carrier R*
shows *S* ⊆ *Idl S*
⟨*proof*⟩

lemma (**in** *ring*) *genideal-self'*:
assumes *carr*: *i* ∈ *carrier R*
shows *i* ∈ *Idl* {*i*}
⟨*proof*⟩

genideal generates the minimal ideal

lemma (**in** *ring*) *genideal-minimal*:
assumes *a*: *ideal I R*
and *b*: *S* ⊆ *I*
shows *Idl S* ⊆ *I*
⟨*proof*⟩

Generated ideals and subsets

lemma (**in** *ring*) *Idl-subset-ideal*:
assumes *Iideal*: *ideal I R*
and *Hcarr*: *H* ⊆ *carrier R*
shows (*Idl H* ⊆ *I*) = (*H* ⊆ *I*)
⟨*proof*⟩

lemma (**in** *ring*) *subset-Idl-subset*:
assumes *Icarr*: *I* ⊆ *carrier R*
and *HI*: *H* ⊆ *I*
shows *Idl H* ⊆ *Idl I*
⟨*proof*⟩

lemma (**in** *ring*) *Idl-subset-ideal'*:

assumes *acarr*: $a \in carrier\ R$ **and** *bcarr*: $b \in carrier\ R$
shows $(Idl\ \{a\} \subseteq Idl\ \{b\}) = (a \in Idl\ \{b\})$
⟨*proof*⟩

lemma (**in** *ring*) *genideal-zero*:
$Idl\ \{\mathbf{0}\} = \{\mathbf{0}\}$
⟨*proof*⟩

lemma (**in** *ring*) *genideal-one*:
$Idl\ \{\mathbf{1}\} = carrier\ R$
⟨*proof*⟩

15.5.2 Generation of Principal Ideals in Commutative Rings

constdefs (**structure** R)
$cgenideal :: ('a, 'b)\ monoid\text{-}scheme \Rightarrow 'a \Rightarrow 'a\ set\ \ (PIdl_1\ -\ [80]\ 79)$
$cgenideal\ R\ a \equiv \{\ x \otimes a \mid x.\ x \in carrier\ R\ \}$

genhideal (?) really generates an ideal

lemma (**in** *cring*) *cgenideal-ideal*:
assumes *acarr*: $a \in carrier\ R$
shows $ideal\ (PIdl\ a)\ R$
⟨*proof*⟩

lemma (**in** *ring*) *cgenideal-self*:
assumes *icarr*: $i \in carrier\ R$
shows $i \in PIdl\ i$
⟨*proof*⟩

cgenideal is minimal

lemma (**in** *ring*) *cgenideal-minimal*:
includes $ideal\ J\ R$
assumes *aJ*: $a \in J$
shows $PIdl\ a \subseteq J$
⟨*proof*⟩

lemma (**in** *cring*) *cgenideal-eq-genideal*:
assumes *icarr*: $i \in carrier\ R$
shows $PIdl\ i = Idl\ \{i\}$
⟨*proof*⟩

lemma (**in** *cring*) *cgenideal-eq-rcos*:
$PIdl\ i = carrier\ R\ \#>\ i$
⟨*proof*⟩

lemma (**in** *cring*) *cgenideal-is-principalideal*:
assumes *icarr*: $i \in carrier\ R$
shows $principalideal\ (PIdl\ i)\ R$
⟨*proof*⟩

15.6 Union of Ideals

lemma (**in** *ring*) *union-genideal*:
assumes *idealI*: *ideal I R*
and *idealJ*: *ideal J R*
shows *Idl* $(I \cup J) = I <+> J$
⟨*proof*⟩

15.7 Properties of Principal Ideals

0 generates the zero ideal

lemma (**in** *ring*) *zero-genideal*:
shows *Idl* $\{\mathbf{0}\} = \{\mathbf{0}\}$
⟨*proof*⟩

1 generates the unit ideal

lemma (**in** *ring*) *one-genideal*:
shows *Idl* $\{\mathbf{1}\} =$ *carrier R*
⟨*proof*⟩

The zero ideal is a principal ideal

corollary (**in** *ring*) *zeropideal*:
shows *principalideal* $\{\mathbf{0}\}$ *R*
⟨*proof*⟩

The unit ideal is a principal ideal

corollary (**in** *ring*) *onepideal*:
shows *principalideal* (*carrier R*) *R*
⟨*proof*⟩

Every principal ideal is a right coset of the carrier

lemma (**in** *principalideal*) *rcos-generate*:
includes *cring*
shows $\exists x \in I.\ I =$ *carrier R* $\#> x$
⟨*proof*⟩

15.8 Prime Ideals

lemma (**in** *ideal*) *primeidealCD*:
includes *cring*
assumes *notprime*: $\neg$ *primeideal I R*
shows *carrier R* $= I \vee (\exists a\ b.\ a \in$ *carrier R* $\wedge\ b \in$ *carrier R* $\wedge\ a \otimes b \in I \wedge a \notin I \wedge b \notin I)$
⟨*proof*⟩

lemma (**in** *ideal*) *primeidealCE*:
includes *cring*
assumes *notprime*: $\neg$ *primeideal I R*

obtains *carrier* $R = I$
| $\exists\, a\ b.\ a \in$ *carrier* $R \wedge b \in$ *carrier* $R \wedge a \otimes b \in I \wedge a \notin I \wedge b \notin I$
⟨*proof*⟩

If $\{\mathbf{0}\}$ is a prime ideal of a commutative ring, the ring is a domain

lemma (**in** *cring*) *zeroprimeideal-domainI*:
assumes *pi*: *primeideal* $\{\mathbf{0}\}$ R
shows *domain* R
⟨*proof*⟩

corollary (**in** *cring*) *domain-eq-zeroprimeideal*:
shows *domain* R = *primeideal* $\{\mathbf{0}\}$ R
⟨*proof*⟩

15.9 Maximal Ideals

lemma (**in** *ideal*) *helper-I-closed*:
assumes *carr*: $a \in$ *carrier* R $x \in$ *carrier* R $y \in$ *carrier* R
and *axI*: $a \otimes x \in I$
shows $a \otimes (x \otimes y) \in I$
⟨*proof*⟩

lemma (**in** *ideal*) *helper-max-prime*:
includes *cring*
assumes *acarr*: $a \in$ *carrier* R
shows *ideal* $\{x \in$ *carrier* $R.\ a \otimes x \in I\}$ R
⟨*proof*⟩

In a cring every maximal ideal is prime

lemma (**in** *cring*) *maximalideal-is-prime*:
includes *maximalideal*
shows *primeideal* I R
⟨*proof*⟩

15.10 Derived Theorems Involving Ideals

— A non-zero cring that has only the two trivial ideals is a field
lemma (**in** *cring*) *trivialideals-fieldI*:
assumes *carrnzero*: *carrier* $R \neq \{\mathbf{0}\}$
and *haveideals*: $\{I.$ *ideal* I $R\} = \{\{\mathbf{0}\},$ *carrier* $R\}$
shows *field* R
⟨*proof*⟩

lemma (**in** *field*) *all-ideals*:
shows $\{I.$ *ideal* I $R\} = \{\{\mathbf{0}\},$ *carrier* $R\}$
⟨*proof*⟩
lemma (**in** *cring*) *trivialideals-eq-field*:
assumes *carrnzero*: *carrier* $R \neq \{\mathbf{0}\}$
shows $(\{I.$ *ideal* I $R\} = \{\{\mathbf{0}\},$ *carrier* $R\}) =$ *field* R

⟨*proof*⟩

Like zeroprimeideal for domains

lemma (**in** *field*) *zeromaximalideal*:
  maximalideal {**0**} *R*
⟨*proof*⟩

lemma (**in** *cring*) *zeromaximalideal-fieldI*:
  assumes *zeromax*: *maximalideal* {**0**} *R*
  shows *field R*
⟨*proof*⟩

lemma (**in** *cring*) *zeromaximalideal-eq-field*:
  maximalideal {**0**} *R* = *field R*
⟨*proof*⟩

end

theory *RingHom*
imports *Ideal*
begin

16 Homomorphisms of Non-Commutative Rings

Lifting existing lemmas in a *ring-hom-ring* locale

locale *ring-hom-ring* = *ring R* + *ring S* + *var h* +
  assumes *homh*: *h* ∈ *ring-hom R S*
  notes *hom-mult* [*simp*] = *ring-hom-mult* [*OF homh*]
    and *hom-one* [*simp*] = *ring-hom-one* [*OF homh*]

interpretation *ring-hom-cring* ⊆ *ring-hom-ring*
  ⟨*proof*⟩

interpretation *ring-hom-ring* ⊆ *abelian-group-hom R S*
⟨*proof*⟩

lemma (**in** *ring-hom-ring*) *is-ring-hom-ring*:
  includes *struct R* + *struct S*
  shows *ring-hom-ring R S h*
⟨*proof*⟩

lemma *ring-hom-ringI*:
  includes *ring R* + *ring S*
  assumes
        hom-closed: !!*x*. *x* ∈ *carrier R* ==> *h x* ∈ *carrier S*

and *compatible-mult*: !!x y. [| x : *carrier* R; y : *carrier* R |] ==> h ($x \otimes y$) = h x $\otimes_S$ h y
and *compatible-add*: !!x y. [| x : *carrier* R; y : *carrier* R |] ==> h ($x \oplus y$) = h x $\oplus_S$ h y
and *compatible-one*: h $\mathbf{1}$ = $\mathbf{1}_S$
shows *ring-hom-ring* R S h
⟨*proof*⟩

lemma *ring-hom-ringI2*:
includes *ring* R + *ring* S
assumes h: $h \in$ *ring-hom* R S
shows *ring-hom-ring* R S h
⟨*proof*⟩

lemma *ring-hom-ringI3*:
includes *abelian-group-hom* R S + *ring* R + *ring* S
assumes *compatible-mult*: !!x y. [| x : *carrier* R; y : *carrier* R |] ==> h ($x \otimes y$) = h x $\otimes_S$ h y
and *compatible-one*: h $\mathbf{1}$ = $\mathbf{1}_S$
shows *ring-hom-ring* R S h
⟨*proof*⟩

lemma *ring-hom-cringI*:
includes *ring-hom-ring* R S h + *cring* R + *cring* S
shows *ring-hom-cring* R S h
⟨*proof*⟩

16.1 The kernel of a ring homomorphism

— the kernel of a ring homomorphism is an ideal
lemma (**in** *ring-hom-ring*) *kernel-is-ideal*:
shows *ideal* (*a-kernel* R S h) R
⟨*proof*⟩

Elements of the kernel are mapped to zero

lemma (**in** *abelian-group-hom*) *kernel-zero* [*simp*]:
$i \in$ *a-kernel* R S h $\Longrightarrow$ h i = $\mathbf{0}_S$
⟨*proof*⟩

16.2 Cosets

Cosets of the kernel correspond to the elements of the image of the homomorphism

lemma (**in** *ring-hom-ring*) *rcos-imp-homeq*:
assumes *acarr*: $a \in$ *carrier* R
and *xrcos*: $x \in$ *a-kernel* R S h +> a
shows h x = h a
⟨*proof*⟩

lemma (**in** *ring-hom-ring*) *homeq-imp-rcos*:
assumes *acarr*: $a \in carrier\ R$
and *xcarr*: $x \in carrier\ R$
and *hx*: $h\ x = h\ a$
shows $x \in a\text{-}kernel\ R\ S\ h +> a$
⟨*proof*⟩

corollary (**in** *ring-hom-ring*) *rcos-eq-homeq*:
assumes *acarr*: $a \in carrier\ R$
shows $(a\text{-}kernel\ R\ S\ h) +> a = \{x \in carrier\ R.\ h\ x = h\ a\}$
⟨*proof*⟩

end

17 QuotRing: Quotient Rings

theory *QuotRing*
imports *RingHom*
begin

17.1 Multiplication on Cosets

constdefs (**structure** R)
$rcoset\text{-}mult :: [('a, -)\ ring\text{-}scheme, 'a\ set, 'a\ set, 'a\ set] \Rightarrow 'a\ set$
$([mod\ -:] - \bigotimes_1 - [81,81,81]\ 80)$
$rcoset\text{-}mult\ R\ I\ A\ B \equiv \bigcup a{\in}A.\ \bigcup b{\in}B.\ I +> (a \otimes b)$

rcoset-mult fulfils the properties required by congruences

lemma (**in** *ideal*) *rcoset-mult-add*:
$[\![x \in carrier\ R;\ y \in carrier\ R]\!] \Longrightarrow [mod\ I:]\ (I +> x) \bigotimes (I +> y) = I +> (x \otimes y)$
⟨*proof*⟩

17.2 Quotient Ring Definition

constdefs (**structure** R)
$FactRing :: [('a,'b)\ ring\text{-}scheme, 'a\ set] \Rightarrow ('a\ set)\ ring$
(**infixl** *Quot 65*)
$FactRing\ R\ I \equiv$
$(\!|carrier = a\text{-}rcosets\ I, mult = rcoset\text{-}mult\ R\ I, one = (I +> \mathbf{1}), zero = I, add = set\text{-}add\ R|\!)$

17.3 Factorization over General Ideals

The quotient is a ring

lemma (**in** *ideal*) *quotient-is-ring*:
shows $ring\ (R\ Quot\ I)$

⟨*proof*⟩

This is a ring homomorphism

lemma (**in** *ideal*) *rcos-ring-hom*:
(*op +> I*) ∈ *ring-hom R* (*R Quot I*)
⟨*proof*⟩

lemma (**in** *ideal*) *rcos-ring-hom-ring*:
ring-hom-ring R (*R Quot I*) (*op +> I*)
⟨*proof*⟩

The quotient of a cring is also commutative

lemma (**in** *ideal*) *quotient-is-cring*:
includes *cring*
shows *cring* (*R Quot I*)
⟨*proof*⟩

Cosets as a ring homomorphism on crings

lemma (**in** *ideal*) *rcos-ring-hom-cring*:
includes *cring*
shows *ring-hom-cring R* (*R Quot I*) (*op +> I*)
⟨*proof*⟩

17.4 Factorization over Prime Ideals

The quotient ring generated by a prime ideal is a domain

lemma (**in** *primeideal*) *quotient-is-domain*:
shows *domain* (*R Quot I*)
⟨*proof*⟩

Generating right cosets of a prime ideal is a homomorphism on commutative rings

lemma (**in** *primeideal*) *rcos-ring-hom-cring*:
shows *ring-hom-cring R* (*R Quot I*) (*op +> I*)
⟨*proof*⟩

17.5 Factorization over Maximal Ideals

In a commutative ring, the quotient ring over a maximal ideal is a field. The proof follows "W. Adkins, S. Weintraub: Algebra – An Approach via Module Theory"

lemma (**in** *maximalideal*) *quotient-is-field*:
includes *cring*
shows *field* (*R Quot I*)
⟨*proof*⟩

end

theory *IntRing*
imports *QuotRing Int*
begin

18 The Ring of Integers

18.1 Some properties of *int*

lemma *dvds-imp-abseq*:
$[\![l\ dvd\ k;\ k\ dvd\ l]\!] \Longrightarrow abs\ l = abs\ (k::int)$
⟨*proof*⟩

lemma *abseq-imp-dvd*:
assumes *a-lk*: $abs\ l = abs\ (k::int)$
shows $l\ dvd\ k$
⟨*proof*⟩

lemma *dvds-eq-abseq*:
$(l\ dvd\ k \wedge k\ dvd\ l) = (abs\ l = abs\ (k::int))$
⟨*proof*⟩

18.2 The Set of Integers as Algebraic Structure

18.2.1 Definition of $\mathcal{Z}$

constdefs
int-ring :: *int ring* ($\mathcal{Z}$)
$int\text{-}ring \equiv (\!|carrier = UNIV,\ mult = op *,\ one = 1,\ zero = 0,\ add = op +|\!)$

lemma *int-Zcarr* [*intro*!, *simp*]:
$k \in carrier\ \mathcal{Z}$
⟨*proof*⟩

lemma *int-is-cring*:
cring $\mathcal{Z}$
⟨*proof*⟩

18.2.2 Interpretations

Since definitions of derived operations are global, their interpretation needs to be done as early as possible — that is, with as few assumptions as possible.

interpretation *int*: *monoid* [$\mathcal{Z}$]
where $carrier\ \mathcal{Z} = UNIV$
and $mult\ \mathcal{Z}\ x\ y = x * y$
and $one\ \mathcal{Z} = 1$
and $pow\ \mathcal{Z}\ x\ n = x\hat{}n$

⟨*proof*⟩

interpretation *int*: *comm-monoid* [$\mathcal{Z}$]
  where *finprod* $\mathcal{Z}$ *f A* = (*if finite A then setprod f A else arbitrary*)
⟨*proof*⟩

interpretation *int*: *abelian-monoid* [$\mathcal{Z}$]
  where *zero* $\mathcal{Z}$ = *0*
    and *add* $\mathcal{Z}$ *x y* = *x* + *y*
    and *finsum* $\mathcal{Z}$ *f A* = (*if finite A then setsum f A else arbitrary*)
⟨*proof*⟩

interpretation *int*: *abelian-group* [$\mathcal{Z}$]
  where *a-inv* $\mathcal{Z}$ *x* = − *x*
    and *a-minus* $\mathcal{Z}$ *x y* = *x* − *y*
⟨*proof*⟩

interpretation *int*: *domain* [$\mathcal{Z}$]
  ⟨*proof*⟩

Removal of occurrences of *UNIV* in interpretation result — experimental.

lemma *UNIV*:
  x ∈ *UNIV* = *True*
  A ⊆ *UNIV* = *True*
  (*ALL x* : *UNIV*. *P x*) = (*ALL x*. *P x*)
  (*EX x* : *UNIV*. *P x*) = (*EX x*. *P x*)
  (*True* −−> *Q*) = *Q*
  (*True* ==> *PROP R*) == *PROP R*
  ⟨*proof*⟩

interpretation *int* [*unfolded UNIV*]:
  partial-order [(| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |)]
  where *carrier* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) = *UNIV*
    and *le* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) *x y* = (*x* ≤ *y*)
    and *lless* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) *x y* = (*x* < *y*)
⟨*proof*⟩

interpretation *int* [*unfolded UNIV*]:
  lattice [(| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |)]
  where *join* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) *x y* = *max x y*
    and *meet* (| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |) *x y* = *min x y*
⟨*proof*⟩

interpretation *int* [*unfolded UNIV*]:
  total-order [(| *carrier* = *UNIV*::*int set*, *le* = *op* ≤ |)]
  ⟨*proof*⟩

18.2.3 Generated Ideals of $\mathcal{Z}$

lemma *int-Idl*:
$Idl_{\mathcal{Z}}\ \{a\} = \{x * a \mid x.\ True\}$
⟨*proof*⟩

lemma *multiples-principalideal*:
principalideal $\{x * a \mid x.\ True\}\ \mathcal{Z}$
⟨*proof*⟩

lemma *prime-primeideal*:
assumes *prime*: *prime* (*nat p*)
shows *primeideal* $(Idl_{\mathcal{Z}}\ \{p\})\ \mathcal{Z}$
⟨*proof*⟩

18.2.4 Ideals and Divisibility

lemma *int-Idl-subset-ideal*:
$Idl_{\mathcal{Z}}\ \{k\} \subseteq Idl_{\mathcal{Z}}\ \{l\} = (k \in Idl_{\mathcal{Z}}\ \{l\})$
⟨*proof*⟩

lemma *Idl-subset-eq-dvd*:
$(Idl_{\mathcal{Z}}\ \{k\} \subseteq Idl_{\mathcal{Z}}\ \{l\}) = (l\ dvd\ k)$
⟨*proof*⟩

lemma *dvds-eq-Idl*:
$(l\ dvd\ k \wedge k\ dvd\ l) = (Idl_{\mathcal{Z}}\ \{k\} = Idl_{\mathcal{Z}}\ \{l\})$
⟨*proof*⟩

lemma *Idl-eq-abs*:
$(Idl_{\mathcal{Z}}\ \{k\} = Idl_{\mathcal{Z}}\ \{l\}) = (abs\ l = abs\ k)$
⟨*proof*⟩

18.2.5 Ideals and the Modulus

constdefs
ZMod :: *int* => *int* => *int set*
$ZMod\ k\ r == (Idl_{\mathcal{Z}}\ \{k\})\ +>_{\mathcal{Z}}\ r$

lemmas *ZMod-defs* =
ZMod-def genideal-def

lemma *rcos-zfact*:
assumes *kIl*: $k \in ZMod\ l\ r$
shows $EX\ x.\ k = x * l + r$
⟨*proof*⟩

lemma *ZMod-imp-zmod*:
assumes *zmods*: $ZMod\ m\ a = ZMod\ m\ b$
shows $a\ mod\ m = b\ mod\ m$

⟨*proof*⟩

lemma *ZMod-mod*:
shows *ZMod m a* = *ZMod m* (*a mod m*)
⟨*proof*⟩

lemma *zmod-imp-ZMod*:
assumes *modeq*: *a mod m* = *b mod m*
shows *ZMod m a* = *ZMod m b*
⟨*proof*⟩

corollary *ZMod-eq-mod*:
shows (*ZMod m a* = *ZMod m b*) = (*a mod m* = *b mod m*)
⟨*proof*⟩

18.2.6 Factorization

constdefs
ZFact :: *int* ⇒ *int set ring*
ZFact k == $\mathcal{Z}$ *Quot* ($Idl_{\mathcal{Z}}$ {*k*})

lemmas *ZFact-defs* = *ZFact-def FactRing-def*

lemma *ZFact-is-cring*:
shows *cring* (*ZFact k*)
⟨*proof*⟩

lemma *ZFact-zero*:
carrier (*ZFact 0*) = ($\bigcup a.$ {{*a*}})
⟨*proof*⟩

lemma *ZFact-one*:
carrier (*ZFact 1*) = {*UNIV*}
⟨*proof*⟩

lemma *ZFact-prime-is-domain*:
assumes *pprime*: *prime* (*nat p*)
shows *domain* (*ZFact p*)
⟨*proof*⟩

end

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